



On Eigenmeasures Under Fourier Transform

Michael Baake¹ · Timo Spindeler¹ · Nicolae Strungaru²

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Abstract

Several classes of tempered measures are characterised that are eigenmeasures of the Fourier transform, the latter viewed as a linear operator on (generally unbounded) Radon measures on $\mathbb{R}^d$. In particular, we classify all periodic eigenmeasures on $\mathbb{R}$, which gives an interesting connection with the discrete Fourier transform and its eigenvectors, as well as all eigenmeasures on $\mathbb{R}$ with uniformly discrete support. An interesting subclass of the latter emerges from the classic cut and project method for aperiodic Meyer sets. Finally, we construct a large class of eigenmeasures with locally finite support that is not uniformly discrete and has large gaps around 0.

Keywords Fourier eigenmeasures · Poisson summation formula · Quasicrystals

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1 Introduction

It is a well-known fact that the Fourier transform on $\mathbb{R}$, more precisely its extension to the Fourier–Plancherel transform on the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$, is a unitary operator of order 4, with eigenvalues $\{1, i, -1, -i\}$. A complete basis of eigenfunctions in $\mathcal{H}$ can be given in terms of the classic Hermite functions [53, Thm. 57]. In physics,

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✉ Michael Baake
mbaake@math.uni-bielefeld.de

Timo Spindeler
tspindel@math.uni-bielefeld.de

Nicolae Strungaru
strungarun@macewan.ca

¹ Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany

² Department of Mathematical Sciences, MacEwan University, 10700 104 Avenue, Edmonton, AB T5J 4S2, Canada

these functions are well known as the eigenfunctions of the classic harmonic oscillator in quantum mechanics, which is a self-adjoint operator on $\mathcal{H}$ that commutes with the Fourier transform; see [13, Sect. 34] as well as [14], and Sect. 2 for details and our conventions.

The corresponding statement for $L^2(\mathbb{R}^d)$ can easily be derived from here, and is helpful in many applications of the Fourier transform [14, 54], both in mathematics and in physics. More generally, eigenfunctions of Fourier cosine and sine transforms as well as Hankel, Mellin and various other integral transforms have been studied, see [53, Ch. IX], in the setting of different function spaces. Clearly, these results can be reformulated in the realm of *finite*, absolutely continuous measures, and can be extended by taking limits in suitable topologies. This point of view obviously harvests the self-duality of the locally compact Abelian group $\mathbb{R}^d$, while the important analogues for mutually dual pairs of groups, such as the integer lattice $\mathbb{Z}^d$ and the corresponding torus $\mathbb{T}^d$, lead to the theory of Fourier series and its applications [14, 23, 42], as well as to the discrete Fourier transform.

There is one famous and well-known extension to measures as follows. If δ_x denotes the normalised Dirac (or point) measure at x , the lattice Dirac comb $\delta_{\mathbb{Z}^d} := \sum_{m \in \mathbb{Z}^d} \delta_m$ satisfies

$$\widehat{\delta_{\mathbb{Z}^d}} = \delta_{\mathbb{Z}^d}, \quad (1)$$

where $\widehat{\cdot}$ denotes Fourier transform. This formula means that

$$\sum_{x \in \mathbb{Z}^d} f(x) = \sum_{y \in \mathbb{Z}^d} \widehat{f}(y) \quad (2)$$

holds for all Schwartz functions f , which is nothing but the Poisson summation formula (PSF) for $\mathbb{Z}^d$, written in terms of a lattice Dirac comb; see [5, 10] as well as [2, Sect. 9.2] and references therein. In fact, (2) holds for the Feichtinger algebra $S_0(\mathbb{R}^d)$ [17, Thm. 4.2], which gives a stronger version of the PSF because $S_0(\mathbb{R}^d)$ contains all Schwarz functions [16, Thm. 9]. The PSF says that $\delta_{\mathbb{Z}^d}$ is an eigenmeasure of the Fourier transform with eigenvalue 1. More precisely, in contrast to the function case mentioned above, it is an example of an *unbounded*, but still translation-bounded, Radon eigenmeasure with uniformly discrete support.

In mathematics, the PSF (for $d = 1$) is related with the functional equation of Riemann's zeta function, via a Mellin transform, while its relevance in physics comes from its role in the understanding of the diffraction theory of crystals. By the latter, we mean lattice-periodic arrangements of a fundamental motif, which shows a pure point (or pure Bragg) diffraction spectrum in scattering experiments with X-rays or other particle beams. The spectrum can then be understood, both qualitatively and quantitatively, via the Fourier transform of lattice-periodic measures of the form $\varrho * \delta_\Gamma$ with a finite motif ϱ and a general lattice Γ in $\mathbb{R}^d$; see [2, Sect. 9.2] for a detailed exposition. The crucial point here is that the lattice can be extracted from the support of the spectrum, while details of the motif need the solution of the difficult inverse problem of crystallographic structure analysis [11].

More generally, and perhaps more importantly as well, the PSF also underlies the pure point nature of the diffraction spectra of perfect quasicrystals [2, 20, 25, 44], which is an important branch in the theory of aperiodic order. This property emerges from the description of such quasicrystals as the partial projection of a higher-dimensional lattice (or its generalisation in the setting of locally compact Abelian groups [34, 37]), where the embedding lattice and its PSF drives the spectral nature. While the original approach did not use the PSF, it was suspected to be the key ingredient [25], and later established as such [2, 41]. Crystals and quasicrystals together form an interesting class of structures within the larger universe of almost periodic measures; see [38] for a recent survey. Moreover, one can embed crystals and quasicrystals into the larger class of modulated (quasi-)crystals, whose spectral structure is still determined by an underlying PSF. A unified treatment of this entire class has recently been achieved in [26].

In this context, it is thus a natural question whether other unbounded eigenmeasures of the Fourier transform exist, which eigenvalues occur, and what they might mean. An interesting example, based on work by Guinand [19], is discussed in [34]. In our terminology, it is a tempered eigenmeasure with eigenvalue i , but the measure does not have a uniformly discrete support. This is a particularly striking example of what is now known as a ‘crystalline’ measure [24, 31, 34], and first results indicate that the underlying measure class deserves further study. In particular, it has an interesting overlap with the class of mathematical quasicrystals, which includes the measure $\delta_{\mathbb{Z}^d}$ mentioned earlier. This observation, in conjunction with the general questions put forward in [25] and the potential relevance to quasicrystals, inspired us to take a systematic look at translation-bounded eigenmeasures, in particular with uniformly discrete support, though we also look beyond. The connection between the diffraction and the dynamical spectrum for translation bounded measures, see [4, 12] and references therein, has potential implications of our findings for the spectral theory of dynamical systems, but we leave this to future work. The paper is organised as follows.

After some preliminaries in Sect. 2, we determine the possible eigenvalues and some first solutions, in the setting of Radon measures, in Sect. 3. Here, we also derive a simple characterisation of eigenmeasures in terms of 4-cycles under the Fourier transform (Theorem 3.5). While this is classic material, and can thus be viewed as a condensed summary, we are paying more attention to the difference between distributions and measures, which is necessary outside the class of positive measures [6]. Then, for $d = 1$, we consider lattice-periodic solutions (Sect. 4) more closely, which are related with eigenvectors of the *discrete Fourier transform* (DFT), that is, the Fourier transform on the finite cyclic group C_n , which is also self-dual. In this periodic case, all eigenmeasures are automatically pure point measures that are supported in a lattice (Theorem 4.3).

Next, in Sect. 5, we consider solutions that implicitly emerge from the cut and project method of aperiodic order (Theorem 5.3 and Corollary 5.7). Then, we characterise the general class of eigenmeasures with uniformly discrete support in Sect. 6, leading to Theorem 6.3, which builds on recent results from [30, 32]. Finally, we construct lattice-based eigenmeasures with a large gap around 0, which is possible via the connection with the DFT. The crucial point here is that this construction, via suitable linear combinations, paves the way to Fourier eigenmeasures with locally finite, but

no longer uniformly discrete, support (Theorem 7.5 and Corollary 7.6). We close with a brief outlook, and provide an Appendix with a streamlined and tailored approach to the DFT.

2 Preliminaries

The *Fourier transform* of a function $g \in L^1(\mathbb{R})$ is defined by

$$\widehat{g}(y) = \int_{\mathbb{R}} e^{-2\pi ixy} g(x) \, dx,$$

which is bounded and continuous, while the inverse transform is then provided by $\check{g}(y) = \widehat{g}(-y)$. If $\check{g} \in L^1(\mathbb{R})$, one has $\widehat{\check{g}} = g$. For the (complex) Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ with inner product

$$\langle g|f \rangle := \int_{\mathbb{R}} \overline{g(x)} f(x) \, dx$$

and norm $\|f\|_2 := \sqrt{\langle f|f \rangle}$, the Fourier transform on $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ has a unique extension to a unitary operator $\mathcal{F}$ on $L^2(\mathbb{R}) = \mathcal{H}$, which is often called the *Fourier–Plancherel transform*; see [8, Thm. 2.5]. The corresponding statements hold for $\mathbb{R}^d$ as well. Here, $\mathcal{F}$ satisfies the relation $\mathcal{F}^2 = I$, where I is the involution defined by $(Ig)(x) = g(-x)$. Consequently, one also has $\mathcal{F}^4 = \text{id}$, which implies that any eigenvalue of $\mathcal{F}$ must be a fourth root of unity.

It is well known that $\mathcal{H}$ possesses an ON-basis of eigenfunctions, which can be given via the Hermite functions. They naturally also appear as the eigenfunctions of the harmonic oscillator in quantum mechanics, which is a self-adjoint operator that acts on $\mathcal{H}$ and commutes with $\mathcal{F}$; compare [13, Sect. 34] and [14]. Concretely, for $n \in \mathbb{N}_0$, one can employ the Hermite polynomials H_n defined by the recursion

$$H_{n+1}(x) = 2x H_n(x) - 2n H_{n-1}(x)$$

for $n \in \mathbb{N}$ together with $H_0(x) = 1$ and $H_1(x) = 2x$. Then, the (normalised) Hermite functions

$$h_n(x) := \left(\frac{\sqrt{2}}{2^n n!} \right)^{1/2} H_n(\sqrt{2\pi} x) e^{-\pi x^2}$$

satisfy the orthonormality relations $\langle h_m|h_n \rangle = \delta_{m,n}$ for $m, n \in \mathbb{N}_0$.

Moreover, for $n \geq 0$, one has

$$\mathcal{F}(h_n) = \widehat{h_n} = (-i)^n h_n,$$

which means that the Hermite functions are eigenfunctions of $\mathcal{F}$; see [2, Rem. 8.3] as well as [54, §6] or [14, Sect. 2.5]. In Dirac's intuitive bra-c-ket notation [13, Sect. 14], one thus obtains the spectral theorem for the unitary operator $\mathcal{F}$ as

$$\mathcal{F} = \sum_{n=0}^{\infty} |h_n\rangle (-i)^n \langle h_n|,$$

which also entails the relation $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, with the four subspaces $\mathcal{H}_\ell = \text{span}_{\mathbb{C}} \{h_{4m+\ell} : m \in \mathbb{N}_0\}$.

Next, let xy denote the inner product of $x, y \in \mathbb{R}^d$ and let μ be a finite measure on $\mathbb{R}^d$. Then, its Fourier transform [42] is the continuous function on $\mathbb{R}^d$ given by

$$\widehat{\mu}(y) := \int_{\mathbb{R}^d} e^{-2\pi i xy} d\mu(x).$$

As the Fourier transform of a finite measure is a continuous function, nothing much of interest happens beyond what we summarised above. All this can be understood by taking limits of absolutely continuous measures (on the basis of sums of Hermite functions as Radon–Nikodym densities) in suitable topologies. Consequently, we proceed with the extension to *unbounded* measures, where the very notion of Fourier transformability is more delicate. The simplest approach works via distribution theory, which we shall adopt here. From now on, unless specified otherwise, the term *measure* will always mean a Radon measure, the latter viewed as a linear functional on $C_c(\mathbb{R}^d)$.

Since $C_c^\infty(\mathbb{R}^d) \subset C_c(\mathbb{R}^d)$, every measure defines a distribution. Now, a measure on $\mathbb{R}^d$ is *tempered* if this distribution is tempered, that is, a continuous linear functional on Schwartz space, $\mathcal{S}(\mathbb{R}^d)$. A tempered measure μ is called *Fourier transformable*, or *transformable* for short, if it is transformable as a distribution and if the transform is again a measure. The distributional transform of μ will interchangeably be denoted by $\mathcal{F}(\mu)$ and by $\widehat{\mu}$. When the measure μ is transformable in this sense, its transform $\nu = \widehat{\mu}$ is a tempered measure, and hence transformable as a distribution, then with $\widehat{\nu} = \mathcal{F}^2(\mu) = I_*\mu$, where the latter means the push-forward of the space inversion defined by $I(x) = -x$, to be discussed in more detail later. As a consequence, $\widehat{\nu}$ is again a measure, and we have the following useful property.

Fact 2.1 *If a tempered measure is transformable, it is also multiply transformable.* $\square$

Since $\mathbb{R}^d$ is a special case of a locally compact Abelian group, let us also mention the classic approach to the transformability of a measure, which completely avoids the use of distribution theory, and is not restricted to tempered measures.

Definition 2.2 A Radon measure μ is called *transformable in the strict sense*, or *strictly transformable* for short, if it is transformable as a measure, in the sense of [1, 8]. This means that there exists some measure $\widehat{\mu}$ on $\mathbb{R}^d$ with the property that, for all $\varphi \in C_c(\mathbb{R}^d)$ with $|\check{\varphi}| \in L^1(\mathbb{R}^d)$, we have $\mu(\varphi) = \widehat{\mu}(\check{\varphi})$.

We refer to [38, 49] for the systematic exposition of recent developments and results. Note that a transformable, tempered measure need not be strictly transformable. By slight abuse of notation, we shall use $\mathcal{F}$ for both versions, as the context will always be clear. As the domain of two successive applications of $\mathcal{F}$ in the strict sense is now generally smaller than that of $\mathcal{F}$, one also needs the notion of double transformability

here. A measure μ is called *doubly* (or *twice*) transformable in the strict sense if μ is strictly transformable as a measure and its transform, $\mathcal{F}(\mu)$, is again transformable in the strict sense.

A measure μ on $\mathbb{R}^d$ is called *translation bounded* if $\sup_{t \in \mathbb{R}^d} |\mu|(t + K) < \infty$ holds for some (and then any) compact set $K \subset \mathbb{R}^d$ with non-empty interior, where $|\mu|$ is the total variation of μ . This notion is crucial in many ways. In particular, every translation-bounded measure is tempered [1, Thm. 7.1]. Further, strict transformability of tempered measures can most easily be decided via an application of [50, Thm. 5.2] as follows; see also [52].

Fact 2.3 *A tempered measure μ on $\mathbb{R}^d$ is strictly transformable as a measure if and only if it is transformable as a tempered distribution such that $\widehat{\mu}$ is a translation-bounded measure. Moreover, the two transforms agree in this case.* $\square$

One important consequence of Fact 2.3 is that a strictly transformable eigenmeasure is automatically translation bounded. However, this type of approach via distributions is only available over $\mathbb{R}^d$. A characterisation of transformable measures on locally compact Abelian groups can be found (without proof) in [15, Thm. C1]. For some subtleties around tempered measures, once one goes beyond the class of positive measures, we also refer to [6].

Remark 2.4 Recall that a measure μ is *slowly increasing* if

$$\int_{\mathbb{R}^d} \frac{d|\mu|(x)}{1 + |P(x)|} < \infty$$

holds for some polynomial P . By [43, p. 242], a positive measure is tempered if and only if it is slowly increasing. A slowly increasing measure is always tempered, but the converse is generally not true; compare [1, Prop. 7.1].

Some authors [22] call a Radon measure μ *tempered* when

$$\int_{\mathbb{R}^d} |f(x)| d|\mu|(x) < \infty$$

holds for all $f \in \mathcal{S}(\mathbb{R}^d)$ and when $f \mapsto \int_{\mathbb{R}^d} f(x) d\mu(x)$ defines a tempered distribution. Note that a measure is tempered in this sense if and only if it is slowly increasing [6, Thm. 2.7]. The somewhat subtle but relevant issue with this definition is that the example from [1] mentioned above provides an example of a Radon measure μ that is not slowly increasing, but whose restriction to $C_c^\infty(\mathbb{R}^d)$ can be extended to a tempered distribution. Let us note in passing that, for this measure, there exists $f \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} |f(x)| d|\mu|(x) = \infty;$$

see [6] for details. $\diamond$

By [41, Thm. 3.12], a strictly transformable measure μ on $\mathbb{R}^d$ is doubly transformable in the strict sense if and only if μ is translation bounded. In this case, now by [38, Thm. 4.9.28], one actually gets $\mathcal{F}^2(\mu) = I.\mu$ as before.

Definition 2.5 A non-trivial tempered measure μ on $\mathbb{R}^d$ is called an *eigenmeasure* of $\mathcal{F}$, or an eigenmeasure under Fourier transform, if it is transformable together with $\mathcal{F}(\mu) = \lambda\mu$ for some $\lambda \in \mathbb{C}$, which is the corresponding *eigenvalue*.

Further, we call an eigenmeasure μ *strict* when it is transformable in the strict sense and satisfies $\mathcal{F}(\mu) = \lambda\mu$ in the sense of measures.

When the support of a measure μ has special discreteness properties, one can profitably employ the following equivalence.

Lemma 2.6 *If $\mu \neq 0$ is a tempered, pure point measure on $\mathbb{R}^d$ with uniformly discrete support, the following properties are equivalent.*

- (1) *The measure μ is strictly transformable, and $\widehat{\mu} = \lambda\mu$ holds as an equation of measures for some $0 \neq \lambda \in \mathbb{C}$.*
- (2) *The measure μ satisfies $\widehat{\mu} = \lambda\mu$ in the distributional sense for some $0 \neq \lambda \in \mathbb{C}$.*

Proof Let us first note that $\lambda = 0$ implies $\mu = 0$, so we must indeed have $\lambda \neq 0$.

To verify (1) $\Rightarrow$ (2), we simply observe that strict transformability of μ implies its (distributional) transformability by Fact 2.3, and the two transforms agree.

Conversely, [7, Lemma 6.3] implies that $\widehat{\mu}$ is translation bounded, and the claim follows again from Fact 2.3. $\square$

These characterisations will cover essentially all situations we encounter below. Let us first show that the eigenmeasure notion is a consistent one, and determine the possible eigenvalues.

3 Eigenvalues and Structure of Solutions

Recall that $\mathcal{S}(\mathbb{R}^d)$ denotes Schwartz space [43], and that I is the reflection in 0, so $I(x) = -x$. As above, we let I also act on functions, via $I(g) := g \circ I$. Accordingly, when μ is a measure, we define the push-forward $I.\mu$ by $(I.\mu)(g) = \mu(g \circ I)$.

Let $\mu \neq 0$ be a tempered measure, so $\mu \in \mathcal{S}'(\mathbb{R}^d)$, where the latter denotes the space of tempered distributions on $\mathbb{R}^d$. If μ is an eigenmeasure of $\mathcal{F}$, hence $\mathcal{F}(\mu) = \lambda\mu$, we get $\mathcal{F}^4(\mu) = \lambda^4\mu$ by repetition and Fact 2.1. On the other hand, given any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and observing the relation $\mathcal{F}^2(\varphi) = \varphi \circ I$, one has

$$\mathcal{F}^2(\mu)(\varphi) = \mu(\mathcal{F}^2(\varphi)) = \mu(\varphi \circ I) = (I.\mu)(\varphi),$$

hence $\mathcal{F}^2(\mu) = I.\mu$ and thus $\mathcal{F}^4(\mu) = \mu$ due to $I^2 = \text{id}$. This implies $\lambda^4 = 1$ as in the case of functions, so λ must be a fourth root of unity. Note that we also get $I.\mu = \mathcal{F}^2(\mu) = \lambda^2\mu$, for any of the possible eigenvalues. This has an immediate consequence as follows.

Fact 3.1 Let μ be any tempered measure on $\mathbb{R}^d$ that is an eigenmeasure of $\mathcal{F}$, hence $\widehat{\mu} = \lambda\mu \neq 0$. Then, $\lambda \in \{1, i, -1, -i\}$, and one has $I.\mu = \lambda^2\mu$. In particular, the support of μ is symmetric, $\text{supp}(\mu) = -\text{supp}(\mu)$, and μ satisfies $\mu(\{-x\}) = \lambda^2\mu(\{x\})$ for all $x \in \mathbb{R}^d$. $\square$

Let us verify that all $\lambda \in \{1, i, -1, -i\}$ actually occur. To this end, we use $\mathcal{F}^4 = \text{id}$ and consider measures $\nu_m = \mu + i^m \mathcal{F}(\mu) + i^{2m} \mathcal{F}^2(\mu) + i^{3m} \mathcal{F}^3(\mu)$, for $0 \leq m \leq 3$. Then, one has

$$\mathcal{F}(\nu_m) = (-i)^m \nu_m, \quad (3)$$

which establishes the existence of eigenmeasures for all possible $\lambda \in \{1, i, -1, -i\}$, provided we start from a transformable, tempered measure μ such that the ν_m with $0 \leq m \leq 3$ are non-trivial. On $\mathbb{R}$, this can be achieved with $\mu = \delta_\alpha * \delta_{\mathbb{Z}}$ for any $\alpha \notin \mathbb{Q}$.

Let us continue with a slightly more complex example that will give additional insight. To this end, we set $d = 1$ and start with an arbitrary real eigenmeasure for $\lambda = 1$, so $\widehat{\mu} = \mu = \overline{\mu}$, such as $\mu = \delta_{\mathbb{Z}}$ from the PSF in (1). Now, let $\chi_s: \mathbb{R} \rightarrow \mathbb{S}^1$ be a character, so $\chi_s(x) = e^{2\pi i s x}$ for some fixed $s \in \mathbb{R}$. Then, for $t \in \mathbb{R}$, a simple calculation shows that

$$\delta_t * (\chi_s \delta_{\mathbb{Z}}) = \overline{\chi_s(t)} \chi_s \cdot (\delta_t * \delta_{\mathbb{Z}}) = \overline{\chi_s(t)} (\chi_s \delta_{\mathbb{Z}+t}). \quad (4)$$

This can be done more generally for any real eigenmeasure, with the following consequence.

Lemma 3.2 Let χ_s be a character on $\mathbb{R}$, and let μ be a transformable, tempered measure on $\mathbb{R}$ that satisfies $\widehat{\mu} = \mu = \overline{\mu}$. Then, the measure $\omega_s := \chi_s \cdot (\delta_s * \mu)$ is a tempered measure that is transformable as well, with

$$\widehat{\omega_s} = e^{2\pi i s^2} \overline{\omega_s}.$$

For $\mu = \omega_0$, this relation reduces to $\widehat{\mu} = \mu$.

Moreover, if μ is transformable in the strict sense, then so is ω_s , and both μ and ω_s are translation-bounded measures.

Proof Under our assumptions, ω_s clearly is a tempered measure, while a simple calculation with the convolution theorem shows that the distribution $\widehat{\omega_s}$ is indeed a measure, so ω_s is transformable by definition.

With μ , also the shifted measure $\delta_s * \mu$ is real. Observing $\widetilde{\delta_s} = \chi_s$, we can simply calculate

$$\mathcal{F}(\omega_s) = \widehat{\chi_s} * (\chi_{-s} \widehat{\mu}) = \delta_s * (\overline{\chi_s} \mu) = \chi_s(s) (\overline{\chi_s} \cdot (\delta_s * \mu)) = e^{2\pi i s^2} \overline{\omega_s},$$

where the third step is the obvious generalisation of (4). This establishes the main claim, while $s = 0$ clearly gives the eigenmeasure relation stated.

When μ is strictly transformable, $\mu = \widehat{\mu}$ is translation bounded by Fact 2.3. It follows immediately that ω_s is translation bounded as well. $\square$

With the ω_s from Lemma 3.2, for generic s , we get the cycle

$$\omega_s \xrightarrow{\mathcal{F}} e^{2\pi i s^2} \overline{\omega_s} \xrightarrow{\mathcal{F}} I.\omega_s \xrightarrow{\mathcal{F}} e^{2\pi i s^2} I.\overline{\omega_s} \xrightarrow{\mathcal{F}} \omega_s, \quad (5)$$

as can be checked by an explicit calculation that we leave to the interested reader. Unless s takes special values, the four measures are distinct, and thus form a 4-cycle. Excluding a few more values for s , the measures ν_m from (3) with $\mu = \omega_s$ are non-trivial, so we see a plethora of eigenmeasures of $\mathcal{F}$, for all possible eigenvalues.

Next, for each $\lambda \in \{\pm 1, \pm i\}$, we define an operator $P_\lambda: \mathcal{S}'(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$ via

$$P_\lambda(\mu) := \frac{1}{4} \sum_{j=0}^3 \lambda^{-j} \mathcal{F}^j(\mu).$$

Its properties can be summarised as follows.

Proposition 3.3 *The operators P_λ with $\lambda \in \{1, i, -1, -i\}$ satisfy the following properties.*

- (1) *One has $P_1 + P_i + P_{-1} + P_{-i} = \text{id}$.*
- (2) *For any $\lambda, \lambda' \in \{\pm 1, \pm i\}$, one has $P_\lambda \circ P_{\lambda'} = \delta_{\lambda, \lambda'} P_\lambda$ and $\mathcal{F} \circ P_\lambda = \lambda P_\lambda$.*
- (3) *A subspace $X \subseteq \mathcal{S}'(\mathbb{R}^d)$ is $\mathcal{F}$ -invariant if and only if $P_\lambda(X) \subseteq X$ holds for all $\lambda \in \{\pm 1, \pm i\}$.*
- (4) *Let $X \subseteq \mathcal{S}'(\mathbb{R}^d)$ be $\mathcal{F}$ -invariant and let $\lambda \in \{\pm 1, \pm i\}$. Then, the restriction of P_λ to X is a projection operator onto the eigenspace*

$$\mathbb{E}_\lambda(X) := \{\omega \in X : \mathcal{F}(\omega) = \lambda\omega\}.$$

In particular, one has $X = \mathbb{E}_1(X) \oplus \mathbb{E}_i(X) \oplus \mathbb{E}_{-1}(X) \oplus \mathbb{E}_{-i}(X)$.

Proof Note first that, for all $\omega \in \mathcal{S}'(\mathbb{R}^d)$, we have

$$\sum_{k=0}^3 P_{i^k}(\omega) = \frac{1}{4} \sum_{k=0}^3 \sum_{j=0}^3 i^{-kj} \mathcal{F}^j(\omega) = \frac{1}{4} \sum_{j=0}^3 \left(\sum_{k=0}^3 i^{-kj} \right) \mathcal{F}^j(\omega).$$

Now, the bracketed sum vanishes unless $j = 0$, which gives

$$\sum_{k=0}^3 P_{i^k}(\omega) = \omega \quad (6)$$

and establishes Property (1).

For Property (2), for all $\lambda \in \{\pm 1, \pm i\}$ and all $\omega \in \mathcal{S}'(\mathbb{R}^d)$, we have

$$\begin{aligned}\mathcal{F}(P_\lambda(\omega)) &= \mathcal{F}\left(\frac{1}{4} \sum_{j=0}^3 \lambda^{-j} \mathcal{F}^j(\omega)\right) = \frac{1}{4} \sum_{j=0}^3 \lambda^{-j} \mathcal{F}^{j+1}(\omega) \\ &= \frac{\lambda}{4} \sum_{j=0}^3 \lambda^{-j-1} \mathcal{F}^{j+1}(\omega) = \lambda P_\lambda(\omega),\end{aligned}$$

with the last equality following from $\mathcal{F}^4 = \text{id}$ and $\lambda^4 = 1$. Furthermore,

$$\begin{aligned}P_\lambda(P_{\lambda'}(\omega)) &= \left(\frac{1}{4} \sum_{j=0}^3 \lambda^{-j} \mathcal{F}^j(P_{\lambda'}(\omega))\right) = \frac{1}{4} \sum_{j=0}^3 \lambda^{-j} (\lambda'^j P_{\lambda'}(\omega)) \\ &= \frac{1}{4} \sum_{j=0}^3 (\lambda^{-1} \lambda')^j P_{\lambda'}(\omega).\end{aligned}$$

Since $\lambda^{-1} \lambda'$ is a fourth root of unity, we have $\sum_{j=0}^3 (\lambda^{-1} \lambda')^j = 4\delta_{\lambda, \lambda'}$, which proves this part.

The direct implication in Property (3) follows immediately from the definition of P_λ , while the converse follows from Properties (1) and (2).

Finally, for Property (4), observe first that we have $P_\lambda(X) \subseteq \mathbb{E}_\lambda(X)$ from Property (2). Moreover, for all $\omega \in \mathbb{E}_\lambda(X)$, we have

$$P_\lambda(\omega) = \frac{1}{4} \sum_{j=0}^3 \lambda^{-j} \mathcal{F}^j(\omega) = \frac{1}{4} \sum_{j=0}^3 \lambda^{-j} (\lambda^j \omega) = \omega,$$

so $P_\lambda : X \rightarrow \mathbb{E}_\lambda(X)$ is onto. The claim now follows from Properties (2) and (3). $\square$

Remark 3.4 Using the work of Simon [45], see also [9, Thm. 1], one can decompose $P_\lambda(\omega)$ as a series of Hermite functions, which are eigenfunctions for λ . Below, we are primarily interested in the case where both ω and $\widehat{\omega}$ are tempered measures with locally finite support. In this case, the Hermite function expansion does not seem to give further insight. $\diamond$

Now, let us denote by $\mathcal{M}_{\text{ttm}}(\mathbb{R}^d)$ the space of transformable tempered measures on $\mathbb{R}^d$, which by definition is $\mathcal{F}$ -invariant. Therefore, we get the following result.

Theorem 3.5 *For each $\lambda \in \{\pm 1, \pm i\}$, the mapping P_λ induces a projection operator from $\mathcal{M}_{\text{ttm}}(\mathbb{R}^d)$ onto the eigenspace $\mathbb{E}_\lambda(\mathcal{M}_{\text{ttm}}(\mathbb{R}^d))$, with $P_1 + P_i + P_{-1} + P_{-i} = \text{id}$. In particular, we have the decomposition $\mathcal{M}_{\text{ttm}}(\mathbb{R}^d) = \mathbb{E}_1(\mathcal{M}_{\text{ttm}}(\mathbb{R}^d)) \oplus \mathbb{E}_i(\mathcal{M}_{\text{ttm}}(\mathbb{R}^d)) \oplus \mathbb{E}_{-1}(\mathcal{M}_{\text{ttm}}(\mathbb{R}^d)) \oplus \mathbb{E}_{-i}(\mathcal{M}_{\text{ttm}}(\mathbb{R}^d))$. $\square$*

This shows that there is an abundance of eigenmeasures, which makes a meaningful classification difficult unless certain subclasses are specified. In this framework, for

$d = 1$, we shall later derive a more explicit characterisation of eigenmeasures that are periodic or are pure point measures with uniformly discrete support.

Remark 3.6 Let us give some further connections.

- (1) Proposition 3.3 and Theorem 3.5 hold for the space of all measures that are strictly transformable, as well as for $L^2(\mathbb{R}^d)$, as mentioned earlier.
- (2) As the authors learned from a correspondence with Feichtinger after submitting the first version of this paper, this type of result holds for any $\mathcal{F}$ -invariant subspace of $S'_0(\mathbb{R}^d)$, where $S'_0(\mathbb{R}^d)$ is the dual of the Feichtinger algebra $S_0(\mathbb{R}^d)$; see [39] and references therein for background and further details. $\diamond$

Let us return to $d = 1$. For $r, s \in \mathbb{R}$ and $\alpha > 0$, we now define

$$\mathcal{Z}_{r,s,\alpha} := \chi_r \cdot (\delta_s * \delta_{\alpha\mathbb{Z}}), \quad (7)$$

which is a translation-bounded (hence tempered) complex measure on $\mathbb{R}$ that is also transformable, as well as strictly transformable.

Lemma 3.7 For arbitrary $r, s \in \mathbb{R}$ and $\alpha > 0$, the measure $\mathcal{Z}_{r,s,\alpha}$ from (7) is tempered and (strictly) transformable, and has the following properties.

- (1) $\mathcal{Z}_{r,s+m\alpha,\alpha} = \mathcal{Z}_{r,s,\alpha}$ holds for all $m \in \mathbb{Z}$.
- (2) $\mathcal{Z}_{r+\frac{m}{\alpha},s,\alpha} = e^{2\pi i \frac{ms}{\alpha}} \mathcal{Z}_{r,s,\alpha}$ holds for all $m \in \mathbb{Z}$.
- (3) $\mathcal{F}(\mathcal{Z}_{r,s,\alpha}) = \delta_r * (\chi_{-s} \cdot \frac{1}{\alpha} \delta_{\mathbb{Z}/\alpha}) = \frac{1}{\alpha} e^{2\pi i rs} \mathcal{Z}_{-s,r,\frac{1}{\alpha}}$.
- (4) $\mathcal{F}^2(\mathcal{Z}_{r,s,\alpha}) = I. \mathcal{Z}_{r,s,\alpha} = \mathcal{Z}_{-r,-s,\alpha}$.

Moreover, when $\alpha^2 = \frac{1}{n}$ for some $n \in \mathbb{N}$, then $\mathcal{Z}_{r,s,\alpha} = \sum_{m=0}^{n-1} \mathcal{Z}_{r,s+m\alpha,1/\alpha}$ holds for all $r, s \in \mathbb{R}$. More generally, for $\alpha^2 = \frac{p}{q}$ with $p, q \in \mathbb{N}$ coprime and $\beta = \sqrt{pq}$, one has

$$\mathcal{Z}_{r,s,\alpha} = \sum_{m=0}^{q-1} \mathcal{Z}_{r,s+m\alpha,\beta}.$$

Proof The first relation is obvious, while the second follows from the fact that $\mathbb{Z}/\alpha$ is dual to $\alpha\mathbb{Z}$ as a lattice. It is clear that $\mathcal{Z}_{r,s,\alpha}$ is translation bounded and (strictly) transformable. Property (3) follows from the convolution theorem [2, Thm. 8.5], applied twice, and an extension of Eq. (4) from $\mathbb{Z}$ to $\alpha\mathbb{Z}$. Next, observing $I.\delta_x = \delta_{-x}$ and $\chi_r(-x) = \chi_{-r}(x)$, Property (4) follows from a simple calculation.

When $\alpha^2 = \frac{1}{n}$, we have $\alpha\mathbb{Z} = \mathbb{Z}/\sqrt{n}$ and $\mathbb{Z}/\alpha = \sqrt{n}\mathbb{Z}$. As $n\mathbb{Z}$ is a sublattice of $\mathbb{Z}$ of index n , the summation formula follows by a standard coset decomposition of $\alpha\mathbb{Z}$ modulo $\mathbb{Z}/\alpha$.

In the general case, one has $\alpha = \beta/q$, which implies $\alpha\mathbb{Z} = \bigcup_{m=0}^{q-1} (\beta\mathbb{Z} + \frac{m}{q}\beta)$. This gives

$$\begin{aligned}
\mathcal{Z}_{r,s,\alpha} &= \chi_r \cdot (\delta_s * \delta_{\alpha\mathbb{Z}}) = \chi_r \cdot \left(\delta_s * \sum_{m=0}^{q-1} \delta_{\beta\mathbb{Z}+m\alpha} \right) \\
&= \sum_{m=0}^{q-1} \chi_r \cdot (\delta_{s+m\alpha} * \delta_{\beta\mathbb{Z}}) = \sum_{m=0}^{q-1} \mathcal{Z}_{r,s+m\alpha,\beta}
\end{aligned}$$

for any $r, s \in \mathbb{R}$ as claimed. $\square$

Remark 3.8 The measure $\mathcal{Z}_{r,s,\alpha}$ is closely related to the widely-used Zak transform of [21], which is given by $(Zf)(\tau, \Omega) = \sum_{k \in \mathbb{Z}} f(\tau + k) e^{-2\pi i k \Omega}$, for all $\tau, \Omega \in \mathbb{R}$. A short computation reveals

$$\mathcal{Z}_{r,s,1}(f) = e^{2\pi i r s} \cdot (Zf)(s, -r).$$

Some of the results of Lemma 3.7 (at least for $\alpha = 1$) can be found in [21], formulated in terms of the Zak transform.

There is a large body of literature on this subject, also in relation to the Weyl–Heisenberg transform and their discrete versions. However, to the best of our knowledge, these works neither address general questions around the transformability of unbounded measures nor is the transform used to construct eigenmeasures under $\mathcal{F}$. $\diamond$

For $\mu = \delta_{\mathbb{Z}}$, the measure ω_s from Lemma 3.2 is $\mathcal{Z}_{s,s,1}$, and gives the 4-cycle from (5) as

$$\mathcal{Z}_{s,s,1} \xrightarrow{\mathcal{F}} e^{2\pi i s^2} \mathcal{Z}_{-s,s,1} \xrightarrow{\mathcal{F}} \mathcal{Z}_{-s,-s,1} \xrightarrow{\mathcal{F}} e^{2\pi i s^2} \mathcal{Z}_{s,-s,1} \xrightarrow{\mathcal{F}} \mathcal{Z}_{s,s,1}.$$

With this, one can choose an integer $m \in \{0, 1, 2, 3\}$ and consider the measure

$$v_m := e^{-\pi i s^2} \mathcal{Z}_{s,s,1} + i^m e^{\pi i s^2} \mathcal{Z}_{-s,s,1} + i^{2m} e^{-\pi i s^2} \mathcal{Z}_{-s,-s,1} + i^{3m} e^{\pi i s^2} \mathcal{Z}_{s,-s,1},$$

which satisfies $\widehat{v_m} = (-i)^m v_m$. When $s \in \mathbb{Q}$, so $s = \frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, the example is (at least) q -periodic, while s irrational provides aperiodic examples.

Proposition 3.9 *Let $\alpha > 1$ be fixed, and let K and J be bounded fundamental domains for the lattices $\alpha\mathbb{Z}$ and $\frac{1}{\alpha}\mathbb{Z}$, respectively. Then, $B_\alpha := \{\mathcal{Z}_{r,s,\alpha} : r \in J, s \in K\}$ is an algebraic basis for $V_\alpha := \text{span}_{\mathbb{C}}\{\mathcal{Z}_{r,s,\alpha} : r, s \in \mathbb{R}\}$.*

Proof By Lemma 3.7(1) and (2), it is clear that the elements of B_α are distinct and that B_α is a spanning set for V_α , which is the complex vector space of all *finite* linear combinations of measures $\mathcal{Z}_{r,s,\alpha}$ with fixed α . It thus remains to show linear independence of the elements of B_α . So, let $n \in \mathbb{N}$ and assume

$$c_1 \mathcal{Z}_{r_1,s_1,\alpha} + \cdots + c_n \mathcal{Z}_{r_n,s_n,\alpha} = 0 \quad (8)$$

with $s_i \in K$ and $r_j \in J$ such that the n pairs (s_i, r_i) are distinct. Since K is a proper fundamental domain for $\alpha\mathbb{Z}$, for any $1 \leq k, \ell \leq n$, we have either $s_k = s_\ell$ or $(s_k + \alpha\mathbb{Z}) \cap (s_\ell + \alpha\mathbb{Z}) = \emptyset$.

Select some $1 \leq k \leq n$ and set $F_k = \{1 \leq \ell \leq n : s_\ell = s_k\}$. Restricting (8) to $s_k + \alpha\mathbb{Z}$ gives

$$\sum_{\ell \in F_k} c_\ell \mathcal{Z}_{r_\ell, s_k, \alpha} = 0.$$

Taking the Fourier transform turns this relation into

$$\sum_{\ell \in F_k} \frac{c_\ell}{\alpha} e^{2\pi i r_\ell s_k} \mathcal{Z}_{-s_k, r_\ell, \frac{1}{\alpha}} = 0. \quad (9)$$

Next, we note that, for all $\ell \in F_k$ with $\ell \neq k$, we have $r_\ell \neq r_k$, as $s_\ell = s_k$ and $(s_\ell, r_\ell) \neq (s_k, r_k)$. Consequently, as $r_k, r_\ell \in J$, we have

$$(r_k + \frac{1}{\alpha}\mathbb{Z}) \cap (r_\ell + \frac{1}{\alpha}\mathbb{Z}) = \emptyset.$$

Now, restricting (9) to $r_k + \frac{1}{\alpha}\mathbb{Z}$ gives

$$\frac{c_k}{\alpha} e^{2\pi i r_k s_k} \mathcal{Z}_{-s_k, r_k, \frac{1}{\alpha}} = 0,$$

which implies $c_k = 0$. Since k was arbitrary, we are done. $\square$

Let us now look into classes of periodic eigenmeasures more systematically.

4 Lattice-Periodic Eigenmeasures

If Γ is a lattice in $\mathbb{R}^d$, and δ_Γ the corresponding Dirac comb, the PSF from (1) becomes

$$\widehat{\delta_\Gamma} = \text{dens}(\Gamma) \delta_{\Gamma^*}, \quad (10)$$

where $\text{dens}(\Gamma)$ is the (uniformly existing) density of Γ , and Γ^* denotes the dual lattice, as given by $\Gamma^* = \{x \in \mathbb{R}^d : xy \in \mathbb{Z} \text{ for all } y \in \Gamma\}$; compare [2, Thm. 9.1].

Fact 4.1 *If $\Gamma \subset \mathbb{R}^d$ is a lattice, the Dirac comb δ_Γ is an eigenmeasure for $\mathcal{F}$ if and only if Γ is self-dual, that is, $\Gamma^* = \Gamma$, which also implies $\text{dens}(\Gamma) = 1$.*

Proof $\widehat{\delta_\Gamma} = \lambda \delta_\Gamma$ means $\Gamma^* = \Gamma$ and $\lambda = \text{dens}(\Gamma)$, via the PSF, together with $\lambda^4 = 1$. As $\lambda = 1$ is the only positive, real solution, we get $\text{dens}(\Gamma) = 1$ and $\Gamma^* = \Gamma$ as claimed. $\square$

To go beyond this case, let us start with $d = 1$ and assume μ to be a measure on $\mathbb{R}$ that is α -periodic, for some $\alpha > 0$, which is to say that $\delta_\alpha * \mu = \mu$. Such a measure is of the form

$$\mu = \varrho * \delta_{\alpha\mathbb{Z}}, \quad (11)$$

where ϱ is a *finite* measure; compare [2, Sect. 9.2.3]. Without loss of generality, we may assume that $\text{supp}(\varrho) \subseteq [0, \alpha]$ with $\varrho(\{\alpha\}) = 0$, for instance by setting $\varrho = \mu|_{[0, \alpha)}$. As the periodic repetition of a finite motif, μ obviously is translation bounded, hence also tempered. Any such measure is (strictly) transformable [1, Cor. 6.1].

Lemma 4.2 *Let $\alpha > 0$ be fixed, and let $\mu \neq 0$ be a tempered measure on $\mathbb{R}$ that is α -periodic. If $\widehat{\mu} = \lambda\mu$, one has $\lambda \in \{1, i, -1, -i\}$ together with $\alpha^2 = n$ for some $n \in \mathbb{N}$. In particular, any such measure must be of the form (11), where the finite measure ϱ can be chosen to have support in $[0, \alpha) \cap \mathbb{Z}/\alpha$, that is, in the natural coset representatives of $\mathbb{Z}/\alpha$ modulo $\alpha\mathbb{Z}$.*

Proof Under the assumption, we may use the representation from (11), together with the choice of ϱ . From here, we see that μ is strictly transformable, and the convolution theorem [2, Thm. 8.5] in conjunction with the general PSF from Eq. (10) gives

$$\widehat{\mu} = \alpha^{-1} \widehat{\varrho} \delta_{\mathbb{Z}/\alpha}. \quad (12)$$

Since $\widehat{\varrho}$ is a continuous function on $\mathbb{R}$, this entails that $\text{supp}(\widehat{\mu}) \subseteq \mathbb{Z}/\alpha$, where $\mathbb{Z}/\alpha$ is a lattice and thus uniformly discrete as a point set.

If $\widehat{\mu} = \lambda\mu$, where μ is non-trivial, we know that $\lambda^4 = 1$. Now, comparing the representation of μ from (11) with Eq. (12) we get $\text{supp}(\mu) = \text{supp}(\varrho) + \alpha\mathbb{Z} \subseteq \mathbb{Z}/\alpha$, hence

$$\alpha \text{supp}(\varrho) + \alpha^2\mathbb{Z} \subseteq \mathbb{Z},$$

where $A + B$ denotes the Minkowski sum of two sets. This inclusion implies $\alpha \text{supp}(\varrho) \subseteq \mathbb{Z}$ because $0 \in \alpha^2\mathbb{Z}$. Since $\alpha \text{supp}(\varrho) \neq \emptyset$, it contains some integer, m say, which now results in $\alpha^2\mathbb{Z} \subseteq \mathbb{Z}$. But this means that $\alpha^2 = n$ must be an integer, and $\text{supp}(\varrho) \subset \mathbb{Z}/\alpha$.

Under our assumptions, and with our choice of ϱ , we get $\text{supp}(\varrho) \subseteq [0, \alpha) \cap \mathbb{Z}/\alpha$, so ϱ must be a pure point measure with support in the finite set $\{\frac{m}{\alpha} : 0 \leq m < n\}$, which establishes the last claim. $\square$

This provides a class of Dirac combs with pure point diffraction [5], with the additional property that they are doubly sparse in the sense of [7, 30, 32], see also [24].

4.1 Connection with Discrete Fourier Transform

Let $\alpha = \sqrt{n}$ with $n \in \mathbb{N}$ be fixed, and consider the product pure point measure

$$\mu = \varrho * \delta_{\alpha\mathbb{Z}} \quad \text{with} \quad \varrho = \sum_{m=0}^{n-1} c_m \delta_{m/\alpha}. \quad (13)$$

Clearly, μ is *crystallographic* in the sense of [2]. Indeed, it is α -periodic, but might also have smaller periods, depending on the coefficients c_m . Moreover, μ is transformable, with

$$\widehat{\mu} = \widehat{\varrho} \widehat{\delta_{\alpha\mathbb{Z}}} = \alpha^{-1} \sum_{m=0}^{n-1} c_m e^{-2\pi i \frac{m}{\alpha}(\cdot)} \delta_{\mathbb{Z}/\alpha}.$$

Observing that $\delta_{\mathbb{Z}/\alpha} = \delta_{\alpha\mathbb{Z}} * \sum_{\ell=0}^{n-1} \delta_{\ell/\alpha}$, one sees that the value of the character in the above sum only depends on the coset of $\alpha\mathbb{Z}$, thus giving the simplification

$$\widehat{\mu} = \delta_{\alpha\mathbb{Z}} * \sum_{\ell=0}^{n-1} \left(\frac{1}{\sqrt{n}} \sum_{m=0}^{n-1} e^{-2\pi i \frac{m\ell}{n}} c_m \right) \delta_{\ell/\alpha}. \quad (14)$$

The crucial observation to make here is that the term in brackets is the *discrete Fourier transform* (DFT) of the vector $c = (c_0, \dots, c_{n-1})^T$.

If U_n denotes the unitary DFT or Fourier matrix, it is well known that its eigenvalues satisfy $\lambda \in \{1, i, -1, -i\}$. Their multiplicities, in closed terms, were already known to Gauss, while the eigenvectors are difficult,¹ and still not known in closed form; see [18] for a version that relates to the Hermite functions of the continuous case.

Theorem 4.3 *Let $\alpha > 0$ be fixed, and let λ be a fourth root of unity. Further, let $\mu \neq 0$ be a measure on $\mathbb{R}$ that is α -periodic. Then, the following properties are equivalent.*

- (1) μ is an eigenmeasure of $\mathcal{F}$ with eigenvalue λ .
- (2) μ is a strict eigenmeasure of $\mathcal{F}$ with eigenvalue λ .
- (3) The measure μ has the form (13) with $\alpha = \sqrt{n}$, for some $n \in \mathbb{N}$ and some eigenvector $c = (c_0, \dots, c_{n-1})^T \in \mathbb{C}^n$ of the corresponding DFT, meaning $U_n c = \lambda c$.

Proof As pointed out above, any α -periodic measure is translation bounded. Then, (1) $\Leftrightarrow$ (2) is a consequence of Fact 2.3.

The equivalence (1) $\Leftrightarrow$ (3) follows from Lemma 4.2 by a comparison of coefficients in (13) and (14). $\square$

Example 4.4 When $\alpha^2 = 1$ in Theorem 4.3, the only solution we get (up to a constant) is $\mu = \delta_{\mathbb{Z}}$, with $\lambda = 1$, as expected.

¹ A good summary with all the relevant details and references can be found in the WIKIPEDIA entry on the DFT, which is recommended for background. Some results are collected in our Appendix.

When $\alpha^2 = 2$, so $\alpha = \sqrt{2}$, the measure ϱ in (11) has the form $\varrho = c_0 \delta_0 + c_1 \delta_{1/\alpha}$. Here, with $c_0 = 1 \pm \sqrt{2}$ and $c_1 = 1$, one gets eigenmeasures with $\lambda = \pm 1$, which are the only possibilities (up to an overall constant). Examples with larger values of α can be constructed with the material given in the Appendix. $\diamond$

Remark 4.5 Note that Fact 3.1 has consequences on the structure of the eigenvectors c of the DFT. Indeed, it is clear that we get $c_0 = \lambda^2 c_0$ together with

$$c_{n-k} = \lambda^2 c_k, \quad \text{for } 1 \leq k \leq n-1.$$

In particular, for $\lambda = \pm 1$, the (partial) vector $(c_1, \dots, c_{n-1})$ is palindromic, while $\lambda = \pm i$ implies $c_0 = 0$ together with $(c_1, \dots, c_{n-1})$ being skew-palindromic. This is clearly visible in the above examples and in those of the Appendix. $\diamond$

Let us next consider another class of eigenmeasures that implicitly emerge from a *cut and project scheme* (CPS) in the theory of aperiodic order; see [2] for background. This will provide non-periodic examples of pure point eigenmeasures.

5 Shadows of Product Measures

Let us return to the simple statement from Fact 4.1, and extend it into a different direction. Here and below, we always assume $\mathbb{R}^d$ to be equipped with Lebesgue measure as its (standard) Haar measure, that is, the unique translation-invariant measure which gives volume 1 to the unit cube. Clearly, Γ can only be self-dual if $\text{dens}(\Gamma) = 1$, which is to say that its fundamental domain has unit volume. Note that this is not a sufficient criterion for $d > 1$.

More generally, when B denotes a basis matrix for Γ , with the columns being the basis vectors in Cartesian coordinates, $B^* := (B^{-1})^T$ is the *dual matrix*, and the fitting basis matrix for Γ^* . In this formulation, self-duality of Γ is equivalent with orthogonality of the matrix B . It is now easy to check that the only self-dual lattices in $\mathbb{R}^2$ are the square lattices, that is, $\mathbb{Z}^2$ and its rotated siblings. A natural choice for the basis matrix thus is

$$B = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = (u_\theta, v_\theta), \quad (15)$$

where u_θ and v_θ are the column vectors. We denote the corresponding lattice by Γ_θ . Note that the parameter can be restricted to $0 \leq \theta < \frac{\pi}{2}$, which covers all cases once, due to the fourfold rotational symmetry of the square lattice. Given this choice of basis, we can always uniquely write $z \in \Gamma_\theta$ as $z = mu_\theta + nv_\theta$ with $m, n \in \mathbb{Z}$. We thus map $z \in \Gamma_\theta$ to a pair $(m_z, n_z) \in \mathbb{Z}^2$ via this choice of basis.

The next result is standard [43, Thm. VII.XIV], and follows from a simple Fubini-type calculation with the product Lebesgue measure on $\mathbb{R}^{p+q} = \mathbb{R}^p \times \mathbb{R}^q$, where the function $f \otimes g : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{C}$ is defined by $f \otimes g(x, y) := f(x)g(y)$ as usual.

Fact 5.1 Let $p, q \in \mathbb{N}$, $f \in L^1(\mathbb{R}^p)$, $g \in L^1(\mathbb{R}^q)$, and consider the product function $h := f \otimes g$. Then, $h \in L^1(\mathbb{R}^{p+q})$, and its Fourier transform reads $\widehat{h} = \widehat{f} \otimes \widehat{g}$. In particular, this applies to $f \in \mathcal{S}(\mathbb{R}^p)$ and $g \in \mathcal{S}(\mathbb{R}^q)$, with $h \in \mathcal{S}(\mathbb{R}^{p+q})$. $\square$

Now, let $\Gamma \subset \mathbb{R}^{d+m}$ be a lattice, let $g \in \mathcal{S}(\mathbb{R}^m)$ be fixed, and consider

$$\delta_{g,\Gamma}^*(f) := \delta_\Gamma(f \otimes g). \quad (16)$$

Since the mapping $\mathcal{S}(\mathbb{R}^d) \ni f \mapsto f \otimes g \in \mathcal{S}(\mathbb{R}^{d+m})$ is continuous with respect to the topology of Schwartz space, and since δ_Γ is a tempered distribution, it is clear that $\delta_{g,\Gamma}^*$ is a tempered distribution as well. In fact, we have more as follows.

Proposition 5.2 The measure $\delta_{g,\Gamma}^*$ from (16) is translation bounded and transformable, with the transform $\widehat{\delta_{g,\Gamma}^*} = \text{dens}(\Gamma) \delta_{g,\Gamma^{**}}^*$.

Proof If $\Gamma \subset \mathbb{R}^{d+m}$ is a lattice and $h \in \mathcal{S}(\mathbb{R}^{m+d})$, one has $\sum_{z \in \Gamma} |h(z)| < \infty$ by standard arguments [43], hence

$$\sum_{(x,y) \in \Gamma} |f(x)g(y)| < \infty \quad (17)$$

holds for all $f \in \mathcal{S}(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^m)$. Consequently, also for all $\varphi \in C_c(\mathbb{R}^d)$, we have

$$\sum_{(x,y) \in \Gamma} |\varphi(x)g(y)| < \infty.$$

We can thus define $\mu: C_c(\mathbb{R}^d) \rightarrow \mathbb{C}$ by

$$\mu(\varphi) = \sum_{(x,y) \in \Gamma} g(y) \delta_x(\varphi) = \sum_{(x,y) \in \Gamma} \varphi(x) g(y),$$

with the sum being absolutely convergent. Moreover, by Eq. (17), $\mu(f)$ is well defined for all $f \in \mathcal{S}(\mathbb{R}^d)$ and satisfies $\mu(f) = \delta_{g,\Gamma}^*(f)$.

Now, we show that μ is also a measure, which implies that $\delta_{g,\Gamma}^*$ is a measure as well. It is obvious that μ is a linear mapping. Next, let $K \subset \mathbb{R}^d$ be a fixed compact set, select $f \in C_c^\infty(\mathbb{R}^d)$ so that $f \geq 1_K$, and consider

$$C_K := \sum_{(x,y) \in \Gamma} |f(x)g(y)| < \infty.$$

Then, if $\varphi \in C_c(\mathbb{R}^d)$ satisfies $\text{supp}(\varphi) \subseteq K$, we get

$$\begin{aligned}
|\mu(\varphi)| &= \left| \sum_{(x,y) \in \Gamma} \varphi(x) g(y) \right| \leq \sum_{(x,y) \in \Gamma} |\varphi(x) g(y)| \leq \|\varphi\|_\infty \sum_{(x,y) \in \Gamma} |f(x) g(y)| \\
&\leq C_K \|\varphi\|_\infty
\end{aligned}$$

which shows that μ is a measure.

Next, by applying the proof of [52, Cor. 2.1] to $h = f \otimes g$, we see that there is a constant C such that $(|\delta_\Gamma| * |I.h|)(x) \leq C$ holds for all $x \in \mathbb{R}^{d+m}$. In particular, for all $t \in \mathbb{R}^d$, we have

$$\begin{aligned}
|\mu|(t+K) &= \sum_{\substack{(x,y) \in \Gamma \\ x \in t+K}} |g(y)| \leq \sum_{(x,y) \in \Gamma} |f(x-t) g(y)| \\
&= \sum_{(x,y) \in \Gamma} |h(x-t, y)| = (|\delta_\Gamma| * |I.h|)(t, 0) \leq C,
\end{aligned}$$

which shows that μ is translation bounded.

The verification of the last claim is similar to an argument from [41]. Indeed, the PSF implies that, for all $f \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\widehat{\delta_{g,\Gamma}^*}(f) = \delta_{g,\Gamma}^*(\widehat{f}) = \delta_\Gamma(\widehat{f} \otimes g) = \text{dens}(\Gamma) \delta_{\Gamma^*}(f \otimes \check{g}) = \text{dens}(\Gamma) \delta_{g,\Gamma^*}^*(f)$$

with the last distribution being a measure by the first part of the proof. $\square$

Now, we can proceed as follows. Let λ be any fixed fourth root of unity, and select a Schwartz function $g \in \mathcal{S}(\mathbb{R})$ such that $\widehat{g} = \lambda g$, which we know to exist via the Hermite functions. Clearly, this implies $\check{g} = \bar{\lambda} g$, because we have $\widehat{\widehat{g}} = g \circ I$ together with $\lambda^4 = 1$.

Next, fix a parameter $0 \leq \theta < \frac{\pi}{2}$ and consider the lattice Γ_θ in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Let $f \otimes g$ refer to the product function with respect to this (Cartesian) splitting, so

$$\begin{aligned}
f \otimes g(z) &= f \otimes g(m_z u_\theta + n_z v_\theta) \\
&= f(m_z \cos(\theta) - n_z \sin(\theta)) \cdot g(m_z \sin(\theta) + n_z \cos(\theta)),
\end{aligned}$$

which elucidates the role of the angle θ . Now, for a fixed lattice Γ_θ , we consider the translation-bounded (and hence tempered) measure ω_g on $\mathbb{R}$ defined by

$$\omega_g(\varphi) := \sum_{z \in \Gamma_\theta} \varphi \otimes g(z) = \delta_{g,\Gamma_\theta}^*(\varphi), \quad (18)$$

where φ is an arbitrary Schwartz function. In fact, (18) is well-defined for $\varphi \in C_c(\mathbb{R})$, too. Then, the (distributional) transform gives

$$\begin{aligned}
\widehat{\omega}_g(\varphi) &= \omega_g(\widehat{\varphi}) = \sum_{z \in \Gamma_\theta} \widehat{\varphi} \otimes g(z) = \sum_{z \in \Gamma_\theta} \widehat{\varphi \otimes \check{g}}(z) \\
&= \sum_{z \in \Gamma_\theta} \varphi \otimes \check{g}(z) = \bar{\lambda} \sum_{z \in \Gamma_\theta} \varphi \otimes g(z) = \bar{\lambda} \omega_g(\varphi).
\end{aligned}$$

Here, we have used Fact 5.1 for $p = q = 1$ in the upper line, while the first equality in the second line follows from the PSF for the self-dual lattice Γ_θ . Since φ is arbitrary, this implies the relation $\widehat{\omega}_g = \bar{\lambda} \omega_g$. Invoking [51, Lemma 5.2 and Thm. 5.2], we see that the measure ω_g from (18) is translation bounded and that $\widehat{\omega}_g = \bar{\lambda} \omega_g$ also holds in the strict sense. Consequently, we can construct eigenmeasures of $\mathcal{F}$ for any fourth root of unity this way. Let us summarise this derivation as follows.

Theorem 5.3 *Let Γ_θ be the self-dual, planar lattice defined by the basis matrix B from (15), and let $g \neq 0$ be a Schwartz function with $\widehat{g} = \lambda g$ for some $\lambda \in \{1, i, -1, -i\}$. Then, the tempered measure ω_g defined by (18) is a (strict) eigenmeasure of $\mathcal{F}$, with $\widehat{\omega}_g = \bar{\lambda} \omega_g$. $\square$*

Remark 5.4 As is clear from [47, Cor. VII.2.6], this approach also works when g is continuous and decays such that $\varphi \otimes g(z) = \mathcal{O}((1 + |z|)^{-2-\epsilon})$ as $|z| \rightarrow \infty$, for some $\epsilon > 0$. $\diamond$

Depending on the parameter θ , the lattice Γ_θ may be in rational or irrational position relative to the horizontal line. The rational case means $\tan(\theta) = \frac{p}{q}$ with integers $0 \leq p \leq q$ that can be chosen coprime, with $q = 1$ when $p = 0$. Rationality implies $\sin(\theta)^2 = p^2/(p^2 + q^2)$ and $\cos(\theta)^2 = q^2/(p^2 + q^2)$. In any such case, the intersection of Γ_θ with the horizontal line is a one-dimensional lattice. Observing that $q \sin(\theta) - p \cos(\theta) = 0$, this lattice is $\alpha\mathbb{Z}$, with

$$\alpha = \frac{q \cos(\theta) + p \sin(\theta)}{q} = \frac{\sqrt{p^2 + q^2}}{q}$$

when $q \neq 0$, and $\alpha = 1$ when $q = 0$. Consequently, ω_g is α -periodic in this case, and we are back to the class discussed in Sect. 4.

Example 5.5 When $\theta = 0$, we get $\Gamma_0 = \mathbb{Z}^2$, and our Radon measure simplifies to $\omega_g = c \delta_{\mathbb{Z}}$, with $c = \sum_{m \in \mathbb{Z}} g(m)$. Since this measure is 1-periodic, it is either an eigenmeasure for eigenvalue 1, or it must be trivial. If g was chosen as a Hermite function, $g = h_n$ say, we get an extra condition for any $n \not\equiv 0 \pmod{4}$, namely

$$\sum_{k \in \mathbb{Z}} h_n(k) = 0. \quad (19)$$

This is clear by the anti-symmetry of h_n for all odd n , while it looks a little less obvious for $n \equiv 2 \pmod{4}$. However, this sum rule follows directly from the classic PSF for functions, via

$$\sum_{k \in \mathbb{Z}} h_n(k) = \sum_{k \in \mathbb{Z}} \widehat{h}_n(k) = (-i)^n \sum_{k \in \mathbb{Z}} h_n(k),$$

which implies (19) for any $n \not\equiv 0 \pmod{4}$. $\diamond$

Example 5.6 Consider $p = q = 1$, so $\theta = \frac{\pi}{4}$. Here, the intersection of the lattice $\Gamma_{\pi/4}$ with the horizontal line is $\sqrt{2}\mathbb{Z}$ and the measure becomes $\omega_g = c_0\delta_{\sqrt{2}\mathbb{Z}} + c_1\delta_{\beta+\sqrt{2}\mathbb{Z}}$, with $\beta = 1/\sqrt{2}$ and coefficients

$$c_0 = \sum_{m \in \mathbb{Z}} g(m\sqrt{2}) \quad \text{and} \quad c_1 = \sum_{m \in \mathbb{Z}} g(m\sqrt{2} + \beta).$$

Since we have α -periodicity with $\alpha^2 = 2$, there are eigenmeasures for $\lambda = \pm 1$. When $\widehat{g} = \pm g$, this is only consistent if the coefficients satisfy $c_0/c_1 = 1 \pm \sqrt{2}$, in line with Example 4.4. This covers the cases of $g = h_n$ with n even. For odd n , the measure ω_g must be trivial, hence $c_0 = c_1 = 0$, which is clear from the anti-symmetry of the Hermite functions in this case.

More generally, extending this observation to other rational values of θ , one obtains further conditions from the structure of the eigenvectors of the DFT. $\diamond$

When $\tan(\theta)$ is *irrational*, we obtain aperiodic extensions, which can be understood by viewing the setting as a CPS of the form $(\mathbb{R}, \mathbb{R}, \Gamma_\theta)$; see [2] for more on this concept, and [33, 36, 37] for general background. The CPS here can be summarised in the diagram

$$\begin{array}{ccccc} \mathbb{R} & \xleftarrow{\pi} & \mathbb{R} \times \mathbb{R} & \xrightarrow{\pi_{\text{int}}} & \mathbb{R} \\ \cup & & \cup & & \cup \text{ dense} \\ \pi(\mathcal{L}) & \xleftarrow{1-\cdot} & \mathcal{L} = \Gamma_\theta & \longrightarrow & \pi_{\text{int}}(\mathcal{L}) \\ \parallel & & & & \parallel \\ L & \xrightarrow{\star} & & & L^\star \end{array} \quad (20)$$

where $\star$ is the star map of the CPS. Clearly, this diagram is not restricted to the case $\mathcal{L} = \Gamma_\theta$, and $\mathcal{L}$ can be any lattice in $\mathbb{R}^2$ subject to the conditions encoded in the CPS.

Combining Theorem 5.3 and Fact 2.3, we can reformulate our previous result as follows.

Corollary 5.7 *Let $(\mathbb{R}, \mathbb{R}, \mathcal{L})$ be a CPS and let $g \in \mathcal{S}(\mathbb{R})$ be arbitrary, but fixed. If the lattice $\mathcal{L}$ is self-dual and if $\check{g} = \lambda g$, then necessarily with $\lambda \in \{1, i, -1, -i\}$, the tempered measure $\omega_g = \delta_{g, \mathcal{L}}^\star$ is an eigenmeasure of $\mathcal{F}$ with eigenvalue λ , also in the strict sense.* $\square$

This result has a nice consequence as follows, still within the CPS (20). If the function $g \in L^2(\mathbb{R})$ is continuous, the measure ω_g is well defined. Now, expanding g as a series in Hermite functions (in the sense explained in Sect. 2), ω_g can be written as a series in eigenmeasures. This has the potential to give a new tool for studying the diffraction of weighted Dirac combs with Meyer set support, which could potentially work also beyond regular model sets or weak model sets of maximal density.

Remark 5.8 Studying the proofs of Lemmas 5.2 and 5.3 in [51] with care, one can see that various of our previous arguments remain true for $g \in C_0(\mathbb{R})$, as long as g asymptotically satisfies $g(x) = \mathcal{O}((1 + |x|)^{2+\epsilon})$ for some $\epsilon > 0$ as $|x| \rightarrow \infty$, while Proposition 5.2 still holds if $g \in C_0(\mathbb{R})$ has the property that both g and $\check{g}$ satisfy such an asymptotic behaviour. $\diamond$

6 Eigenmeasures with Uniformly Discrete Support

Let us next characterise eigenmeasures on $\mathbb{R}$ with uniformly discrete support. Note that each tempered measure with uniformly discrete support is strongly tempered, see [6, Thm. 4.4], which allows us to use the results of [30]. We recall Eq. (7) and begin with another consequence of the cycle structure induced by $\mathcal{F}^4 = \text{id}$.

Lemma 6.1 *Let $r, s \in \mathbb{R}$ and $n \in \mathbb{N}$ be fixed, and let λ be a fourth root of unity. Then,*

$$\mathcal{Y}_{r,s,\sqrt{n},\lambda} := P_\lambda(\mathcal{Z}_{r,s,\sqrt{n}}) = \frac{1}{4}(\text{id} + \lambda^{-1}\mathcal{F} + \lambda^{-2}\mathcal{F}^2 + \lambda^{-3}\mathcal{F}^3)\mathcal{Z}_{r,s,\sqrt{n}}$$

is either trivial or an eigenmeasure of $\mathcal{F}$ for the eigenvalue λ , also in the strict sense. Further, it is supported inside $\sqrt{n}\mathbb{Z} + F$ for some finite set $F \subset \mathbb{R}$. In particular, the support of the measure $\mathcal{Y}_{r,s,\sqrt{n},\lambda}$ is uniformly discrete.

Proof The eigenmeasure property is clear from Theorem 3.5. Next, set $\theta = \sqrt{n}$. Then, by Lemma 3.7, we have

$$\begin{aligned} \mathcal{Y}_{r,s,\theta,\lambda} &= \frac{1}{4} \left(\mathcal{Z}_{r,s,\theta} + \frac{e^{2\pi i r s}}{\lambda \theta} \mathcal{Z}_{-s,r,\frac{1}{\theta}} + \lambda^{-2} \mathcal{Z}_{-r,-s,\theta} + \frac{e^{2\pi i r s}}{\lambda^3 \theta} \mathcal{Z}_{s,-r,\frac{1}{\theta}} \right) \\ &= \frac{1}{4} \mathcal{Z}_{r,s,\theta} + \frac{1}{4\lambda^2} \mathcal{Z}_{-r,-s,\theta} + \frac{e^{2\pi i r s}}{4\lambda \theta} \sum_{m=0}^{n-1} \left(\mathcal{Z}_{-s,r+\frac{m}{\theta},\theta} + \frac{1}{\lambda^2} \mathcal{Z}_{s,-r+\frac{m}{\theta},\theta} \right). \end{aligned}$$

This implies $\text{supp}(\mathcal{Y}_{r,s,\sqrt{n},\lambda}) \subseteq \sqrt{n}\mathbb{Z} + \left\{ \pm s, \pm r + \frac{m}{\sqrt{n}} : 0 \leq m \leq n-1 \right\}$, which clearly is a uniformly discrete subset of $\mathbb{R}$. $\square$

For fixed $n \in \mathbb{N}$ and $\lambda \in \{1, i, -1, -i\}$, we now define a space of tempered measures via

$$\mathcal{E}_\lambda(n) := \text{span}_{\mathbb{C}} \{ \mathcal{Y}_{r,s,\sqrt{n},\lambda} : r, s \in \mathbb{R} \}. \quad (21)$$

Clearly, any element from $\mathcal{E}_\lambda(n)$ is an eigenmeasure of $\mathcal{F}$ with eigenvalue λ and uniformly discrete support.

Let us return to the general class of tempered measures with uniformly discrete support. If such a measure μ is an eigenmeasure of $\mathcal{F}$, the support of $\hat{\mu}$ must be the same, so μ is doubly sparse; see [7, 30, 32] and references therein for background on such measures. This has a strong consequence as follows.

Proposition 6.2 *Let $\mu \neq 0$ be an eigenmeasure of $\mathcal{F}$ on $\mathbb{R}$ such that $\text{supp}(\mu) \subset \mathbb{R}$ is uniformly discrete. Then, there exist integers $n, k \in \mathbb{N}$ such that, with $\beta = \sqrt{n}$,*

$$\mu = \sum_{i=1}^k c_i \mathcal{Z}_{r_i, s_i, \beta}$$

holds for suitable $c_1, \dots, c_k \in \mathbb{C}$ and $r_1, \dots, r_k, s_1, \dots, s_k \in \mathbb{R}$.

Proof Clearly, since $\text{supp}(\widehat{\mu}) = \text{supp}(\mu)$, the measure μ is doubly sparse, and an application of [30, Thms. 1 and 3] guarantees the existence of an integer $\kappa \in \mathbb{N}$ and some number $\alpha > 0$ such that $\mu = \sum_{j=1}^{\kappa} P_j \delta_{\alpha\mathbb{Z}+s_j}$, where the P_j are trigonometric polynomials. Expanding the latter and rewriting μ in terms of our measures $\mathcal{Z}_{r,s,\alpha}$ from (7) gives

$$\mu = \sum_{i=1}^{\ell} c_i \mathcal{Z}_{r_i, s_i, \alpha} \quad (22)$$

for some $\ell \in \mathbb{N}$, which might be larger than κ , and numbers $c_1, \dots, c_{\ell} \in \mathbb{C}$ together with $r_1, \dots, r_{\ell}, s_1, \dots, s_{\ell} \in \mathbb{R}$, which need not be distinct.

With Lemma 3.7(3), we see that

$$\text{supp}(\mu) \subseteq \alpha\mathbb{Z} + F_s \quad \text{and} \quad \text{supp}(\widehat{\mu}) \subseteq \frac{1}{\alpha}\mathbb{Z} + F_r$$

with the finite sets $F_s = \{s_1, \dots, s_{\ell}\}$ and $F_r = \{r_1, \dots, r_{\ell}\}$. By [49, Lemma 5.5.1], there exists a finite set $G \subset \mathbb{R}$ such that $\frac{1}{\alpha}\mathbb{Z} + F_r \subseteq \text{supp}(\widehat{\mu}) + G$, which implies

$$\frac{1}{\alpha}\mathbb{Z} \subseteq \text{supp}(\widehat{\mu}) + G - F_r = \text{supp}(\mu) + G - F_r \subseteq \alpha\mathbb{Z} + F_s + G - F_r = \alpha\mathbb{Z} + F',$$

where F' is still a finite set. Consequently, there are integers $m, n \in \mathbb{N}$ with $m < n$ and some element $t \in F'$ such that $\frac{m}{\alpha}$ and $\frac{n}{\alpha}$ both lie in $\alpha\mathbb{Z} + t$. This implies $\frac{n-m}{\alpha} \in \alpha\mathbb{Z}$, hence $0 \neq n-m \in \alpha^2\mathbb{Z}$, and $\alpha^2 \in \mathbb{Q}$.

Now, let $\alpha^2 = \frac{p}{q}$ with $p, q \in \mathbb{N}$, and set $\beta = \sqrt{pq}$, so $\beta = \alpha q$ and $\alpha = \beta/q$. Invoking the last relation from Lemma 3.7, since $\beta^2 = pq$ is an integer, we can rewrite μ from (22) as

$$\mu = \sum_{i=1}^{\ell} \sum_{j=0}^{q-1} c_i \mathcal{Z}_{r_i, s_i + j\alpha, \beta},$$

which, after expanding the double sum and relabelling the parameters, proves the claim. $\square$

Recall that Lemma 3.7(3), applied several times, leads to the cycle

$$\mathcal{Z}_{r,s,\alpha} \xrightarrow{\mathcal{F}} \frac{1}{\alpha} e^{2\pi i r s} \mathcal{Z}_{-s, r, \frac{1}{\alpha}} \xrightarrow{\mathcal{F}} \mathcal{Z}_{-r, -s, \alpha} \xrightarrow{\mathcal{F}} \frac{1}{\alpha} e^{2\pi i r s} \mathcal{Z}_{s, -r, \frac{1}{\alpha}} \xrightarrow{\mathcal{F}} \mathcal{Z}_{r,s,\alpha}$$

of length at most 4, where $\mathcal{Z}_{-r,-s,\alpha} = I.\mathcal{Z}_{r,s,\alpha}$. Ignoring the actual cycle length, which can be 1, 2 or 4, we get the following general result.

Theorem 6.3 *Let $\mu \neq 0$ be a transformable measure on $\mathbb{R}$ and λ a fourth root of unity. Then, the following properties are equivalent.*

- (1) *The measure μ is an eigenmeasure of $\mathcal{F}$ for λ with uniformly discrete support, either in the distribution or in the strict sense.*
- (2) *There are $n, k \in \mathbb{N}$ and numbers $c_1, \dots, c_k \in \mathbb{C}$ and $r_1, \dots, r_k, s_1, \dots, s_k \in \mathbb{R}$ such that one has $\mu = P_\lambda(v)$ for $v = \sum_{i=1}^k c_i \mathcal{Z}_{r_i, s_i, \beta}$ with $\beta = \sqrt{n}$.*
- (3) *There is some $n \in \mathbb{N}$ such that $\mu \in \mathcal{E}_\lambda(n)$, as defined in Eq. (21).*

Proof By Lemma 2.6, the two notions of transformability are equivalent in this case, as claimed under (1).

(1) $\Rightarrow$ (2): By Proposition 6.2, we know that μ must be of the form

$$\mu = \sum_{i=1}^k c_i \mathcal{Z}_{r_i, s_i, \beta}$$

for suitable integers n, k and with suitable numbers r_i, s_j and c_ℓ . Then, since $\mu \in \mathbb{E}_\lambda(X)$, where X is the space of measures that are transformable, either in the distribution or in the strict sense, we have $\mu = P_\lambda(\mu)$ by Proposition 3.3. The claim follows by setting $v = \mu$.

(2) $\Rightarrow$ (3): Since $v = \sum_{i=1}^k c_i \mathcal{Z}_{r_i, s_i, \sqrt{n}}$, we get

$$\mu = P_\lambda(v) = \sum_{i=1}^k c_i P_\lambda(\mathcal{Z}_{r_i, s_i, \sqrt{n}}) = \sum_{i=1}^k c_i \mathcal{Y}_{r_i, s_i, \sqrt{n}, \lambda} \in \mathcal{E}_\lambda(n).$$

The implication (3) $\Rightarrow$ (1) is clear from Lemma 6.1. $\square$

For concrete calculations, given an eigenmeasure μ with uniformly discrete support, we can decompose μ into its even and odd part relative to I by

$$\mu = \frac{\mu + I.\mu}{2} + \frac{\mu - I.\mu}{2} =: \mu_+ + \mu_-.$$

Then, one derives the following conditions for $\widehat{\mu} = \lambda\mu$,

$$\begin{aligned} \lambda = \pm 1: \quad \mu_- &= 0 \quad \text{and} \quad \widehat{\mu}_+ = \pm \mu_+, \\ \lambda = \pm i: \quad \mu_+ &= 0 \quad \text{and} \quad \widehat{\mu}_- = \pm \mu_-, \end{aligned} \tag{23}$$

which formed the basis for Lemma 6.1. It is clear from Proposition 6.2 in conjunction with Proposition 3.9 that it suffices to consider measures

$$\mu = \sum_{i=1}^k c_i \mathcal{Z}_{r_i, s_i, \alpha} \tag{24}$$

with fixed $\alpha = \sqrt{n}$ and parameters $r_i \in (-\frac{1}{2\alpha}, \frac{1}{2\alpha}] =: J_r$ and $s_i \in (-\frac{\alpha}{2}, \frac{\alpha}{2}] =: J_s$. Invoking Lemma 3.7(4), we get $\mu = \mu_+ + \mu_-$ with

$$\mu_{\pm} = \frac{1}{2} \sum_{i=1}^k c_i (\mathcal{Z}_{r_i, s_i, \alpha} \pm \mathcal{Z}_{-r_i, -s_i, \alpha}),$$

where $-r_i$ and $-s_i$ are taken modulo $\frac{1}{\alpha}$ and α , respectively, when $r_i = \frac{1}{2\alpha}$ or $s_i = \frac{\alpha}{2}$. Now, one can use the fundamental domains to rewrite the possible eigenmeasures with restricted parameters, and without ambiguities. Some care has to be exercised with the 2-division points

$$\{(0, 0), (0, \frac{\alpha}{2}), (\frac{1}{2\alpha}, 0), (\frac{1}{2\alpha}, \frac{\alpha}{2})\} \subset J$$

under the inversion I in such calculations, the details of which are left to the interested reader.

7 Eigenmeasures with Locally Finite Support

Following [7], we say that a measure is *sparse* if its support is locally finite (that is, discrete and closed). We say that μ is *doubly sparse*² if μ is a transformable tempered measure such that both μ and $\hat{\mu}$ are sparse. Finally, when μ is a doubly sparse measure which is an eigenmeasure for $\mathcal{F}$, we will simply call it a *doubly sparse eigenmeasure*.

Clearly, linear combinations of measures $\mu_i \in \mathcal{E}_{\lambda}(n_i)$, with fixed λ , are eigenmeasures with discrete support. The latter is uniformly discrete if and only if, for all $i \neq j$, one has $\sqrt{m_i/m_j} \in \mathbb{Q}$, which is to say that there are many more possibilities. In particular, for each fourth root of unity, there are doubly sparse eigenmeasures with non-uniformly discrete support, for instance of the type studied by Meyer in [34, 35].

Notice that, since the union of finitely many locally finite sets is locally finite, the set X of doubly sparse measures on $\mathbb{R}^d$ is a subspace of $\mathcal{M}_{\text{ttm}}(\mathbb{R}^d)$, which is by definition $\mathcal{F}$ -invariant. Therefore, Proposition 3.3(4) yields the following result.

Theorem 7.1 *Let μ be a transformable tempered measure. Then, μ is a doubly sparse measure if and only if $P_{\lambda}(\mu)$ is a doubly sparse eigenmeasure for all $\lambda \in \{\pm 1, \pm i\}$.*

In particular, a measure is a doubly sparse measure if and only if it is a linear combination of doubly sparse eigenmeasures. $\square$

Next, we are going to construct doubly sparse eigenmeasures that are supported inside a lattice, but vanish on arbitrarily large intervals.

Let us first fix $n \in \mathbb{N}$ and $\lambda \in \{\pm 1, \pm i\}$. Then, we know from the multiplicities in Table 1 (which is due to Gauss) that the dimension $m = \dim(E_{\lambda})$ of the eigenspace E_{λ} of U_n satisfies

$$4m + 4 \geq n.$$

² Meyer calls such a measure *crystalline* [34], which we prefer to avoid because ‘crystals’ are often connected with atomic arrangements in solids that are uniformly discrete.

Therefore, we get the following useful result.

Fact 7.2 For each $n \in \mathbb{N}$ and $\lambda \in \{\pm 1, \pm i\}$, we have $\dim(E_\lambda) \geq \max(0, \frac{n}{4} - 1)$. $\square$

This gives us the following simple, but powerful lemma.

Lemma 7.3 Let $12 \leq n \in \mathbb{N}$ and $\lambda \in \{\pm 1, \pm i\}$ be fixed. Let $k \in \mathbb{N}$ be chosen such that $1 \leq k \leq \frac{n}{4} - 2$. Then, for each $0 \leq j_1 < j_2 < \dots < j_k \leq n - 1$, there exists some measure μ that is $\sqrt{n}$ -periodic, supported inside $\mathbb{Z}/\sqrt{n}$, and satisfies $\widehat{\mu} = \lambda\mu$ together with the vanishing condition $\mu(\{j_\ell/\sqrt{n}\}) = 0$ for all $1 \leq \ell \leq k$.

Proof Let X be the space of measures that are $\sqrt{n}$ -periodic, supported inside $\mathbb{Z}/\sqrt{n}$, and let

$$\begin{aligned} Y &:= \{\mu \in X : \widehat{\mu} = \lambda\mu\}, \\ Z &:= \{\mu \in X : \mu(\{j_\ell/\sqrt{n}\}) = 0 \text{ for all } 1 \leq \ell \leq k\}. \end{aligned}$$

By Theorem 4.3, we have

$$\dim(Y) = \dim(E_\lambda) \geq \frac{n}{4} - 1 \geq k + 1.$$

On the other hand, we have $\dim(Z) = n - k$, which shows that

$$\dim(Y) + \dim(Z) > \dim(X),$$

and hence $Y \cap Z \neq \emptyset$. This implies our claim. $\square$

When $j_1, j_2, \dots, j_k$ are consecutive numbers, we get the following consequence.

Corollary 7.4 Let $12 \leq n \in \mathbb{N}$ and $\lambda \in \{\pm 1, \pm i\}$ be fixed, and let $1 \leq k \leq \frac{n}{4} - 3$. Then, for each $0 \leq j \leq n - k - 1$, there exists a non-zero measure μ of the form given in Theorem 4.3 that satisfies $\widehat{\mu} = \lambda\mu$ together with $\text{supp}(\mu) \subset \mathbb{Z}/\sqrt{n}$ and the vanishing condition $\mu(\{m/\sqrt{n}\}) = 0$ for all $j \leq m \leq j + k$.

Further, for the case $j = 0$, the vanishing condition can be improved to $\mu(\{\ell/\sqrt{n}\}) = 0$ for all $|\ell| \leq k$.

Proof The first claim is clear. Now, choosing $j = 0$ gives the existence of an eigenmeasure μ that satisfies $\widehat{\mu} = \lambda\mu$ together with $\text{supp}(\mu) \subset \mathbb{Z}/\sqrt{n}$ and $\mu(\{m/\sqrt{n}\}) = 0$ for all $0 \leq m \leq k$.

Finally, since μ is an eigenmeasure, Fact 3.1 gives

$$\mu(\{-m/\sqrt{n}\}) = \lambda^2 \mu(\{m/\sqrt{n}\}) = 0,$$

which holds for all $0 \leq m \leq k$. $\square$

This shows that doubly sparse measures with large gaps around 0 exist that are Fourier eigenmeasures. For $12 \leq n \in \mathbb{N}$, setting $k = \lfloor \frac{n}{4} \rfloor - 3$, such measures μ vanish on the open interval $(-\frac{k+1}{\sqrt{n}}, \frac{k+1}{\sqrt{n}})$, so μ has a gap of length $\frac{k+1}{\sqrt{n}}$. Since

$$\frac{\sqrt{n}}{4} \geq \frac{\lfloor \frac{n}{4} \rfloor}{\sqrt{n}} \geq \frac{\lfloor \frac{n}{4} \rfloor - 2}{\sqrt{n}} = \frac{k+1}{\sqrt{n}} \geq \frac{\frac{n}{4} - 3}{\sqrt{n}} = \frac{\sqrt{n}}{4} - \frac{3}{\sqrt{n}},$$

the total length of the gap grows like $\frac{\sqrt{n}}{2}$ as $n \rightarrow \infty$.

Let us quickly look at an example for $\lambda \in \{\pm 1, \pm i\}$ arbitrary, but fixed. For every sufficiently large $n \in \mathbb{N}$, select a non-zero measure

$$\mu_n = \delta_{\sqrt{n}\mathbb{Z}} * \sum_{m=0}^{n-1} c_m^{(n)} \delta_{m/\sqrt{n}} \quad (25)$$

where $c_m^{(n)} \leq 1$ with $c_0^{(n)} = c_1^{(n)} = \dots = c_k^{(n)} = 0$ and $c_{n-k} = \dots = c_{n-1} = 0$ constitute an eigenvector of the corresponding DFT. Such a measure exists by Corollary 7.4 when $k = \lfloor \frac{n}{4} \rfloor - 3$, and is translation bounded by construction. Next, such measures can be combined as follows.

Theorem 7.5 *Let $(k_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{N}$ with $\lim_n k_n = \infty$. Let $r \in \mathbb{N}$ be fixed and let $(a_n)_{n \geq 12}$ be a sequence in $\mathbb{C}$ such that $|a_n| \leq (k_n)^r$ holds for all sufficiently large $n \in \mathbb{N}$. Then, with the measures μ_n from (25),*

$$\mu := \sum_{n=12}^{\infty} a_n \mu_{k_n}$$

is a Radon measure with locally finite support that is tempered and satisfies $\widehat{\mu} = \lambda \mu$.

Proof Since each μ_{k_n} vanishes on a gap around 0 that asymptotically grows like $\frac{\sqrt{k_n}}{2}$, the evaluation of μ on a test function of compact support effectively means the evaluation of a finite sum. This shows vague convergence.

When the test function is from the Schwartz class, the growth restriction on the a_n guarantees that the sum also converges in the topology of Schwartz space, so μ is a tempered distribution, and thus a tempered measure.

Since transformability of μ as a tempered distribution is clear, which bypasses the discontinuity issue discussed in [46], the eigenmeasure property follows by construction. $\square$

This result constructively shows that lots of Fourier eigenmeasures with locally finite support exist that fail to be translation bounded, and also fail to be uniformly discrete. Let us formulate two obvious consequences as follows.

Corollary 7.6 *Let $12 \leq n \in \mathbb{N}$ and $\lambda \in \{\pm 1, \pm i\}$ be fixed. Select some integers $0 \leq j \leq n - k - 1$, where $1 \leq k \leq \frac{n}{4} - 3$. Then, there exists a non-zero vector $c = (c_0, c_1, \dots, c_{n-1})^T \in E_\lambda$ such that $c_{j+\ell} = 0$ holds for all $0 \leq \ell \leq k$.*

In particular, for $j = 0$, the symmetry constraints then give $c_0 = c_1 = \dots = c_k = 0$ together with $c_{n-1} = c_{n-2} = \dots = c_{n-k} = 0$. $\square$

Let us briefly discuss how one can explicitly construct such an eigenvector, and hence an eigenmeasure as in the second claim of Corollary 7.4.

Remark 7.7 Let $n \geq 12$ and let $\{v_1, \dots, v_m\}$ be a basis for the eigenspace E_λ . Let A be the matrix with vectors $v_1^T, \dots, v_m^T$ as rows and $c^T = (c_0, \dots, c_n)$ be the last row in the reduced row echelon form of A .

Then, $c \in E_\lambda$ with $c_0 = \dots = c_{m-2} = 0$, and Remark 4.5 now gives the relations $c_{n-1} = \dots = c_{n-m+2} = 0$. Consequently, the eigenvector c satisfies the second conclusion of Corollary 7.6. Then, the corresponding eigenmeasure given by (13) is an eigenmeasure with a large gap at 0. $\diamond$

8 Outlook

The result of Theorem 7.1 implies that the characterisation of all doubly sparse measures, an important open problem in the area of Aperiodic Order as well as in Harmonic Analysis, is equivalent to the characterisation of all doubly sparse eigenmeasures. This is an interesting problem which is made difficult by the fact that, besides linear combination of eigenmeasures with uniformly discrete support, this class contains many other examples [24, 34, 35], for instance with a support of the form $\{\pm\sqrt{m+1/r} : m \in \mathbb{N}_0\}$ with $r \in \mathbb{N}$, which is not uniformly discrete. Two particularly nice ones derive from the work of Guinand [19].

Note that all measures in Theorem 7.5 are supported inside $\bigcup_n \mathbb{Z}/\sqrt{n}$. One can go beyond this situation by selecting any bounded sequence $(s_n)_{n \in \mathbb{N}}$ and defining

$$\mu := P_\lambda \left(\sum_n a_n \delta_{s_n} * \mu_{k_n} \right).$$

It is then natural to ask whether the measure in [34], or even any Fourier eigenmeasure with locally finite support, can be obtained this way.

Another interesting question, in a rather different direction, is to what extent, if any, our results can be used in the spectral theory of dynamical systems. While the equivalence of pure point dynamical spectrum and pure point diffraction spectrum is well understood, the full meaning of the eigenmeasure property seems open.

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Appendix: Discrete Fourier Transform

Here, we recall some well-known properties of the DFT, where we refer the reader to the rather informative WIKIPEDIA entry on the DFT for further details and references. Given $n \in \mathbb{N}$, the unitary Fourier matrix $U_n \in \text{Mat}(n, \mathbb{C})$ has matrix entries $\omega^{k\ell}/\sqrt{n}$ with $\omega = e^{-2\pi i/n}$ and $0 \leq k, \ell \leq n-1$, where we label the entries starting with 0. Since $n=1$ is trivial, we only consider $n \geq 2$. While U_2 is an involution, U_n has order 4 for all $n \geq 3$, which follows from the relation

$$U_n^2 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \mathbf{0} & & 1 \\ \vdots & & \ddots & \\ 0 & 1 & & \mathbf{0} \end{pmatrix}.$$

The eigenvalues and their multiplicities show a structure modulo 4, as already known to Gauss. They are summarised in Table 1.

In line with Proposition 3.3, for $\lambda \in \{\pm 1, \pm i\}$, we define

$$P_\lambda^{(n)} := \frac{1}{4} \sum_{\ell=0}^3 \lambda^{-\ell} U_n^\ell,$$

which are projectors with $P_\lambda^{(n)} P_{\lambda'}^{(n)} = \delta_{\lambda, \lambda'} P_\lambda^{(n)}$. They also satisfy the relations $U_n P_\lambda^{(n)} = P_\lambda^{(n)} U_n = \lambda P_\lambda^{(n)}$ and $P_1^{(n)} + P_i^{(n)} + P_{-1}^{(n)} + P_{-i}^{(n)} = \mathbb{1}_n$. In particular, the eigenspace of U_n for the eigenvalue λ is then simply $E_\lambda = P_\lambda^{(n)} \mathbb{C}^n$, and the rank of $P_\lambda^{(n)}$ is the dimension of the eigenspace, as given by Table 1.

Table 1 Eigenvalues of U_n and their multiplicities

n	$\lambda = 1$	$\lambda = i$	$\lambda = -1$	$\lambda = -i$
$4m$	$m+1$	$m-1$	m	m
$4m+1$	$m+1$	m	m	m
$4m+2$	$m+1$	m	$m+1$	m
$4m+3$	$m+1$	m	$m+1$	$m+1$

Concretely, we have

$$U_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

with eigenvalues ± 1 and eigenvectors $(1 \pm \sqrt{2}, 1)$. Since the DFT matrices are symmetric, left and right eigenvectors map to each other under transposition. We state them in row form. The projectors read

$$P_{\pm 1}^{(2)} = \frac{1}{2}(\mathbb{1}_2 \pm U_2) = \frac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{2} \pm 1 & \pm 1 \\ \pm 1 & \sqrt{2} \mp 1 \end{pmatrix} \quad \text{and} \quad P_{\pm i}^{(2)} = \mathbb{0}_2,$$

where the first rows of the symmetric projectors $P_{\pm 1}^{(2)}$ correspond to the above eigenvectors.

For $n = 3$, setting $\omega = e^{-2\pi i/3}$, one finds

$$U_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \bar{\omega} \\ 1 & \bar{\omega} & \omega \end{pmatrix},$$

with eigenvectors $(1 \pm \sqrt{3}, 1, 1)$ for $\lambda = \pm 1$ and $(0, -1, 1)$ for $\lambda = -i$. Here, we have $P_i^{(3)} = \mathbb{0}_3$ together with

$$P_{\pm 1}^{(3)} = \frac{1}{4} \begin{pmatrix} \frac{2(3 \pm \sqrt{3})}{3} & \pm \frac{2}{\sqrt{3}} & \pm \frac{2}{\sqrt{3}} \\ \pm \frac{2}{\sqrt{3}} & \frac{3 \mp \sqrt{3}}{3} & \frac{3 \mp \sqrt{3}}{3} \\ \pm \frac{2}{\sqrt{3}} & \frac{3 \mp \sqrt{3}}{3} & \frac{3 \mp \sqrt{3}}{3} \end{pmatrix} \quad \text{and} \quad P_{-i}^{(3)} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

For $n = 4$, one has

$$U_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix},$$

where the eigenspace for $\lambda = 1$ is two-dimensional, spanned by $(1, 0, 1, 0)$ and $(2, 1, 0, 1)$. The other eigenvectors are $(-1, 1, 1, 1)$ for $\lambda = -1$ and $(0, -1, 0, 1)$ for $\lambda = -i$. Here, the projectors are $P_i^{(4)} = \mathbb{0}_4$ together with

$$P_{\pm 1}^{(4)} = \frac{1}{4} \begin{pmatrix} 2 \pm 1 \pm 1 & \pm 1 & \pm 1 \\ \pm 1 & 1 & \mp 1 & 1 \\ \pm 1 & \mp 1 & 2 \pm 1 & \mp 1 \\ \pm 1 & 1 & \mp 1 & 1 \end{pmatrix} \quad \text{and} \quad P_{-i}^{(4)} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix},$$

where the rank of $P_1^{(4)}$ is two, as it must.

Next, $n = 5$ is the first case where all possible eigenvalues occur. With $\omega = e^{-2\pi i/5}$, we get

$$U_5 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \bar{\omega}^2 & \bar{\omega} \\ 1 & \omega^2 & \bar{\omega} & \omega & \bar{\omega}^2 \\ 1 & \bar{\omega}^2 & \omega & \bar{\omega} & \omega^2 \\ 1 & \bar{\omega} & \bar{\omega}^2 & \omega^2 & \omega \end{pmatrix},$$

where $(\tau, 1, 0, 0, 1)$ and $(\tau, 0, 1, 1, 0)$ span the eigenspace of $\lambda = 1$, with the golden ratio $\tau = \frac{1}{2}(1 + \sqrt{5})$. The other eigenvectors are $(0, -1, \tau \pm \eta, -(\tau \pm \eta), 1)$ with $\eta = \sqrt{\tau+2}$ for $\lambda = \pm i$, and $(-\frac{2}{\tau}, 1, 1, 1, 1)$ for $\lambda = -1$. Here, the projectors are

$$P_{\pm 1}^{(5)} = \frac{1}{20} \begin{pmatrix} 10 \pm (4\tau-2) & \pm(4\tau-2) & \pm(4\tau-2) & \pm(4\tau-2) & \pm(4\tau-2) \\ \pm(4\tau-2) & 5 \pm (3-\tau) & \mp(\tau+2) & \mp(\tau+2) & 5 \pm (3-\tau) \\ \pm(4\tau-2) & \mp(\tau+2) & 5 \pm (3-\tau) & 5 \pm (3-\tau) & \mp(\tau+2) \\ \pm(4\tau-2) & \mp(\tau+2) & 5 \pm (3-\tau) & 5 \pm (3-\tau) & \mp(\tau+2) \\ \pm(4\tau-2) & 5 \pm (3-\tau) & \mp(\tau+2) & \mp(\tau+2) & 5 \pm (3-\tau) \end{pmatrix}$$

and

$$P_{\pm i}^{(5)} = \frac{1}{40} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 10 \mp \sqrt{20}\eta & \mp\sqrt{60-20\tau} & \pm\sqrt{60-20\tau} & -10 \pm \sqrt{20}\eta \\ 0 & \mp\sqrt{60-20\tau} & 10 \pm \sqrt{20}\eta & -10 \mp \sqrt{20}\eta & \pm\sqrt{60-20\tau} \\ 0 & \pm\sqrt{60-20\tau} & -10 \mp \sqrt{20}\eta & 10 \pm \sqrt{20}\eta & \mp\sqrt{60-20\tau} \\ 0 & -10 \pm \sqrt{20}\eta & \pm\sqrt{60-20\tau} & \mp\sqrt{60-20\tau} & 10 \mp \sqrt{20}\eta \end{pmatrix},$$

where the rank of $P_1^{(5)}$ is two, while all others have rank one.

As a last case, let us consider $n = 6$. Here, with $\omega = e^{-\pi i/3}$, we have

$$U_6 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & -1 & \bar{\omega}^2 & \bar{\omega} \\ 1 & \omega^2 & \bar{\omega}^2 & 1 & \omega^2 & \bar{\omega}^2 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \bar{\omega}^2 & \omega^2 & 1 & \bar{\omega}^2 & \omega^2 \\ 1 & \bar{\omega} & \bar{\omega}^2 & -1 & \omega^2 & \omega \end{pmatrix}.$$

The eigenspace for $\lambda = 1$ is two-dimensional, spanned by $(\beta, 1, 0, 1 - \beta, 0, 1)$ and $(1 + \beta, 0, 1, \beta, 1, 0)$, as is that for $\lambda = -1$, which is spanned by $(-\beta, 1, 0, 1 + \beta, 0, 1)$ and $(1 - \beta, 0, 1, -\beta, 1, 0)$, with $\beta = \sqrt{3}/2$ in both cases. The remaining eigenvectors are given by $(0, -1, 1 \pm \sqrt{2}, 0, -(1 \pm \sqrt{2}), 1)$ for $\lambda = \pm i$, respectively. The projectors read

$$P_{\pm 1}^{(6)} = \frac{1}{4\sqrt{6}} \begin{pmatrix} 2(\sqrt{6} \pm 1) & \pm 2 & \pm 2 & \pm 2 & \pm 2 & \pm 2 \\ \pm 2 & \sqrt{6} \pm 1 & \mp 1 & \mp 2 & \mp 1 & \sqrt{6} \pm 1 \\ \pm 2 & \mp 1 & \sqrt{6} \mp 1 & \pm 2 & \sqrt{6} \mp 1 & \mp 1 \\ \pm 2 & \mp 2 & \pm 2 & 2(\sqrt{6} \mp 1) & \pm 2 & \mp 2 \\ \pm 2 & \mp 1 & \sqrt{6} \mp 1 & \pm 2 & \sqrt{6} \mp 1 & \mp 1 \\ \pm 2 & \sqrt{6} \pm 1 & \mp 1 & \mp 2 & \mp 1 & \sqrt{6} \pm 1 \end{pmatrix}$$

and

$$P_{\pm i}^{(6)} = \frac{1}{8} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 \mp \sqrt{2} & \mp \sqrt{2} & 0 & \pm \sqrt{2} & -2 \pm \sqrt{2} \\ 0 & \mp \sqrt{2} & 2 \pm \sqrt{2} & 0 & -2 \mp \sqrt{2} & \pm \sqrt{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \pm \sqrt{2} & -2 \mp \sqrt{2} & 0 & 2 \pm \sqrt{2} & \mp \sqrt{2} \\ 0 & -2 \pm \sqrt{2} & \pm \sqrt{2} & 0 & \mp \sqrt{2} & 2 \mp \sqrt{2} \end{pmatrix}.$$

Here, $P_{\pm 1}^{(6)}$ have rank two and $P_{\pm i}^{(6)}$ are of rank one.

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