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Noncommutative aspects of Villadsen algebras

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Abstract. In this short article, we survey the importance of Villadsen algebras in the context of the classification theory, with an emphasis on its non-commutative aspects.

1. Introduction

Initially developed for commutative Banach algebras, Gelfand's theory was later extended to the non-commutative setting of C^* -algebras. It provides a powerful tool for understanding the structure of C^* -algebras by associating them with a space of characters, which are certain types of continuous linear functionals. In the commutative case, this space is the maximal ideal space of the commutative C^* -algebra. Hence, an unital commutative C^* -algebra can be viewed as the algebra of continuous functions on a compact space, whereas a non-unital commutative C^* -algebra is the algebra of continuous functions that vanish at infinity on a locally compact space. The one-point compactification of a locally compact space corresponds to the smallest unitization of the algebra, while the largest compactification, the Stone-Čech compactification, corresponds to the multiplier algebra. This duality between topological spaces and the algebras allows us to think about non-commutative C^* -algebras as the study of the algebra of continuous functions on a **noncommutative space**, or more concisely, as the study of noncommutative topological spaces.

A critical feature of commutative C^* -algebra $C(K)$ is the presence of non-trivial closed two-sided ideals. Take any closed subset $F \subset K$. The continuous function in $C(K)$ that vanish on F form an ideal. In particular, if the set F is a single point, the corresponding ideal is maximal. Hence, we could expect that non-commutative C^* -algebra with no proper ideals, i.e., simple C^* -algebras, would be, in some sense, the most noncommutative C^* -algebras. This explains the interest in constructing simple C^* -algebras.

2. Preliminaries

One important tool used to study noncommutative C^* -algebras is K -theory, particularly the K_0 group. Before 1998, all known simple C^* -algebras had the property that their K_0 -group is *weakly unperforated*.

Definition 2.1. An ordered group (G, G_+) is *unperforated* if $nx \geq 0$ for some $n > 0$ implies $x \geq 0$; G is *weakly unperforated* if $nx > 0$ for some $n > 0$ implies $x > 0$.

Remark 2.1. A weakly unperforated group is unperforated if and only if it is torsion-free.



Remark 2.2. An ordered group has a perforation if there is a “hole” in the positive cone: We can think of the case $G = \mathbb{Z}$ and $G_+ = \{0, 2, 3, \dots\}$ with the missing 1 in G_+ representing the “hole.”

The covering dimension is an important invariant used to study classical topological spaces. It is natural to try to extend this concept to the setting of non-commutative topological spaces. Rieffel, in [10], introduced the topological stable rank as a non-commutative version of the covering dimension.

Definition 2.2. Given a C^* -algebra A , the topological stable rank is the least integer n such that the subset $Lg_n(A)$ of A^n , consisting of all n -tuples that generate A as a left ideal, is dense in A^n . If no such n exists, the stable rank of A is infinity.

Before 1999, all simple C^* -algebras had their topological stable rank, either 1 or infinity.

Indeed, in 1998 and 1999, Jasper Villadsen [12] constructed simple C^* -algebras that have a perforation in their K_0 -group, which is Villadsen’s first construction, followed by a second construction, [13], of simple C^* -algebras whose stable rank can take any value from the set $\{1, 2, 3, \dots\}$, labelled as Villadsen’s second construction.

To best understand the importance of Villadsen algebras, we will study them in the context of the classification theory. A significant number of mathematicians have contributed to the program of classifying C^* -algebras. It would be impossible to list them all. So, we will restrict to the ones that made landmark contributions. It all started with Murray and von Neumann [11]. Then J. Glimm [6] classified what is now known as Glimm algebras. Then, O. Bratteli [1] and G. Elliott [3] independently obtained classifications of AF-algebras. Bratteli used diagrams, now known as Bratteli diagrams, while Elliott used K-theory for the first time. The use of K-theory proved to be a significant insight. At the present time, a large class of C^* -algebras has been classified: the simple, separable, nuclear, Z-stable C^* -algebras that satisfy the Universal Coefficient Theorem (the UCT). Some key steps along the way were Kirchberg’s [8] and Phillips’s [9] classification of simple, purely infinite algebras and Elliott, Gong, Li and Niu’s [4] classification of approximately homogeneous, abbreviated AH-algebras with slow dimension growth.

Even though this class of classifiable algebras is vast, the Villadsen algebras do not belong to it by design. Initially known as exotic C^* -algebras, since the completion of the classification of Z-stable C^* -algebras, the study and the classification of Villadsen algebras have become mainstream.

3. Villadsen algebras

Villadsen algebras are a particular class of AH-algebras. The building blocks are cut down by a projection p , $pM_n(C(X)p$, where X is a compact metric space and $p \in C(X) \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators on some infinite dimensional Hilbert space. If p is a trivial projection, or equivalent to a trivial projection, as is the case for contractible spaces X , the building blocks can be assumed to be $M_n(C(X))$.

When we construct inductive limits of C^* -algebras, the maps between building blocks play an important role. For example, if the maps between the building blocks are zero, regardless of what the building blocks are, the inductive limit algebra is trivial.

When constructing Villadsen algebra, the maps are diagonal (in the first construction) and generalized diagonal (in the second).

We will restrict our attention to the first construction. The maps are diagonal, and on the diagonal, we have two kinds of eigenvalue functions: one kind is evaluation points, and a

projection gives the second kind. More explicitly, the entries on the diagonal are either of the form $f \mapsto f(y)$ for a point $y \in X$ or are of the form $f \mapsto f \circ \pi_i$ where π_i is one of the canonical projection maps $\pi_i : X^{\times m} \rightarrow X$.

If the sequence of evaluation points is dense in X , then by [2], we conclude that the inductive limit is a simple C^* -algebra. We note that Elliott, Li and Niu [5] recently showed that the inductive limit does not depend on the sequence of points evaluations as long as the sequence is dense. In their statement, the inductive limit must be simple, and the seed space can be any compact metric space.

Using a different proof, we showed, see [7], the independence of the inductive limit of the points evaluations when restricted to a seed space X , which is strongly homogeneous. Our result does not require the inductive limit to be a simple C^* -algebras.

4. Villadsen idempotents

In [7], we constructed idempotent algebras with a special seed space, namely the strongly homogeneous spaces. One example of such space is the Hilbert cube. Next, we show that more general spaces can be used to construct idempotent algebras. Before presenting the construction, we review a useful lemma that shows how base space homeomorphisms interact with the Villadsen connecting maps. The proof is found in [7].

Lemma 4.1. Suppose that $\phi : M_n \otimes C(X) \rightarrow M_k \otimes C(X^{\times m})$ is a Villadsen connecting map with evaluation points (y_i) . Let $h : X \rightarrow X$ be a given self-homeomorphism of X . Let us denote the pullback action of h on $M_n \otimes C(X)$ by $\hat{h}$. Then

$$\left(\bigoplus_m \hat{h} \right) \circ \phi \circ \hat{h}$$

is a Villadsen connecting map with evaluation points $(h(y_i))$.

Example 4.1. Based on Lemma 4.1, we exhibit a Villadsen algebra whose radius of comparison is not zero; hence, an algebra which is not $\mathcal{Z}$ -stable. For this construction, we use the notation from [5]. In the setting of Villadsen algebras of the first kind, we denote by n_i and k_i the number of projections and evaluation points, respectively, in the building block i . Assume X is the seed space, specifically a Hilbert or finite-dimensional cube. Moreover, we assume that X is a solid space as defined in [5] and we have a single evaluation at each step, i.e. $k_i = 1$, for all i , and $n_i = 4i^2 - 1$. The maps are as described in Lemma 4.1. It is shown in [5], Theorem 3.4, that the radius of comparison is the infinite product;

$$\text{radius of comparison} = \frac{\dim(X)}{2} \times \frac{n_1}{n_1 + k_1} \times \cdots \times \frac{n_i}{n_i + k_i} \times \cdots$$

With our numbers, the radius of comparison equals:

$$\text{radius of comparison} = \frac{\dim(X)}{2} \times \frac{4 \times 1^2 - 1}{4 \times 1^2} \times \cdots \times \frac{4 \times i^2 - 1}{4i^2} \times \cdots = \frac{\dim(X)}{2} \times \frac{\pi}{2};$$

a value which clearly is not zero.

Next, we use Lemma 4.1 to construct a family of Villadsen idempotents. The construction will be very similar to the one from [7]. A key ingredient is the special form of the connecting maps. However, finite-dimensional connected manifolds or infinite-dimensional manifolds are selected as the seed space instead of the strongly homogeneous spaces like the Hilbert cube.

The following lemma was shown to us by Allen Hatcher.

Lemma 4.2. Assume M is a smooth connected manifold. Then, there is a sequence of homeomorphisms $h_n : M \rightarrow M$ such that $\{h_n(x_0)\}_n$ is a dense set in M and $\text{dist}(h_n, h_{n+1}) < \frac{1}{n}$.

Proof. Fix a point x_0 in M , and for any other point $z \in M$ choose a smooth path from x_0 to z defined by a smooth map $f : [0, 1] \rightarrow M$ with $f(0) = x_0$ and $f(1) = z$. We can assume f is injective and has nonzero derivative $f'(t)$ for each t . These derivatives give a vector field along the path, and this can be extended to a smooth vector field on M that is 0 outside a neighbourhood of the path. This vector field generates a flow, a one-parameter group of diffeomorphisms $F_s : M \rightarrow M$ for $s \in \mathbb{R}$, with F_0 the identity and $F_{s+s'}$ equal to the composition $F_s F_{s'}$. After parametrization, we can assume $F_1(x_0) = z$. Then the n -th compositional power of $F_{1/n}$ is F_1 , so z is in the orbit of x_0 under $F_{1/n}$. For large enough n , $F_{1/n}$ will be close to the identity. The upshot is that we have a mapping, $F := F_{1/n}$, that is as close as we like to the identity mapping and has the property that $F^n(x_0) = z$. Thus we obtain a finite orbit $\{F^k(x_0)\}_{k=0}^{k=n}$ where the first point is x_0 and the final point is z . We can then repeat the construction with z instead of x_0 and a newly chosen z' instead of z , obtaining a finite orbit passing through x_0 and through z . We can continue, in effect, inserting arbitrarily large finite sets into the orbit, and hence we can choose the orbit to be dense. $\square$

Next, we consider a sequential inductive limit of Villadsen's form, where each object is the tensor square of the preceding object, and each connecting map is, up to a perturbation, the corresponding doubling of the previous connecting map. Succinctly put, the perturbations are chosen large enough to obtain a simple (Villadsen) C^* -algebra but small enough to get the desired idempotence property.

It is a general fact in analysis that repeated squaring will, when it converges, produce an idempotent.

Example 4.2. We consider the case where the initial object is $C(M)$, where M is a finite-dimensional, connected manifold. Let $A_1 := M_2(C(M))$, and let $\phi_{\{y\}}$ denote a Villadsen connecting map with one evaluation point, $y \in M$. Let $h_n : M \rightarrow M$ be a sequence of homeomorphisms such that:

- The orbit of the point $y \in M$ under the h_n is dense in M
- The homeomorphisms $h_n : M \rightarrow M$ satisfy $\text{dist}(h_n, h_{n+1}) < \frac{1}{n}$.

Now, the first property ensures that, given Lemma 4.1, the inductive limit

$$V_2 = \varinjlim_{i \rightarrow \infty} \left(A^{\otimes 2^{i-1}}, \bigoplus_{2^i} \hat{h}_i \circ \phi_{\{y\}} \circ \hat{h}_i \right) \quad (1)$$

has a dense set of evaluation points $\{h_i(y)\}_{i=1}^\infty$ and is therefore simple. The second property is sufficient to ensure that we obtain the desired idempotence property. To see this in detail, we shall compare the obtained inductive limit decomposition of V_2 with that of $V_2 \otimes V_2$. We exhibit both limits in the rows of the diagram below, where we denote the objects of the inductive limit by A_i and denote the connecting maps by ϕ_i . Then we have:

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\phi_1} & A_2 & \xrightarrow{\phi_2} & A_3 & \xrightarrow{\phi_3} & A_4 & \xrightarrow{\phi_4} & A_5 \cdots & \xrightarrow{\quad \quad \quad} & V_2 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & A_1 \otimes A_1 & \xrightarrow{\phi_1 \oplus \phi_1} & A_2 \otimes A_2 & \xrightarrow{\phi_2 \oplus \phi_2} & A_3 \otimes A_3 & \xrightarrow{\phi_3 \oplus \phi_3} & A_4 \otimes A_4 \cdots & \xrightarrow{\quad \quad \quad} & V_2 \otimes V_2 \end{array} \quad (2)$$

The vertical arrows are the natural isomorphisms of $A_i \otimes A_i$ with A_{i+1} , and the diagram is asymptotically commutative because of how the homeomorphisms were chosen. This shows that

$V_2 \cong V_2 \otimes V_2$, as claimed. Setting the initial object A_1 equal to $M_p(C(M))$, we obtain from the above construction a family of idempotents, V_p .

5. Conclusion

We present a method of constructing idempotent C^* -algebras. This construction appears amenable to extensions to obtain non $\mathcal{Z}$ -stable C^* -algebras and Villadsen algebra of the second kind.

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