Pseudo-symplectic methods for stochastic Hamiltonian systems

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Introduction

- We propose a systematic approach to construct explicit pseudo-symplectic schemes for stochastic Hamiltonian systems.
- This method is based on generating functions, so it is an extension of the techniques used for constructing high-order symplectic schemes for stochastic Hamiltonian systems.
- We study the order of convergence of the proposed explicit pseudo-symplectic schemes.
- The excellent long term performance of the pseudo-symplectic schemes is verified numerically.

Introduction

Stochastic Hamiltonian system (SHS)

$$
\begin{aligned}
dP_i &= -\frac{\partial H^{(0)}(P,Q)}{\partial Q_i}dt - \sum_{r=1}^{m}\frac{\partial H^{(r)}(P,Q)}{\partial Q_i}\circ dw_t^r, \quad P(0)=p,\\
dQ_i &= \frac{\partial H^{(0)}(P,Q)}{\partial P_i}dt + \sum_{r=1}^{m}\frac{\partial H^{(r)}(P,Q)}{\partial P_i}\circ dw_t^r, \quad Q(0)=q,
\end{aligned}
\tag{1}
$$

where P, Q, p, q are n-dimensional vectors with components P_i, Q_i, p_i, q_i, $i = 1, \ldots, n$, and $w_t^r, r = 1, \ldots, m$ are independent standard Wiener processes.

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where P, Q, p, q are n-dimensional vectors with components P_i, Q_i, p_i, q_i, $i = 1, \ldots, n$, and $w_t^r, r = 1, \ldots, m$ are independent standard Wiener processes.

Symplecticity of the solution $\varphi_t(y) = \left(P(t,y)^T, Q(t,y)^T\right)^T$, $y = (p^T, q^T)^T$

$$
\left(\frac{\partial \varphi_t}{\partial y}\right)^T J \left(\frac{\partial \varphi_t}{\partial y}\right) = J \quad \text{with} \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{2}
$$

Introduction

Pseudo-symplectic method of mean square order (M, N), $N > M$

$$\left\| \left(\frac{\partial \Phi_h}{\partial y_0} \right)^T J \left(\frac{\partial \Phi_h}{\partial y_0} \right) - J \right\|_{L^2(\Omega)} = \left(E \left\| \left(\frac{\partial \Phi_h}{\partial y_0} \right)^T J \left(\frac{\partial \Phi_h}{\partial y_0} \right) - J \right\|^2 \right)^{\frac{1}{2}} = \mathcal{O}(h^{N+1}) \tag{3}$$

Here the one-step numerical method $y_1 = \Phi_h(y_0)$ is of mean square order M.

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References

- G. N. Milstein, Y. M. Repin, M. V. Tretyakov, Numerical methods for stochastic systems preserving symplectic structure, SIAM J. Numer. Anal. 40 (2002) 1583-1604.
- X. Niu, J. Cui, J. Hong, Z. Liu, Explicit pseudo-symplectic methods for stochastic Hamiltonian systems, BIT Numer Math 58 (2018) 163-178.
- J. Deng, C. Anton, Y. Wong, High-order symplectic schemes for stochastic Hamiltonian systems, Commun. Cumput. Phys. 16 (2014) 169-200.

Generating functions

Expansions of generating functions

$$
\begin{aligned}
&S^1_\omega(P,q)=\sum_\alpha G^1_\alpha(P,q)J_\alpha, \quad S^2_\omega(p,Q)=\sum_\alpha G^2_\alpha(p,Q)J_\alpha,\\
&S^3_\omega(y)=\sum_\alpha G^3_\alpha(y)J_\alpha, \quad \alpha=(j_1,j_2,\dots,j_l), j_i\in\{0,1,\dots,m\}, i=1,\dots,l
\end{aligned}
\tag{4}
$$

where J_α is the multiple Stratonovich integral

$$
J_\alpha=\int_0^t\int_0^{s_l}\dots\int_0^{s_2}\circ dw^{j_1}_{s_1}\dots\circ dw^{j_{l-1}}_{s_{l-1}}\circ dw^{j_l}_{s_l} \quad dw^0_s=ds.
$$

Similarly, I_α is the multiple Ito stochastic integral

$$
I_\alpha=\int_0^t\int_0^{s_l}\dots\int_0^{s_2} dw^{j_1}_{s_1}\dots dw^{j_{l-1}}_{s_{l-1}}dw^{j_l}_{s_l}.
$$

Symplectic schemes based on truncations of generating functions

Three types of symplectic schemes

- Type 1.

$$P = p - \nabla_q S^1_\omega(P, q), \quad Q = q + \nabla_P S^1_\omega(P, q), \tag{5}$$

- Type 2.

$$Q = q - \nabla_p S^2_\omega(p, Q), \quad P = p + \nabla_Q S^2_\omega(p, Q), \tag{6}$$

- Type 3.

$$Y = y - J\nabla S^3_\omega((y + Y)/2), \tag{7}$$

where $Y = (P^T, Q^T)^T$, $y = (p^T, q^T)^T$.

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Truncations of generating functions

For a symplectic scheme of mean square order $k = 1, 2, \ldots$ we truncate the Stratonovich expansion of S^i_ω, $i = 1, 2, 3$ according to $\mathcal{A}_k = \{\alpha : l(\alpha) + n(\alpha) \leq 2k\}$. For order $k = 0.5, 1.5, \ldots$ we truncate the Ito expansion of S^i_ω, $i = 1, 2, 3$ according to $\mathcal{B}_k = \{\alpha : l(\alpha) + n(\alpha) \leq 2k$ or $l(\alpha) = n(\alpha) = k + 0.5\}$. Here $l(\alpha)$ is the length of the multi-index α and $n(\alpha)$ is the number of zero components of α.

Symplectic scheme of order 0.5 based on S^1_ω

$$P = p - \nabla_q S^1_h(P,q), \quad Q = q + \nabla_P S^1_h(P,q), \tag{8}$$

$$S^1_\omega \approx S^1_h = G^1_{(0)} I_{(0)} + \sum_{r=1}^{m} \left(G^1_{(r)} I^h_{(r)} + G^1_{(r,r)} I^h_{(0)}/2 \right), \tag{9}$$

with $I_{(0)} = J_{(0)} = h$, $I^h_{(r)} = J^h_{(r)} = \sqrt{h}\xi_h(r)$, $r = 1, \ldots, m$, with $\xi_h(r)$ bounded random variables:

$$\xi_h(r) = \begin{cases} -A_h & \text{if } \xi < -A_h \\ \xi(r) & \text{if } |\xi| \le A_h \\ A_h & \text{if } \xi > A_h, \end{cases}, \qquad A_h = \sqrt{2k|\ln h|}, k \ge 1, \tag{10}$$

where $\xi(r)$ are independent random variable $\xi(r) \sim N(0,1)$ and $k = 1$. Here $G^1_{(0)} = H^{(0)}$, $G^1_{(r)} = H^{(r)}$, $r = 1, \ldots, m$,

$$G^1_{(r,r)} = \sum_{k=1}^{n} \frac{\partial H^{(r)}}{\partial q_k} \frac{\partial H^{(r)}}{\partial P_k}, r = 0, \ldots, m \quad G^1_{(i,j)} = \sum_{k=1}^{n} \frac{\partial H^{(j)}}{\partial q_k} \frac{\partial H^{(i)}}{\partial P_k}, i \ne j.$$

Midpoint symplectic scheme of order 0.5 based on S^3_ω

Truncation of expansion of S^3_ω

$$S^3_h = \sum_{\alpha\in\mathcal{B}_{0.5}} G^3_\alpha J^h_\alpha = G^3_{(0)} I_{(0)} + \sum_{r=1}^{m} \left(G^3_{(r)} I^h_{(r)} + G^3_{(r,r)} I^h_{(0)}/2 \right), G^3_{(i)} = H^{(i)}, G^3_{(r,r)} = 0$$

.

Midpoint symplectic scheme of order 0.5 based on S_ω^3

Truncation of expansion of S_ω^3

$$S_h^3 = \sum_{\alpha \in \mathcal{B}_{0.5}} G_\alpha^3 J_\alpha^h = G_{(0)}^3 I_{(0)} + \sum_{r=1}^{m} \left(G_{(r)}^3 I_{(r)}^h + G_{(r,r)}^3 I_{(0)}^h / 2 \right), G_{(i)}^3 = H^{(i)}, G_{(r,r)}^3 = 0$$

.

Symplectic scheme of mean square order 0.5

$$\begin{aligned} Y = y &- J\nabla S_h^3 \left(\frac{y+Y}{2} \right) = y - hJ\nabla H^{(0)} \left(\frac{y+Y}{2} \right) \\ &- \sqrt{h} \sum_{r=1}^{m} J\nabla H^{(r)} \left(\frac{y+Y}{2} \right) \xi_h(r), \end{aligned} \tag{11}$$

where $Y = (P^T, Q^T)^T$, $y = (p^T, q^T)^T$, and $\xi_h(r)$, $r = 1, \ldots, m$ are given in (10) with $k = 1$.

Explicit pseudo-symplectic schemes

- For general SHSs, the symplectic schemes are implicit.
- To construct an explicit pseudo-symplectic schemes of mean square order (k, i), $i > k$, we use Taylor expansions in the implicit equations of the symplectic scheme of order k and we keep only the terms of mean square order $\mathcal{O}(h^i)$.
- Depending which type of one-step approximation we use, we get three types of pseudo-symplectic schemes.
- We focus on the explicit pseudo-symplectic schemes of mean square order $(0.5, 1)$, but similarly we can obtain higher order pseudo-symplectic schemes.

Definitions and notations

- $\mathcal{C}_b^k$ = the functions space consisting of $k-$ times continuous differentiable functions with bounded derivatives up to order k.
- For $f : \mathbb{R}^k \to \mathbb{R}$, $f \in C^2(\mathbb{R}^k)$, the gradient of $f = \nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_k} \right)^T$ and the Hessian matrix of $f = f_{xx} = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j=1,\dots,k} \in \mathcal{M}_{k,k}(\mathbb{R})$
- For any integer $s > 0$, any function $f : \mathbb{R}^k \to \mathbb{R}^l$, $f \in C^s(\mathbb{R}^k, \mathbb{R}^l)$, and any $x \in \mathbb{R}^k$, $t_i = (t_{i1}, \dots, t_{ik})^T \in \mathbb{R}^k$, $i = 1, \dots, s$ we have the differential $D^s f(x) \cdot (t_1, \dots, t_s)$ of order s of f at the point x and in the directions t_i, $i = 1, \dots, s$:

$$D^s f(x) \cdot (t_1, \dots, t_s) = \begin{pmatrix} \sum_{\alpha \in \{1,\dots,k\}^s} \frac{\partial^s f_1(x_1,\dots,x_k)}{\partial x_{\alpha_1} \dots \partial x_{\alpha_s}} t_{1\alpha_1} \dots t_{s\alpha_s} \\ \dots \\ \sum_{\alpha \in \{1,\dots,k\}^s} \frac{\partial^s f_l(x_1,\dots,x_k)}{\partial x_{\alpha_1} \dots \partial x_{\alpha_s}} t_{1\alpha_1} \dots t_{s\alpha_s} \end{pmatrix}.$$

- For $F : \mathbb{R}^k \to \mathcal{M}_{l,j}(\mathbb{R})$, $F = (F_1, \dots, F_j)$, with columns $F_1, \dots, F_j \in C^s(\mathbb{R}^k, \mathbb{R}^l)$, and any $x \in \mathbb{R}^k$, $t_i \in \mathbb{R}^k$, $i = 1, \dots, s$ we define $D^s F(x) \cdot (t_1, \dots, t_s) = (D^s F_1(x) \cdot (t_1, \dots, t_s), \dots, D^s F_j(x) \cdot (t_1, \dots, t_s)) \in \mathcal{M}_{l,j}(\mathbb{R})$.

Pseudo-symplectic schemes based on the generating function S^1_ω

Truncation of expansion of S^1_ω

$$S^1 = hH^{(0)} + \sum_{r=1}^{m} \left(H^{(r)} \sqrt{h} \xi(r) + G^1_{(r,r)} \frac{h}{2} \right),$$

with $\xi(r)$ independent random variables, $\xi(r) \sim N(0,1)$, $r = 1, \ldots, m$.

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Pseudo-symplectic scheme of mean square order (0.5,1)

$$\tilde{p} = p - \nabla_q S^1(p,q) + h \sum_{r_1,r_2=1}^{m} \xi(r_1)\xi(r_2) H^{(r_1)}_{qp}(p,q) \nabla_q H^{(r_2)}(p,q) \tag{12}$$

$$P = p - \nabla_q S^1(\tilde{p},q), \tag{13}$$

$$Q = q + \nabla_p S^1(\tilde{p},q). \tag{14}$$

Pseudo-symplectic schemes based on the generating function S^2_ω

Truncation of expansion of S^2_ω

$$S^2 = hH^{(0)} + \sum_{r=1}^{m} \left(H^{(r)} \sqrt{h} \xi(r) + G^2_{(r,r)} \frac{h}{2} \right),$$

with $\xi(r)$ independent random variables, $\xi(r) \sim N(0,1)$, $r = 1, \ldots, m$,
$G^2_{(0)} = H^{(0)}$, $G^2_{(1)} = H^{(1)}$, and $G^2_{(r,r)} = \sum_{k=1}^{n} \frac{\partial H^{(r)}}{\partial p_k} \frac{\partial H^{(r)}}{\partial Q_k}$, $r = 1, \ldots, m$.

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$G^2_{(0)} = H^{(0)}$, $G^2_{(1)} = H^{(1)}$, and $G^2_{(r,r)} = \sum_{k=1}^{n} \frac{\partial H^{(r)}}{\partial p_k} \frac{\partial H^{(r)}}{\partial Q_k}$, $r = 1, \ldots, m$.

Pseudo-symplectic scheme of mean square order (0.5,1)

$$\tilde{q} = q - \nabla_p S^2(p,q) + h \sum_{r_1, r_2 = 1}^{m} \xi(r_1)\xi(r_2) H^{(r_1)}_{pq}(p,q) \nabla_p H^{(r_2)}(p,q) \tag{15}$$

$$P = p + \nabla_q S^2(p, \tilde{q}), \tag{16}$$

$$Q = q - \nabla_p S^2(p, \tilde{q}), \tag{17}$$

Pseudo-symplectic schemes based on the generating functions S^1_ω and S^2_ω

Theorem 1

If $H^{(r)} \in \mathcal{C}^4_b$, $r = 0, \ldots, m$, then the schemes given by (12)-(14) or by (15)-(17) are pseudo-symplectic methods of mean-square order $(0.5, 1)$.

Pseudo-symplectic schemes based on the generating functions S^1_ω and S^2_ω

Theorem 1

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The idea of the proof

- We can show that the method (12)-(14) is of mean square oder 0.5 by comparing with the symplectic method (8).
- $$\left(\frac{\partial(P,Q)}{\partial(p,q)}\right) = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$
- $$\left(\frac{\partial(P,Q)}{\partial(p,q)}\right)^T J \left(\frac{\partial(P,Q)}{\partial(p,q)}\right) = \begin{pmatrix} \Sigma_{11}^T\Sigma_{21} - \Sigma_{21}^T\Sigma_{11} & \Sigma_{11}^T\Sigma_{22} - \Sigma_{21}^T\Sigma_{12} \\ \Sigma_{12}^T\Sigma_{21} - \Sigma_{22}^T\Sigma_{11} & \Sigma_{12}^T\Sigma_{22} - \Sigma_{22}^T\Sigma_{12} \end{pmatrix}.$$
- We define $\tilde{S}^1 = S^1(\tilde{p}, q)$. We have $\|\tilde{S}^1_{y_1 y_2}\|^2 \leq h|\tilde{f}_{0,0,y_1,y_2}(|\xi_1|, \dots, |\xi_m|)|$ and $\|S^1_{y_1 y_2}\|^2 \leq h|f_{0,0,y_1,y_2}(|\xi_1|, \dots, |\xi_m|)|$, with $f_{0,0,y_1,y_2}$ and $\tilde{f}_{0,0,y_1,y_2}$ polynomial functions for any $y_1, y_2 \in \{p, q\}$.

The idea of the proof (cont.)

$$\Sigma_{11} = \left(\frac{\partial P}{\partial p}\right) = I_n - \tilde{S}^1_{qp}\left(\frac{\partial \tilde{p}}{\partial p}\right), \quad \Sigma_{12} = \left(\frac{\partial P}{\partial q}\right) = -\tilde{S}^1_{qq} - \tilde{S}^1_{qp}\left(\frac{\partial \tilde{p}}{\partial q}\right)$$

$$\Sigma_{21} = \left(\frac{\partial Q}{\partial p}\right) = \tilde{S}^1_{pp}\left(\frac{\partial \tilde{p}}{\partial p}\right), \quad \Sigma_{22} = \left(\frac{\partial Q}{\partial q}\right) = I_n + \tilde{S}^1_{pq} + \tilde{S}^1_{pp}\left(\frac{\partial \tilde{p}}{\partial q}\right)$$

$$\left(\frac{\partial \tilde{p}}{\partial p}\right) = I_n - S^1_{qp} + h\sum_{r_1,r_2=1}^{m} \xi(r_1)\xi(r_2) D^1_p\left(H^{(r_1)}_{qp}\right)\cdot\left(\nabla_q H^{(r_2)}\right)$$

$$+ h\sum_{r_1,r_2=1}^{m} \xi(r_1)\xi(r_2) H^{(r_1)}_{qp} H^{(r_2)}_{qp}$$

$$\left(\frac{\partial \tilde{p}}{\partial q}\right) = -S^1_{qq} + h\sum_{r_1,r_2=1}^{m} \xi(r_1)\xi(r_2) D^1_p\left(H^{(r_1)}_{qq}\right)\cdot\left(\nabla_q H^{(r_2)}\right)$$

$$+ h\sum_{r_1,r_2=1}^{m} \xi(r_1)\xi(r_2) H^{(r_1)}_{qp} H^{(r_2)}_{qq}.$$

The idea of the proof (cont.)

$$\tilde{S}^1_{qp} = S^1_{qp} + h \sum_{r_1,r_2=1}^{m} \xi(r_1)\xi(r_2) D^1_p \left(H^{(r_1)}_{qp} \right) \cdot \left(\nabla_q H^{(r_2)} \right) + R_{1,2} \tag{18}$$

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$$\tilde{S}^1_{qq} = S^1_{qq} + h \sum_{r_1,r_2=1}^{m} \xi(r_1)\xi(r_2) D^1_p \left(H^{(r_1)}_{qq} \right) \cdot \left(\nabla_q H^{(r_2)} \right) + R_{2,2} \tag{20}$$

where $\|R_{1,2}\|^2 \leq h^3 |f_{1,2}(|\xi_1|, \ldots, |\xi_m|)|$, $\|R_{1,3}\|^2 \leq h^3 |f_{1,3}(|\xi_1|, \ldots, |\xi_m|)|$, and $\|R_{2,2}\|^2 \leq h^3 |f_{2,2}(|\xi_1|, \ldots, |\xi_m|)|$ with $f_{1,2}$, $f_{1,3}$, $f_{2,2}$ polynomial functions.

$$\left\| \left(\frac{\partial(P,Q)}{\partial(p,q)} \right)^T J \left(\frac{\partial(P,Q)}{\partial(p,q)} \right) - J \right\|_{L^2(\Omega)} = \mathcal{O}(h^2). \tag{21}$$

Pseudo-symplectic midpoint schemes

Pseudo-symplectic midpoint scheme of mean square order (0.5,1)

$$\tilde{y} = y - J\nabla S^3(y) + \frac{h}{2}\sum_{r_1,r_2=1}^{m} \xi(r_1)\xi(r_2) J H_{yy}^{(r_1)}(y) J\nabla H^{(r_2)}(y) \tag{22}$$

$$Y = y - J\nabla S^3\left(\frac{y+\tilde{y}}{2}\right), \tag{23}$$

$$S^3 = hH^{(0)} + \sqrt{h}\sum_{r=1}^{m} H^{(r)}\xi(r),$$

with $Y = (P^T, Q^T)^T$, $\tilde{y} = (\tilde{p}^T, \tilde{q}^T)^T$, $y = (p^T, q^T)^T$, and $\xi(r)$ independent random variables, $\xi(r) \sim N(0,1)$, $r = 1, \ldots, m$. .

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Theorem 2

If $H^{(r)} \in \mathcal{C}_b^4$, $r = 0, \ldots, m$, then the scheme given by (22)-(23) is a pseudo-symplectic method of mean-square order $(0.5, 1)$.

Pseudo-symplectic schemes for SHS with additive noise

Stochastic Hamiltonian systems with additive noise

$$
\begin{aligned}
dP_i &= -\frac{\partial H^{(0)}(P,Q)}{\partial Q_i}dt - \sum_{r=1}^{m} \sigma_r \circ dw_t^r, \quad P(t_0) = p, \\
dQ_i &= \frac{\partial H^{(0)}(P,Q)}{\partial P_i}dt + \sum_{r=1}^{m} \tau_r \circ dw_t^r, \quad Q(t_0) = q,
\end{aligned}
\tag{24}
$$

where $i = 1, \ldots, n$.

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\end{aligned}
\tag{24}
$$

where $i = 1, \ldots, n$.

Notice that $H^{(r)} = \sum_{i=1}^{n}(P_i\tau_r + Q_i\sigma_r)$, where σ_r and τ_r are constants, $H^{(r)}_{yy} = 0$, $y = (p^T, q^T)^T$, and $\nabla H^{(r)} = (\tau_r e^T, \sigma_r e^T)^T$, with $e \in \mathbb{R}^n$, $e = (1, \ldots, 1)^T$.

Pseudo-symplectic schemes for SHS with additive noise

Pseudo-symplectic scheme based on the generating function S^1_ω has order (1,1.5)

$$\tilde{p} = p - h\nabla_q H^0(p,q) - \sqrt{h}\sum_{r=1}^{m}\sigma_r\xi(r) \tag{25}$$

$$P = p - h\nabla_q H^0(\tilde{p},q) - \sqrt{h}\sum_{r=1}^{m}\sigma_r\xi(r), \tag{26}$$

$$Q = q + h\nabla_p H^0(\tilde{p},q) + \sqrt{h}\sum_{r=1}^{m}\tau_r\xi(r). \tag{27}$$

Pseudo-symplectic schemes for SHS with additive noise

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Midpoint pseudo-symplectic scheme based on the generating function S^3_ω has order (1,1.5)

$$\tilde{y} = y - J\nabla S^3(y), \quad Y = y - J\nabla S^3\left(\frac{y+\tilde{y}}{2}\right). \tag{28}$$

The Kobo oscillator

$$\begin{aligned} dP &= -aQdt - \sigma Q \circ dw_t^1, \quad P(0) = p, \\ dQ &= aPdt + \sigma P \circ dw_t^2, \quad Q(0) = q, \end{aligned} \tag{29}$$

where a and σ are constants.

The Kobo oscillator

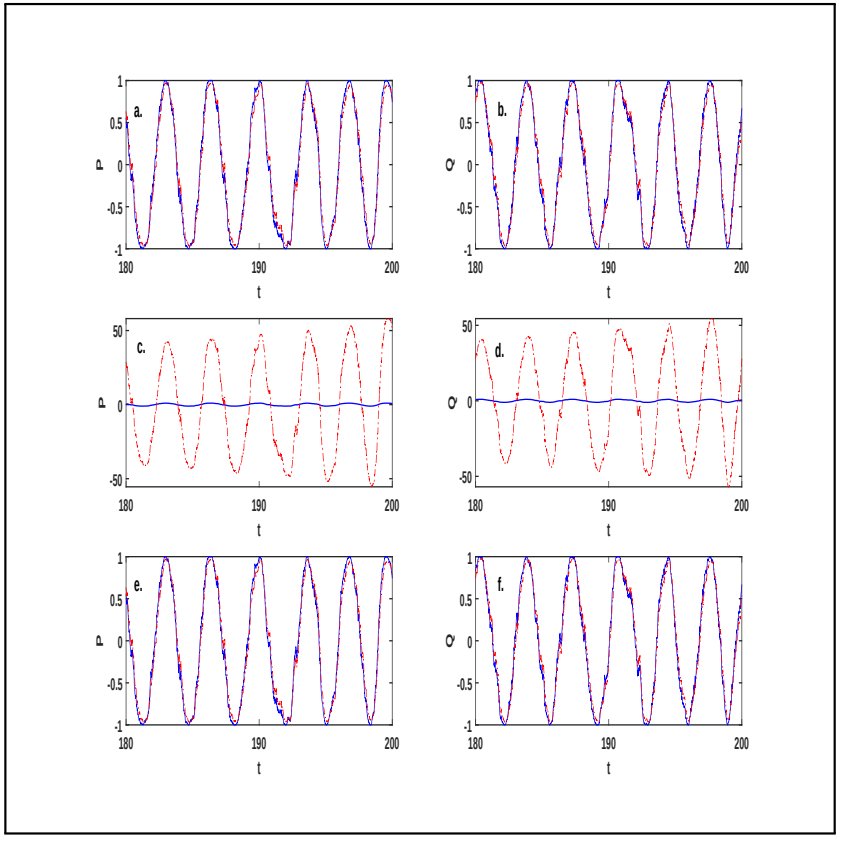
$$\begin{aligned} dP &= -aQdt - \sigma Q \circ dw_t^1, \quad P(0) = p, \\ dQ &= aPdt + \sigma P \circ dw_t^2, \quad Q(0) = q, \end{aligned} \tag{29}$$

where a and σ are constants.

- The phase trajectory of (29) lies on the circle with the center at the origin and the radius $\sqrt{p^2 + q^2}$.
- We can easily find the exact solution of the linear system (29).

Sample trajectories of Kobo oscillator with $a = 2$, $\sigma = 0.3$, $p = 1$ and $q = 0$, time step $h = 0.01$

- (a), (b) the exact solution (blue, solid line) and the S^1_ω symplectic scheme of order 0.5 (red, dot-dashed line);
- (c), (d) the exact solution (blue, solid line) and the Euler scheme (red, dot-dashed line);
- (e), (f) the exact solution (blue, solid line) and the pseudo-symplectic scheme (12)-(14) of order (0.5,1) (red, dot-dashed line)



Numerical simulations-Kobo oscillator

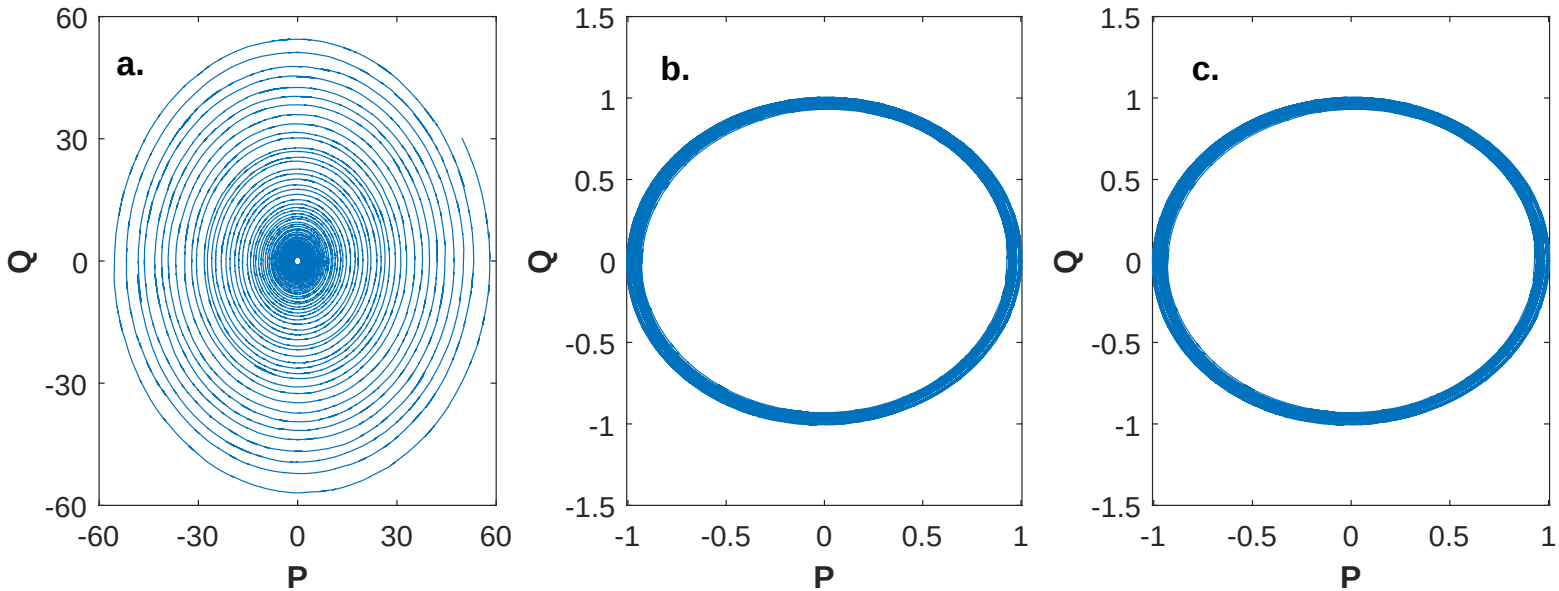


Figure: A sample phase trajectory of Kobo oscillator with $a = 2$, $\sigma = 0.3$, $p = 1$ and $q = 0$: (a) The Euler scheme; (b) S^1_ω symplectic scheme of order 0.5; (c) Pseudo-symplectic scheme (12)-(14) of order (0.5,1); with time step $h = 0.01$ on the time interval $t \leq 200$

Schemes based on the generating function S_ω^1 of mean square order 1

A symplectic scheme of order 1

$$P_{k+1} = P_k - haQ_k - \sqrt{h}\sigma Q_k \xi_k(h) - \frac{h\sigma^2 \xi_k^2(h) P_{k+1}}{2}, \tag{30}$$

$$Q_{k+1} = Q_k + haP_{k+1} + \sqrt{h}\sigma \xi_k(h) P_{k+1} + \frac{h\sigma^2 \xi_k^2(h) Q_k}{2}, \tag{31}$$

Schemes based on the generating function S^1_ω of mean square order 1

A symplectic scheme of order 1

$$P_{k+1} = P_k - haQ_k - \sqrt{h}\sigma Q_k \xi_k(h) - \frac{h\sigma^2 \xi_k^2(h) P_{k+1}}{2}, \tag{30}$$

$$Q_{k+1} = Q_k + haP_{k+1} + \sqrt{h}\sigma \xi_k(h) P_{k+1} + \frac{h\sigma^2 \xi_k^2(h) Q_k}{2}, \tag{31}$$

Pseudo-symplectic scheme of order $(1, 1.5)$: (32)- (34); and and of order $(1, 2)$: (32), (33) and (35)

$$P_{k+1} = P_k - haQ_k - \sqrt{h}\sigma Q_k \xi_k - \frac{h\sigma^2 \xi_k^2 \tilde{P}_{k+1}}{2}, \tag{32}$$

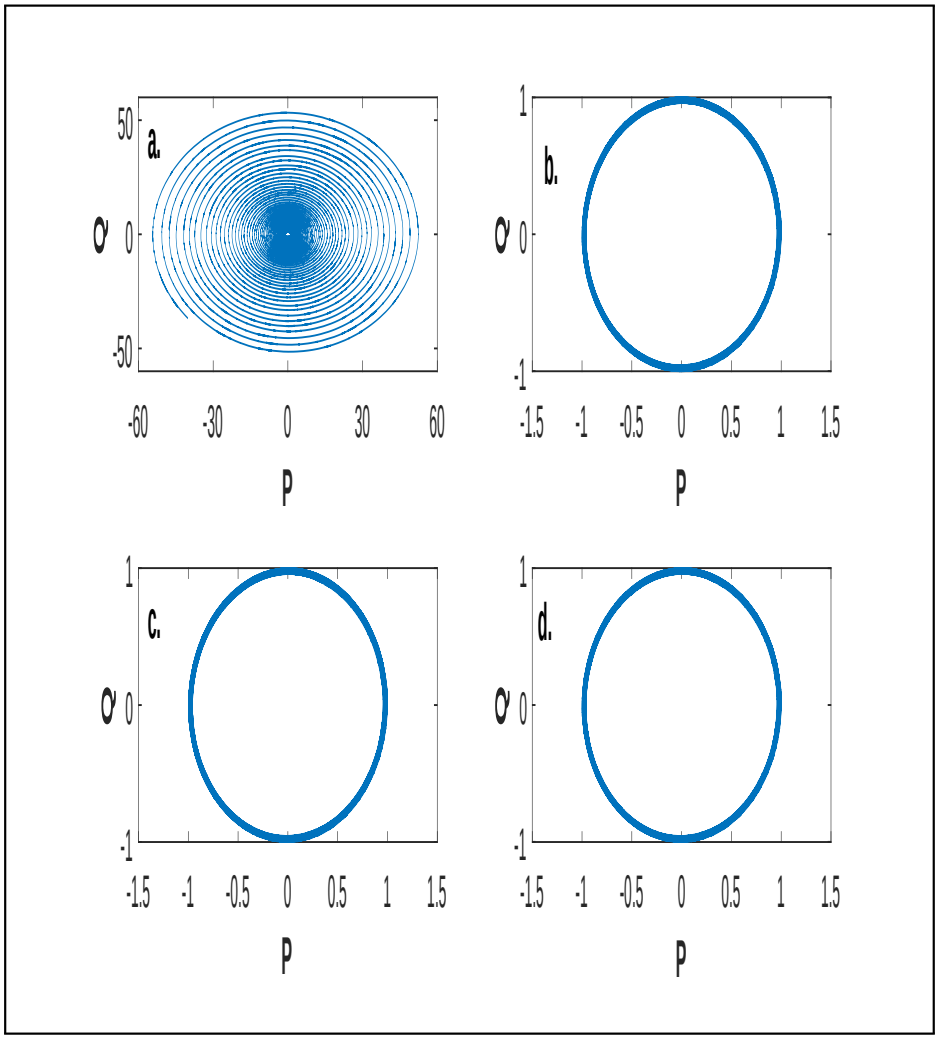
$$Q_{k+1} = Q_k + ha\tilde{P}_{k+1} + \sqrt{h}\sigma \xi_k \tilde{P}_{k+1} + \frac{h\sigma^2 \xi_k^2 Q_k}{2}, \tag{33}$$

$$\tilde{P}_{k+1} = P_k - \sqrt{h}\sigma Q_k \xi_k - haQ_k - \frac{h\sigma^2 \xi_k^2 P_k}{2} + \frac{h\sqrt{h}\sigma^3 \xi_k^3 Q_k}{2}, \tag{34}$$

$$\tilde{P}_{k+1} = P_k - \sqrt{h}\sigma Q_k \xi_k - haQ_k - \frac{h\sigma^2 \xi_k^2 P_k}{2} + \frac{h\sqrt{h}\sigma^3 \xi_k^3 Q_k}{2} + \frac{h^2\sigma^2 a\xi^2 Q_k}{2} + \frac{h^2\sigma^4 \xi^4 P_k}{4}. \tag{35}$$

A sample phase trajectory of Kobo oscillator with $a = 2$, $\sigma = 0.3$, $p = 1$ and $q = 0$, time step $h = 0.01$ on the time interval $t \leq 200$

- (a) The Milstein scheme;
- (b) S^1_ω symplectic scheme of order 1;
- (c) Pseudo-symplectic scheme (32)-(34) of order (1,1.5);
- (d) Pseudo-symplectic scheme (32), (33),(35) of order (1,2);



Midpoint schemes based on the generating function S_ω^3 of mean square order 1

A symplectic scheme of order 1, $Y_k = (P_k^T, Q_k^T)^T$

$$Y_{k+1} = Y_k - \frac{\sqrt{h}\sigma\xi_k(h)}{2} J\left(Y_k + Y_{k+1}\right) - \frac{ha}{2} J\left(Y_k + Y_{k+1}\right) \tag{36}$$

Midpoint schemes based on the generating function S_ω^3 of mean square order 1

A symplectic scheme of order 1, $Y_k = (P_k^T, Q_k^T)^T$

$$Y_{k+1} = Y_k - \frac{\sqrt{h}\sigma\xi_k(h)}{2} J\left(Y_k + Y_{k+1}\right) - \frac{ha}{2} J\left(Y_k + Y_{k+1}\right) \tag{36}$$

Scheme of mean square order 1, $Y_k = (P_k^T, Q_k^T)^T$, $\tilde{Y}_k = (\tilde{P}_k^T, \tilde{Q}_k^T)^T$

$$\tilde{Y}_{k+1} = Y_k - \sqrt{h}\sigma\xi_k JY_k - haJY_k - \frac{h}{2}\sigma^2\xi_k^2 Y_k \tag{37}$$

$$Y_{k+1} = Y_k - \frac{\sqrt{h}\sigma\xi_k}{2} J\left(Y_k + \tilde{Y}_{k+1}\right) - \frac{ha}{2} J\left(Y_k + \tilde{Y}_{k+1}\right), \tag{38}$$

$$\left\| \left(\frac{\partial Y}{\partial y}\right)^T J \left(\frac{\partial Y}{\partial y}\right) - J \right\|_{L^2(\Omega)} = \mathcal{O}(h^2),$$

Pseudo-symplectic midpoint schemes of mean square order 1

A pseudo-symplectic scheme of order (1,1.5)

$$\tilde{Y}_{k+1} = Y_k - \sqrt{h}\sigma\xi_k JY_k - haJY_k - \frac{h}{2}\sigma^2\xi_k^2 Y_k + h\sqrt{h}\left(\frac{\sigma^3\xi_k^3}{4}JY_k - \sigma a\xi_k Y_k\right) \tag{39}$$

$$Y_{k+1} = Y_k - \frac{\sqrt{h}\sigma\xi_k}{2}J\left(Y_k + \tilde{Y}_{k+1}\right) - \frac{ha}{2}J\left(Y_k + \tilde{Y}_{k+1}\right), \tag{40}$$

Pseudo-symplectic midpoint schemes of mean square order 1

A pseudo-symplectic scheme of order (1,1.5)

$$\tilde{Y}_{k+1} = Y_k - \sqrt{h}\sigma\xi_k JY_k - haJY_k - \frac{h}{2}\sigma^2\xi_k^2 Y_k + h\sqrt{h}\left(\frac{\sigma^3\xi_k^3}{4}JY_k - \sigma a\xi_k Y_k\right) \tag{39}$$

$$Y_{k+1} = Y_k - \frac{\sqrt{h}\sigma\xi_k}{2}J\left(Y_k + \tilde{Y}_{k+1}\right) - \frac{ha}{2}J\left(Y_k + \tilde{Y}_{k+1}\right), \tag{40}$$

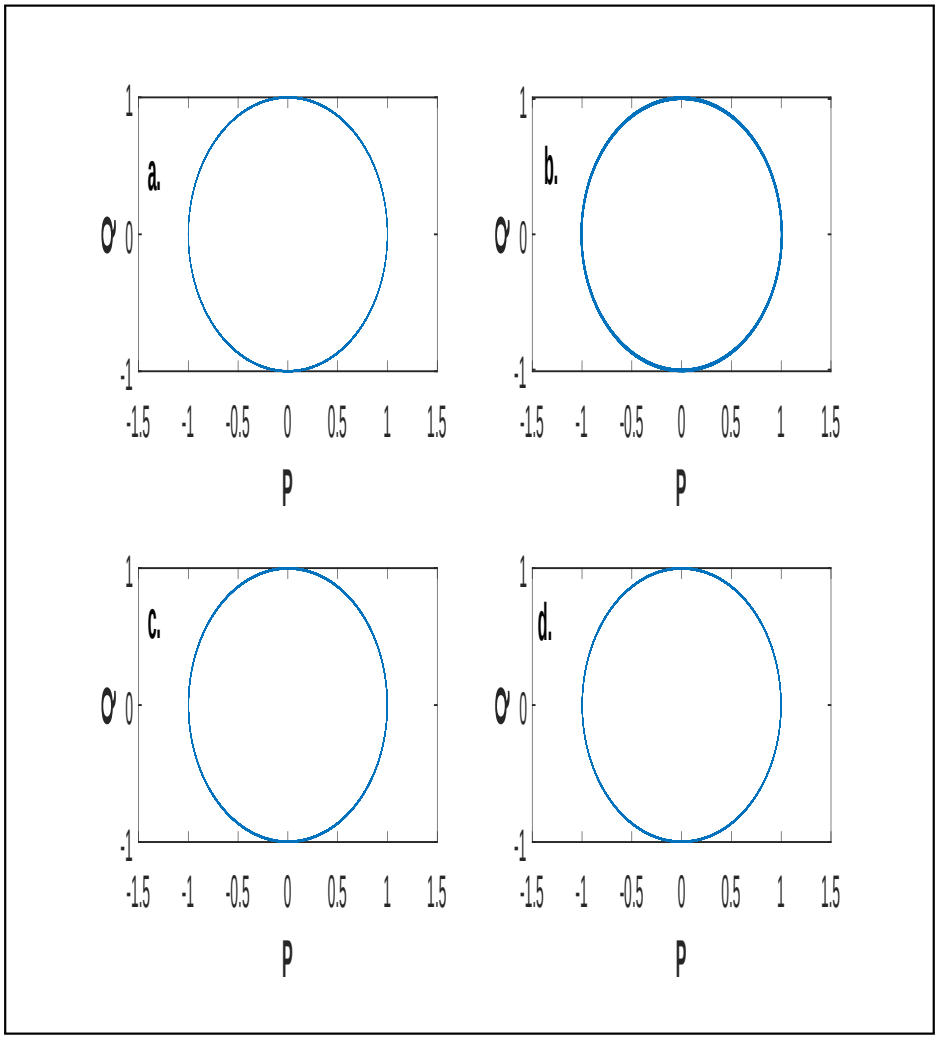
A pseudo-symplectic scheme of order (1,2)

$$\begin{aligned}\tilde{Y}_{k+1} = Y_k &- \sqrt{h}\sigma\xi_k JY_k - haJY_k - \frac{h}{2}\sigma^2\xi_k^2 Y_k + h\sqrt{h}\left(\frac{\sigma^3\xi_k^3}{4}JY_k - \sigma a\xi_k Y_k\right) \\ &+ h^2\left(\frac{3}{4}\sigma^2 a\xi_k^2 JY_k - \frac{a^2}{2}Y_k + \frac{\sigma^4\xi_k^4}{8}Y_k\right)\end{aligned} \tag{41}$$

$$Y_{k+1} = Y_k - \frac{\sqrt{h}\sigma\xi_k}{2}J\left(Y_k + \tilde{Y}_{k+1}\right) - \frac{ha}{2}J\left(Y_k + \tilde{Y}_{k+1}\right), \tag{42}$$

A sample phase trajectory of Kobo oscillator with $a = 2$, $\sigma = 0.3$, $p = 1$ and $q = 0$, time step $h = 0.01$ on the time interval $t \leq 200$

- (a) S^3_ω symplectic scheme of order 1;
- (b) Scheme (37)-(38) of mean square order 1;
- (c) Pseudo-symplectic scheme (39), (40) of order (1,1.5);
- (d) Pseudo-symplectic scheme (41), (42) of order (1,2) ;



A nonlinear oscillator

$$dP = (Q - Q^3)dt - \sigma_1 P dw_t^1 - \sigma_2 Q dw_t^2 \tag{43}$$

$$dQ = Pdt + \sigma_1 Q dw_t^1. \tag{44}$$

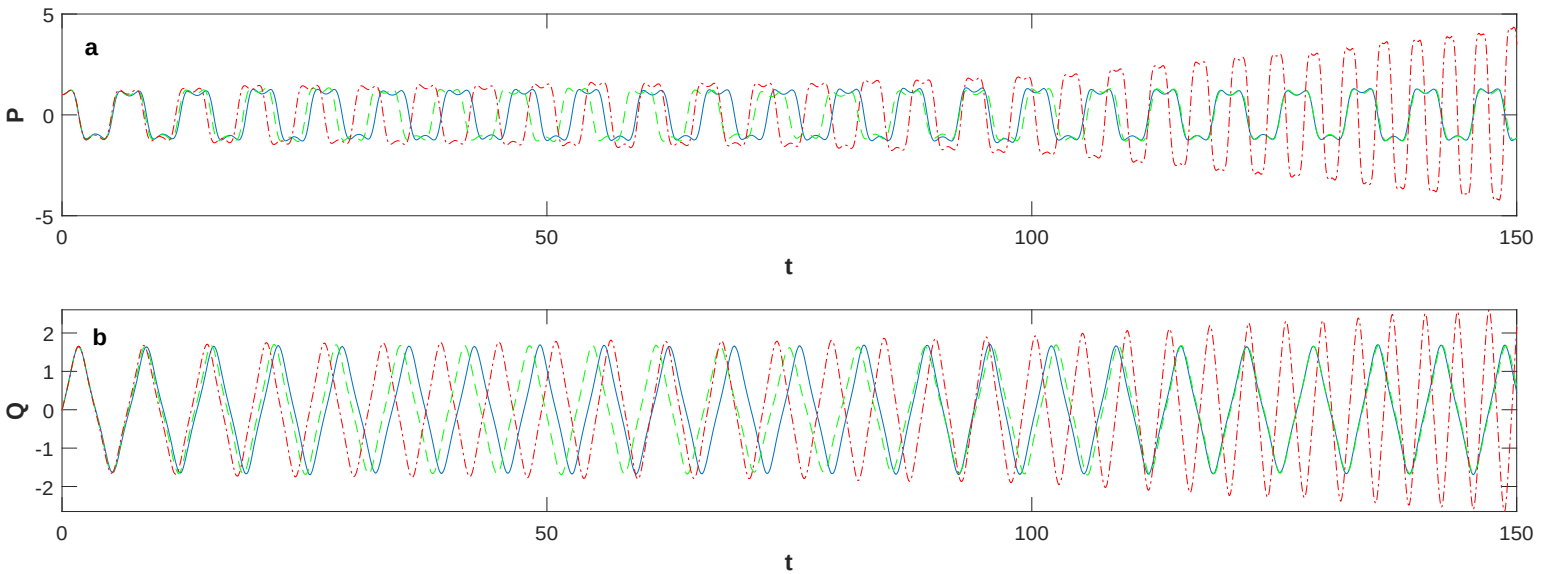


Figure: Sample trajectories for the solution of (43)-(44) obtained for $h = 0.005$, $\sigma_1 = \sigma_2 = 0.01$ with: the symplectic method (9)-(8) (blue plain line), the pseudo-symplectic method (12)-(14) (green, dashed line), and the backward Euler method (red, dot-dashed line)

Conclusions

- We present a systematic way of constructing explicit pseudo-symplectic schemes for general stochastic Hamiltonian systems.
- The explicit pseudo-symplectic schemes of order (M, N) requires the evaluations of more derivatives of the Hamiltonian functions than the explicit strong Taylor approximation method, or the implicit symplectic schemes based on generating functions of mean square order M, and the number of these derivatives increases with the pseudo-symplectic order N.
- Overall the pseudo-symplectic schemes offer a good compromise between computational time and accuracy, because they are more accurate than the the explicit strong Taylor approximation methods and less computationally expensive than the implicit symplectic schemes based on generating functions.
- It is of interest to develop high pseudo-symplectic order pseudo-symplectic Runge-Kutta schemes for general Hamiltonian systems.