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ORIGINAL ARTICLE

Tempered distributions with translation bounded measure as Fourier transform and the generalized Eberlein decomposition

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Abstract
In this paper, we study the class of tempered distributions whose Fourier transform is a translation bounded measure and show that each such distribution in $\mathbb{R}^d$ has order at most $2d$. We show the existence of the generalized Eberlein decomposition within this class of distributions, and its compatibility with all previous Eberlein decompositions. The generalized Eberlein decomposition for Fourier transformable measures and properties of its components are discussed. Lastly, we take a closer look at the absolutely continuous spectrum of measures supported on Meyer sets.

KEYWORDS
almost periodic measures, Fourier transform of measures, Lebesgue decomposition

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1 | INTRODUCTION

Diffraction is one of the main techniques used by physicists to study the atomic structure of materials. It is used in material sciences to determine the arrangements of atoms in metals and crystals and in biology to determine the structure of organic compounds.

Mathematically, the process of diffraction is modeled as follows: Starting with a set $\Lambda \subset \mathbb{R}^d$, representing the position of atoms in the structure, or more generally a (translation bounded) measure μ, one can construct a measure γ called the autocorrelation (or 2-point correlation) of Λ or μ (see [3] for the general background). The measure γ is positive definite and hence Fourier transformable [1, 9, 22], and its Fourier transform $\widehat{\gamma}$ is a positive measure [1, 9, 22]. It is this measure $\widehat{\gamma}$ that models the outcome of a physical diffraction experiment, and for this reason, it is called the diffraction measure of Λ (or μ). It has a decomposition (with respect to the Lebesgue measure $\boldsymbol{\lambda}$)

$$\widehat{\gamma} = (\widehat{\gamma})_{\mathrm{pp}} + (\widehat{\gamma})_{\mathrm{ac}} + (\widehat{\gamma})_{\mathrm{sc}} ,$$

into positive pure point, absolutely continuous, and singular continuous measures, respectively.

The pure point spectrum $(\widehat{\gamma})_{\rm pp}$ is usually associated to the order present in the structure. Structures with pure point diffraction, meaning $\widehat{\gamma} = (\widehat{\gamma})_{\rm pp}$, have been of special interest to physicists and mathematicians. While for a log time pure point diffraction has been associated with periodicity, the discovery of quasicrystals [28] has led to a paradigm shift and showed that there is much we do not understand about physical diffraction. It is now known that pure point diffraction is equivalent to the mean almost periodicity of the structure [16, 17].

Over the years, it has become clear that structures with nontrivial continuous spectrum can also have interesting properties, and hence are of interest. These include examples like the pinwheel tiling, the Thue–Morse chain [3, Theorem 10.1], the Rudin–Shapiro [3, Theorem 10.2] chain, and many random structures [3, Chapter 11]. While the case of structures with pure point diffraction is very well understood now [16, 17], the study of systems with continuous or mixed spectrum is just in its infancy. The singular continuous diffraction spectrum $(\widehat{\gamma})_{\rm sc}$ in particular is very mysterious and very little is known about it in general.

One of the most useful tools in the study of the components of the diffraction is the Eberlein decomposition (see below for definition and properties). Given a measure μ with autocorrelation γ, there exists a unique decomposition (see Theorem 2.17 and Theorem 2.18 below)

$$\gamma = \gamma_{\rm s} + \gamma_0 \,,$$

called the Eberlein decomposition, with the property that both $\gamma_{\rm s}$ and γ_0 are positive definite measures and

$$\widehat{\gamma_{\rm s}} = (\widehat{\gamma})_{\rm pp} \qquad \text{and} \qquad \widehat{\gamma_0} = (\widehat{\gamma})_{\rm c} \,.$$

This allows us to study the pure point diffraction spectrum $(\widehat{\gamma})_{\rm pp}$ and continuous diffraction spectrum $(\widehat{\gamma})_{\rm c}$ of μ, respectively, in the Fourier dual space by studying instead the components $\gamma_{\rm s}$ and γ_0, respectively, of the autocorrelation measure γ in the real space. This approach has proven effective in the study of diffraction of measures with Meyer set support [31–34, 36] and in the study of compatible random substitutions [7]. Because of this, one would like to further decompose

$$\gamma_0 = \gamma_{0a} + \gamma_{0s},$$

such that

$$\widehat{\gamma_{\rm 0s}} = (\widehat{\gamma})_{\rm sc} \qquad \text{and} \qquad \widehat{\gamma_{\rm 0a}} = (\widehat{\gamma})_{\rm ac}.$$

Whenever this is possible, we will refer to

$$\gamma = \gamma_{\rm s} + \gamma_{0a} + \gamma_{0s} \tag{1.1}$$

as the complete (or generalized) Eberlein decomposition of γ.

The generalized Eberlein decomposition exists for Fourier transformable measures with Meyer set support [34, 36]. This result has important consequences for one-dimensional Pisot substitutions, see, for example, [2, 4], and we expect that establishing the existence of the generalized Eberlein decomposition may have important consequences in general. This makes it an important open problem for diffraction theory. Let us state this explicitly.

Question 1.1. Given a positive definite (or more generally a Fourier transformable) measure γ on $\mathbb{R}^d$, can one find a decomposition as in (1.1), which is the Fourier dual to the Lebesgue decomposition of $\widehat{\gamma}$?

It is the main goal of this paper to show that each Fourier transformable measure γ on $\mathbb{R}^d$ has a generalized Eberlein decomposition (1.1) with γ_{0a}, γ_{0s} tempered distributions of order $2d$.

Our approach is as follows. Since for a Fourier transformable measure γ, the measure $\widehat{\gamma}$ is translation bounded [22, Theorem 4.9.23], we restrict our attention to the class $\mathcal{DTBM}(\mathbb{R}^d)$ of tempered distributions whose Fourier transform is a translation bounded measure (see paragraph before Remark 2.1, for the definition of translation bounded measures). The Fourier transform then becomes a bijection between $\mathcal{DTBM}(\mathbb{R}^d)$ and $\mathcal{M}^\infty(\mathbb{R}^d)$. So, we will first study the properties of the space $\mathcal{DTBM}(\mathbb{R}^d)$. In the first main result of the paper, Theorem 3.4, we show that for each $\omega \in \mathcal{DTBM}(\mathbb{R}^d)$, there exists some $f \in C_{\mathrm{u}}(\mathbb{R}^d)$ such that

$$\omega = \left((\partial_{x_1})^{2d} + \cdots + (\partial_{x_d})^{2d} + (-1)^d\right)(f\boldsymbol{\lambda})$$

in the sense of distributions. In particular, it follows that each element $\omega \in \mathcal{DTBM}(\mathbb{R}^d)$ is a distribution of order (at most) $2d$.

Next, we prove in Proposition 4.1 that each distribution $\omega \in \mathcal{DTBM}(\mathbb{R}^d)$ admits a generalized Eberlein decomposition

$$\omega = \omega_{\mathrm{s}} + \omega_{0\mathrm{a}} + \omega_{0\mathrm{s}}\,,$$

with $\omega_{\mathrm{s}}, \omega_{0\mathrm{a}}, \omega_{0\mathrm{s}} \in \mathcal{DTBM}(\mathbb{R}^d)$, which is in Fourier duality with the Lebesgue decomposition. By combining these results, we prove in Theorem 4.3 that each Fourier transformable measure γ on $\mathbb{R}^d$ admits a generalized Eberlein decomposition

$$\gamma = \gamma_{\mathrm{s}} + \omega_{0\mathrm{a}} + \omega_{0\mathrm{s}},$$

where γ_{s} and $\gamma_0 := \omega_{0\mathrm{a}} + \omega_{0\mathrm{s}}$ are Fourier transformable measures and $\omega_{0\mathrm{a}}$, $\omega_{0\mathrm{s}}$ are tempered distributions of order (at most) 2d.

We complete the paper by taking a closer look in Section 5 at those distributions in $\mathcal{DTBM}(\mathbb{R}^d)$ whose Fourier transform is pure point, continuous, or absolutely continuous measure, respectively.

In general, given a positive definite measure γ, one would hope all three components of the generalized Eberlein decomposition are measures. It is not clear if this is the case. More importantly, one can define the diffraction of more general processes (see, e.g., [15] and [16, section 1.4 and Theorem 3.28]) in which case the autocorrelation is not necessarily a measure but the diffraction is a translation bounded measure. In this case, the autocorrelation can be viewed as an element in $\mathcal{DTBM}(\mathbb{R}^d)$, and hence our approach can be used in the study of such more general objects.

Lastly, let us mention here that some of the results of this paper can be extended to the space $\mathcal{S}_{\widehat{\mathcal{M}}}(\mathbb{R}^d)$ of tempered distributions with measure Fourier transform, which was introduced and studied in [37]. However, in diffraction theory, we only deal with measures whose Fourier transform is a translation bounded measure, and as translation boundedness gives more properties and simplifies many proofs (see, e.g., Theorem 3.4), we will restrict to $\mathcal{DTBM}(\mathbb{R}^d)$ in this paper.

2 PRELIMINARIES

Throughout this paper, we will work in the d-dimensional Euclidean space $\mathbb{R}^d$. The Lebesgue measure will be denoted by $\boldsymbol{\lambda}$ or simply $\mathrm{d}x$. We will denote by $C_{\mathrm{u}}(\mathbb{R}^d)$, $C_{\mathrm{c}}(\mathbb{R}^d)$, $\mathsf{S}(\mathbb{R}^d)$, and $C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$, the spaces of uniformly continuous and bounded functions, compactly supported continuous functions, Schwartz functions, and compactly supported functions, which are arbitrarily often differentiable, respectively. For any function g on $\mathbb{R}^d$ and $t \in \mathbb{R}^d$, the functions $T_t g, \widetilde{g}$, and $g^{\dagger}$ are defined by

$$(T_t g)(x) := g(x-t)\,, \qquad \widetilde{g}(x) := \overline{g(-x)}, \qquad \text{and} \qquad g^{\dagger}(x) := g(-x)\,.$$

A *measure* μ on $\mathbb{R}^d$ is a linear functional on $C_{\mathrm{c}}(\mathbb{R}^d)$ such that, for every compact subset $K \subset \mathbb{R}^d$, there is a constant $a_K > 0$ with

$$|\mu(\varphi)| \leqslant a_K\, \|\varphi\|_{\infty}$$

for all $\varphi \in C_{\mathrm{c}}(\mathbb{R}^d)$ with $\mathrm{supp}(\varphi) \subseteq K$. Here, $\|\varphi\|_{\infty}$ denotes the supremum norm of φ. By Riesz' representation theorem, this definition is equivalent to the classical measure theory concept of regular Radon measure.

Similar to functions, for a measure μ on $\mathbb{R}^d$ and $t \in \mathbb{R}^d$, we define $T_t\mu, \widetilde{\mu}$, and $\mu^{\dagger}$ by

$$(T_t\mu)(\varphi) := \mu(T_{-t}\varphi)\,, \qquad \widetilde{\mu}(\varphi) := \overline{\mu(\widetilde{\varphi})}, \qquad \text{and} \qquad \mu^{\dagger}(\varphi) := \mu(\varphi^{\dagger}).$$

Given a measure μ, there exists a positive measure $|\mu|$ such that, for all $\psi \in C_{\mathsf{c}}(\mathbb{R}^d)$ with $\psi \geqslant 0$, we have [23] (compare [25, Appendix])

$$|\mu|(\psi) = \sup\{|\mu(\varphi)| \ : \ \varphi \in C_{\mathsf{c}}(\mathbb{R}^d),\ |\varphi| \leqslant \psi\}.$$

The measure $|\mu|$ is called the *total variation of* μ.

The measure μ is called *translation bounded* if for all compact sets $K \subset \mathbb{R}^d$, we have

$$\|\mu\|_K := \sup_{t\in\mathbb{R}^d} |\mu|(t+K) < \infty .$$

We will denote the space of translation bounded measures by $\mathcal{M}^\infty(\mathbb{R}^d)$.

Remark 2.1.

(a) A measure μ is translation bounded if and only if for all $\varphi \in C_{\mathsf{c}}(\mathbb{R}^d)$, we have [1, 22] $\varphi * \mu \in C_{\mathsf{u}}(\mathbb{R}^d)$. Here,

$$(\varphi * \mu)(x) := \int_{\mathbb{R}^d} \varphi(x-y)\, \mathrm{d}\mu(y) .$$

(b) A measure μ is translation bounded if and only if $\|\mu\|_U < \infty$ for a single precompact Borel set with nonempty interior [6, 30]. Moreover, in this case, two different precompact Borel sets with nonempty interior define equivalent norms.

Since $C_{\mathsf{c}}^\infty(\mathbb{R}^d) \subseteq C_{\mathsf{c}}(\mathbb{R}^d)$, each measure is a distribution of order 0. Recall here that a distribution T on $\mathbb{C}_{\mathsf{c}}^\infty(\mathbb{R}^d)$ is of *finite order* if, for each compact set $K \subseteq \mathbb{R}^d$, there are a number $N \in \mathbb{N}$ and a constant $c < \infty$ such that

$$|T\varphi| \ \leqslant \ c\, \|\varphi\|_N \ =: \ c \max\{|D^\alpha \varphi(x)| \ : \ x \in \mathbb{R}^d,\ |\alpha| \leqslant N\}$$

holds for all $\varphi \in C_{\mathsf{c}}^\infty(\mathbb{R}^d)$ with $\mathrm{supp}(\varphi) \subseteq K$. If T is such that one N will do for all K (but not necessarily with the same c), the smallest such N is called the *order* of T [26]. A measure μ is called a *tempered measure* if $\mu : C_{\mathsf{c}}^\infty(\mathbb{R}^d) \to \mathbb{R}^d$ is the restriction to $C_{\mathsf{c}}^\infty(\mathbb{R}^d)$ of a tempered distribution $\omega : \mathsf{S}(\mathbb{R}^d) \to \mathbb{C}$. It is easy to see that this is equivalent to $\mu : C_{\mathsf{c}}^\infty(\mathbb{R}^d) \to \mathbb{C}$ being continuous with respect to the Schwartz topology induced by the embedding $C_{\mathsf{c}}^\infty(\mathbb{R}^d) \hookrightarrow \mathsf{S}(\mathbb{R}^d)$ (see, e.g., [1]).

Remark 2.2. Let us recall that a measure μ is called

(1) *slowly increasing* if there exists a polynomial $P \in \mathbb{R}[X_1, \dots, X_d]$ such that

$$\int_{\mathbb{R}^d} \frac{1}{1+|P(x)|}\, \mathrm{d}|\mu|(x) < \infty ,$$

(2) *strongly tempered* if $|\mu|$ is tempered.

A measure μ is strongly tempered if and only if it is slowly increasing [8, Theorem 2.6]. In this case, μ is tempered, see, for example, [1, p. 47] or [8, Lemma 2.3]. The converse of the last claim holds only for positive measures: Any positive tempered measure is strongly tempered [27, p. 242] [13, Theorem 2.1], and there are examples of signed tempered measures, which are not strongly tempered [1, 8]. Also note that any translation bounded measure is slowly increasing and hence tempered [1, Theorem 7.1]. For a review of properties and relationships among these concepts, we refer the reader to [8].

The basic tool we will use in this paper is the following estimate. As this estimate gives an alternate proof that every translation bounded measure is slowly increasing, we include it here. Our proof below follows the proof of [1, Theorem 7.1], and can be extended beyond $G = \mathbb{R}^d$.

Lemma 2.3. *Let $\nu \in \mathcal{M}^\infty(\mathbb{R}^d)$. Then, there is a constant $c > 0$ such that*

$$\int_{\mathbb{R}^d} \frac{1}{(2\pi x_1)^{2d} + \cdots + (2\pi x_d)^{2d} + 1} \, \mathrm{d}|\nu|(x) \leqslant c \, \|\nu\|_{[-\frac{1}{2},\frac{1}{2}]^d} < \infty .$$

In particular,

$$\frac{1}{(2\pi \mathrm{i}(\cdot)_1)^{2d} + \cdots + (2\pi \mathrm{i}(\cdot)_d)^{2d} + (-1)^d} \, \nu$$

is a finite measure on $\mathbb{R}^d$.

Proof. We first note that

$$\begin{aligned} &\int_{\mathbb{R}^d} \frac{1}{(2\pi x_1)^{2d} + \cdots + (2\pi x_d)^{2d} + 1} \, \mathrm{d}|\nu|(x) \\ &= \sum_{k \in \mathbb{Z}^d} \int_{k+[-\frac{1}{2},\frac{1}{2}]^d} \frac{1}{(2\pi x_1)^{2d} + \cdots + (2\pi x_d)^{2d} + 1} \, \mathrm{d}|\nu|(x) \\ &\leqslant \|\nu\|_{[-\frac{1}{2},\frac{1}{2}]^d} \sum_{k \in \mathbb{Z}^d} \sup_{x \in k+[-\frac{1}{2},\frac{1}{2}]^d} \frac{1}{(2\pi x_1)^{2d} + \cdots + (2\pi x_d)^{2d} + 1} . \end{aligned} \tag{2.1}$$

Now, let $M := \{k \in \mathbb{Z}^d \ : \ |k_j| \geqslant 2 \text{ for all } 1 \leqslant j \leqslant d\}$. On the one hand, since $\mathbb{Z}^d \setminus M$ is finite, one has

$$c_1 := \sum_{k \in \mathbb{Z}^d \setminus M} \sup_{x \in k+[-\frac{1}{2},\frac{1}{2}]^d} \frac{1}{(2\pi x_1)^{2d} + \cdots + (2\pi x_d)^{2d} + 1} < \infty . \tag{2.2}$$

Also note that for $d > 1$, by Hölder's inequality, we have

$$\sum_{j=1}^d (2\pi x_j)^2 \cdot 1 \leqslant \left(\sum_{j=1}^d (2\pi x_j)^{2d} \right)^{\frac{1}{d}} \left(\sum_{j=1}^d 1^q \right)^{\frac{1}{q}} ,$$

where $q = \frac{d}{d-1}$ is the conjugate of d. Therefore, one has

$$\left((2\pi x_1)^2 + (2\pi x_2)^2 + \cdots + (2\pi x_d)^2\right)^d \leqslant \left((2\pi x_1)^{2d} + (2\pi x_2)^{2d} + \cdots + (2\pi x_d)^{2d}\right) d^{d-1} . \tag{2.3}$$

Moreover, (2.3) trivially holds when $d = 1$. Therefore, we obtain

$$\begin{aligned} c_2 &:= \sum_{k \in M} \sup_{x \in k+[-\frac{1}{2},\frac{1}{2}]^d} \frac{1}{(2\pi x_1)^{2d} + \cdots + (2\pi x_d)^{2d} + 1} \\ &\overset{(2.3)}{\leqslant} \sum_{k \in M} \sup_{x \in k+[-\frac{1}{2},\frac{1}{2}]^d} \frac{d^{d-1}}{((2\pi x_1)^2 + \cdots + (2\pi x_d)^2)^d} \\ &\leqslant \sum_{|k_1| \geqslant 2} \cdots \sum_{|k_d| \geqslant 2} \left(\frac{d}{(2\pi(|k_1|-1))^2 + \cdots + (2\pi(|k_d|-1))^2} \right)^d \\ &\leqslant \left(\sum_{|k_1| \geqslant 1} \frac{d}{(2\pi k_1)^2} \right) \cdot \ldots \cdot \left(\sum_{|k_d| \geqslant 1} \frac{d}{(2\pi k_d)^2} \right) < \infty . \end{aligned} \tag{2.4}$$

Finally, Equations (2.1), (2.2), and (2.4) give

$$\int_{\mathbb{R}^d} \frac{1}{(2\pi x_1)^{2d} + \cdots + (2\pi x_d)^{2d} + 1} \,\mathrm{d}|\nu|(x) \leqslant \|\nu\|_{[-\frac{1}{2},\frac{1}{2}]^d} (c_1 + c_2)\,.$$

The claim follows with $c := c_1 + c_2$. □

As immediate consequences of Lemma 2.3, we obtain the following results.

Corollary 2.4. *Let $\nu \in \mathcal{M}^\infty(\mathbb{R}^d)$. Then,*

$$\frac{1}{\left((2\pi \mathrm{i}(\cdot)_1)^2 + \cdots + (2\pi \mathrm{i}(\cdot)_d)^2\right)^d + (-1)^d}\,\nu$$

is a finite measure on $\mathbb{R}^d$.

Proof. This follows immediately from Lemma 2.3 and the inequality

$$\frac{1}{\left((2\pi x_1)^2 + \cdots + (2\pi x_d)^2\right)^d + 1} \leqslant \frac{1}{(2\pi x_1)^{2d} + \cdots + (2\pi x_d)^{2d} + 1}\,.$$ □

Another consequence is that each translation bounded measure is slowly increasing, compare [1, Theorem 7.1], and thus a tempered distribution. Let us state this explicitly.

Corollary 2.5. *Let $\mu \in \mathcal{M}^\infty(\mathbb{R}^d)$. Then, for all $f \in S(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^d} |f(s)|\, d|\mu|(s) < \infty\,.$$

Moreover, the mapping $f \mapsto \int_{\mathbb{R}^d} f(s)\, d\mu(s)$ is a tempered distribution.

Proof. By the above with $P(x_1, \dots, x_d) := (2\pi x_1)^{2d} + \cdots + (2\pi x_d)^{2d}$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} |f(x)|\,\mathrm{d}|\mu|(x) &\leqslant \int_{\mathbb{R}^d} |f(x)(1+P(x))|\,\mathrm{d}\left(\frac{1}{1+P}|\mu|\right)(x)\\ &\leqslant C\|(1+P)f\|_\infty\end{aligned}$$

for all $f \in \mathrm{S}(\mathbb{R}^d)$, where

$$C = \int_{\mathbb{R}^d} \frac{1}{(2\pi x_1)^{2d} + \cdots + (2\pi x_d)^{2d} + 1}\,\mathrm{d}|\nu|(x) < \infty\,.$$

The claim follows now from [8, Theorem 2.7]. □

Finally, let us introduce some basic concepts about point sets, which we will see in the paper.

Definition 2.6. A point set $\Lambda \subset \mathbb{R}^d$ is called *relatively dense* if there is a compact set $K \subset \mathbb{R}^d$ such that $\Lambda + K = \mathbb{R}^d$.

The set Λ is called *uniformly discrete* if there is an open neighborhood U of 0 such that $(x + U) \cap (y + U) = \varnothing$ for all distinct $x, y \in \Lambda$.

The set Λ is called a *Meyer set* if it is relatively dense and $\Lambda - \Lambda$ is uniformly discrete. Here, $\Lambda - \Lambda$ denotes the *Minkowski difference*

$$\Lambda - \Lambda := \{x - y \;:\; x, y \in \Lambda\}\,.$$

For more properties of Meyer sets and their full classification, we refer the reader to [14, 19, 21].

2.1 | Fourier transformability

Next, we briefly review the notion of Fourier transformability for measures and the connection to the Fourier transform of tempered distributions. Let us recall that the Fourier transform and inverse Fourier transform, respectively, of a function $f \in L^1(\mathbb{R}^d)$ are denoted by $\widehat{f}$ and $\widecheck{f}$. They are defined via

$$\widehat{f}(x) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot y} f(y)\,\mathrm{d}y \qquad \text{and} \qquad \widecheck{f}(x) = \widehat{f}(-x)\,.$$

Next, let us review the definition of Fourier transformability for measures.

Definition 2.7. A measure μ on $\mathbb{R}^d$ is called *Fourier transformable as measure* if there exists a measure $\widehat{\mu}$ on $\mathbb{R}^d$ such that

$$\widecheck{\varphi} \in L^2(|\widehat{\mu}|) \qquad \text{and} \qquad \mu(\varphi * \widetilde{\varphi}) = \widehat{\mu}\big(|\widecheck{\varphi}|^2\big)$$

for all $\varphi \in C_{\mathrm{c}}(\mathbb{R}^d)$. In this case, $\widehat{\mu}$ is called the *measure Fourier transform* of μ.

We will often use the following result.

Theorem 2.8 [35, Theorem 5.2]. *Let μ be a measure on $\mathbb{R}^d$. Then, μ is Fourier transformable as a measure if and only if μ is tempered as a distribution and its Fourier transform as a tempered distribution is a translation bounded measure. Moreover, in this case, the Fourier transform of μ in the measure and distribution sense coincide.*

The following result is easy to prove and will be useful later in the paper. Note that the definition of the Fourier transformability of measures guarantees that (2.5) below holds for all Fourier transformable measures μ and all $\varphi \in K_2(\mathbb{R}^d) = \mathrm{Span}\{\psi * \phi \,:\, \psi, \phi \in C_{\mathrm{c}}(\mathbb{R}^d)\}$. Nevertheless, there are many Fourier transformable measures for which (2.5) fails for functions $\varphi \in C_{\mathrm{c}}(\mathbb{R}^d)\backslash K_2(\mathbb{R}^d)$. The fact that ρ is a finite measure is crucial in Proposition 2.9.

Proposition 2.9. *Let ρ be a finite measure on $\mathbb{R}^d$, let $h = \widecheck{\rho}$, and let $\mu = h\boldsymbol{\lambda}$, where $\boldsymbol{\lambda}$ denotes the Lebesgue measure. Then, for all $\varphi \in C_{\mathrm{c}}(\mathbb{R}^d)$, we have*

$$\mu(\varphi) = \rho(\widecheck{\varphi})\,. \tag{2.5}$$

Proof. Note here that since ν is finite, all integrals below are finite and hence, by Fubini's theorem,

$$\begin{aligned}\mu(\varphi) &= \int_{\mathbb{R}^d} \varphi(x)\, h(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \varphi(x) \int_{\mathbb{R}^d} \mathrm{e}^{2\pi \mathrm{i} x\cdot y}\,\mathrm{d}\rho(y)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)\,\mathrm{e}^{2\pi \mathrm{i} x\cdot y}\,\mathrm{d}x\,\mathrm{d}\rho(y) = \int_{\mathbb{R}^d} \widecheck{\varphi}(y)\,\mathrm{d}\rho(y) = \rho(\widecheck{\varphi})\,,\end{aligned}$$

which establishes the desired equality. □

2.2 | Diffraction theory

In this section, we briefly review the mathematical diffraction theory. For a more detailed review, we recommend [3, Chapter 9].

Consider a translation bounded measure $\mu \in \mathcal{M}^\infty(\mathbb{R}^d)$, and denote by μ_n the restriction of μ to the cube $[-n,n]^d$. Let

$$\gamma_n := \frac{\mu_n * \widetilde{\mu_n}}{(2n)^d}\,.$$

Then, the sequence $(\gamma_n)_{n\in\mathbb{N}}$ has a subsequence converging in the vague topology for measures to some measure γ, compare [3, Proposition 9.1]. Following Hof [12], we introduce the following definition.

Definition 2.10. We will refer to any vague cluster point γ of $(\gamma_n)_n$ as an *autocorrelation measure of* μ.

Remark 2.11.

(1) In the definition of the autocorrelation measure, one can replace the cubes $[-n, n]^d$ by balls of radius n centered in the origin, or more generally by van Hove sequences.
(2) The restriction to translation bounded measures μ is usually imposed to guarantee that $(\gamma_n)_{n\in\mathbb{N}}$ has at least one cluster point. One can replace it with the weaker assumption that $(\gamma_n)_{n\in\mathbb{N}}$ has at least one cluster point γ.
(3) In [37], the authors studied the diffraction of a large class of distribution on $\mathbb{R}^d$.

Since each measure γ_n is positive definite, so is its limit, see [22, section 11] for definition and properties of positive definite measures. It follows that γ is a Fourier transformable measure and its Fourier transform $\widehat{\gamma}$ is a positive measure [22, Theorem 4.11.5].

Definition 2.12. Let μ be a translation bounded measure, and let γ be an autocorrelation of μ. The measure $\widehat{\gamma}$ is called *a diffraction measure of* μ.

If $\Lambda \subseteq \mathbb{R}^d$ is uniformly discrete, then

$$\delta_\Lambda := \sum_{x\in\Lambda} \delta_x$$

is a translation bounded measure. We will refer to any autocorrelation γ and diffraction $\widehat{\gamma}$ of δ_Λ simply as *autocorrelation and diffraction of* Λ.

Example 2.13. Consider $\Lambda := \mathbb{Z} \subseteq \mathbb{R}$. A simple computation shows that $(\gamma_n)_{n\in\mathbb{N}}$ converges vaguely to $\gamma = \delta_{\mathbb{Z}}$. Moreover, Poisson's Summation Formula, see, for example, [3, Proposition 9.4], gives

$$\widehat{\gamma} = \widehat{\delta_{\mathbb{Z}}} = \delta_{\mathbb{Z}}\,.$$

Therefore, the diffraction of $\mathbb{Z}$ is the pure point measure $\delta_{\mathbb{Z}}$.

2.3 | Almost periodicity

Here, we briefly review the concepts of almost periodicity for functions, measures and tempered distributions. For more general overviews, we recommend the review [22] as well as [10, 16–18, 37], just to name a few.

Definition 2.14. A function $f \in C_{\mathsf{u}}(\mathbb{R}^d)$ is *strongly (or Bohr) almost periodic*, if the closure of $\{T_t f \,:\, t \in \mathbb{R}^d\}$ in $(C_{\mathsf{u}}(\mathbb{R}^d), \|\cdot\|_\infty)$ is compact. $f \in C_{\mathsf{u}}(\mathbb{R}^d)$ is *weakly almost periodic*, if the closure of $\{T_t f \,:\, t \in \mathbb{R}^d\}$ in the weak topology of the Banach space $(C_{\mathsf{u}}(\mathbb{R}^d), \|\cdot\|_\infty)$ is compact. The spaces of strongly and weakly almost periodic functions on $\mathbb{R}^d$ are denoted by $SAP(\mathbb{R}^d)$ and $WAP(\mathbb{R}^d)$, respectively.

For every $f \in WAP(\mathbb{R}^d)$, the *mean*

$$M(f) := \lim_{n\to\infty} \frac{1}{(2n)^d} \int_{s+[-n,n]^d} f(x)\,\mathrm{d}x$$

exists uniformly in $s \in \mathbb{R}^d$, see [10, Theorem 14.1], [22, Proposition 4.5.9]. Moreover, we also have $|f| \in WAP(\mathbb{R}^d)$ [10, Theorem 11.2], [22, Proposition 4.3.11].

Definition 2.15. A function $f \in WAP(\mathbb{R}^d)$ is *null weakly almost periodic* if $M(|f|) = 0$. We denote the space of null weakly almost periodic functions by $WAP_0(\mathbb{R}^d)$.

In the spirit of [11], these notions carry over to measures and tempered distributions, respectively, via convolutions with functions in $C_c(\mathbb{R}^d)$ or $\mathsf{S}(\mathbb{R}^d)$, respectively.

Definition 2.16. A measure $\mu \in \mathcal{M}^\infty(\mathbb{R}^d)$ is *strongly*, *weakly,* or *null weakly almost periodic* if $\varphi * \mu$ is a strongly, weakly, or null weakly almost periodic function, for all $\varphi \in C_c(\mathbb{R}^d)$. We will denote by $\mathcal{SAP}(\mathbb{R}^d)$, $\mathcal{WAP}(\mathbb{R}^d)$, and $\mathcal{WAP}_0(\mathbb{R}^d)$ the spaces of strongly, weakly, and null weakly almost periodic measures.

Analogously, a tempered distribution $\omega \in \mathsf{S}'(\mathbb{R}^d)$ is called, respectively, *weakly, strongly,* or *null weakly almost periodic* if, for all $f \in \mathsf{S}(\mathbb{R}^d)$, the function $f * \omega$ is, respectively, weakly, strongly, or null weakly almost periodic. Here,

$$(f * \omega)(x) = \omega(f(x - \cdot)) \in C^\infty(\mathbb{R}^d)\,.$$

We shall denote the corresponding spaces of tempered distributions by $\mathsf{WAP}(\mathbb{R}^d)$, $\mathsf{SAP}(\mathbb{R}^d)$, and $\mathsf{WAP}_0(\mathbb{R}^d)$, respectively.

At the end of this section, we introduce the *Eberlein decomposition* for measures and distributions, and discus its relevance to diffraction theory. Let us first remind the reader of the following results.

Proposition 2.17. *[22, Theorem 4.10.10], [37, Theorem 5.2]*

(a) $\mathcal{WAP}(\mathbb{R}^d) = \mathcal{SAP}(\mathbb{R}^d) \oplus \mathcal{WAP}_0(\mathbb{R}^d)$.
(b) $\mathsf{WAP}(\mathbb{R}^d) = \mathsf{SAP}(\mathbb{R}^d) \oplus \mathsf{WAP}_0(\mathbb{R}^d)$.

Proposition 2.17 says that each $\mu \in \mathcal{WAP}(\mathbb{R}^d)$ has a unique Eberlein decomposition

$$\mu = \mu_s + \mu_0\,, \tag{2.6}$$

with $\mu_s \in \mathcal{SAP}(\mathbb{R}^d)$ and $\mu_0 \in \mathcal{WAP}_0(\mathbb{R}^d)$. Similarly, each $\omega \in \mathsf{WAP}(\mathbb{R}^d)$ has a unique Eberlein decomposition

$$\omega = \omega_s + \omega_0\,, \tag{2.7}$$

with $\omega_s \in \mathsf{SAP}(\mathbb{R}^d)$ and $\omega_0 \in \mathsf{WAP}_0(\mathbb{R}^d)$.

Furthermore, by [37, Theorem 5.3], we have $\mathcal{WAP}(\mathbb{R}^d) \subseteq \mathsf{WAP}(\mathbb{R}^d)$, $\mathcal{SAP}(\mathbb{R}^d) \subseteq \mathsf{SAP}(\mathbb{R}^d)$, and $\mathcal{WAP}_0(\mathbb{R}^d) \subseteq \mathsf{WAP}_0(\mathbb{R}^d)$. In particular, for a measure $\mu \in \mathcal{WAP}(\mathbb{R}^d) \subseteq \mathsf{WAP}(\mathbb{R}^d)$, the decompositions of (2.6) and (2.7) coincide.

The importance of the Eberlein decomposition for diffraction theory is given by the following two results.

Theorem 2.18 [22, Theorem 4.10.4 and Theorem 4.10.12]. *Let $\mu \in \mathcal{M}^\infty(\mathbb{R}^d)$ be Fourier transformable as measure. Then, $\mu \in \mathcal{WAP}(\mathbb{R}^d)$, the measures μ_s, μ_0 are Fourier transformable, and*

$$\widehat{\mu_s} = (\widehat{\mu})_{pp} \qquad \text{and} \qquad \widehat{\mu_0} = (\widehat{\mu})_{c}\,.$$

Theorem 2.19 [37, Theorem 6.1]. *Let $\omega \in \mathcal{S}'(\mathbb{R}^d)$. If $\widehat{\omega}$ is a measure, then $\omega \in \mathsf{WAP}(\mathbb{R}^d)$ and*

$$\widehat{\omega_s} = (\widehat{\omega})_{pp} \qquad \text{and} \qquad \widehat{\omega_0} = (\widehat{\omega})_{c}\,.$$

As we mentioned in the introduction, it is our goal in this paper to show the existence of the generalized Eberlein decomposition, and study some of its properties. To do this, we will work with the larger class $\mathcal{DTBM}(\mathbb{R}^d)$ of tempered distributions whose Fourier transform is a translation bounded measure.

3 | PROPERTIES OF $\mathcal{DTBM}(\mathbb{R}^d)$

As emphasized in the previous section, the following space will play the central role in this section

$$\mathcal{DTBM}(\mathbb{R}^d) := \{\omega \in \mathsf{S}'(\mathbb{R}^d) \ : \ \widehat{\omega} \in \mathcal{M}^\infty(\mathbb{R}^d)\}\,.$$

Note that by Theorem 2.19, we have $\mathcal{DTBM}(\mathbb{R}^d) \subseteq \mathsf{WAP}(\mathbb{R}^d)$. Moreover, by Theorem 2.8, if μ is a measure on $\mathbb{R}^d$, which is Fourier transformable as measure, then $\mu \in \mathcal{DTBM}(\mathbb{R}^d)$.

Let us mention that Meyer [20] introduced an example of a tempered measure τ, which is pure point, has locally finite support, and satisfies

$$\widehat{\tau} = -i\tau\,.$$

This example led to renewed interest in the study of Fourier analysis and diffraction of tempered measures, which are not (necessarily) translation bounded. As the tools we are using in this paper often rely on the translation boundedness of $\widehat{\omega}$, the study of Meyer-type measures is beyond the scope of our paper, but a very interesting and important project, which needs to be addressed in the future.

Let us note that every measure $\mu \in \mathcal{M}^\infty(\mathbb{R}^d)$ is a tempered measure by Corollary 2.5, and hence the Fourier transform of some $\omega \in \mathsf{S}'(\mathbb{R}^d)$, which by definition is in $\mathcal{DTBM}(\mathbb{R}^d)$. Therefore, we get the following simple fact.

Fact 3.1. *The Fourier transform is a bijection from $\mathcal{DTBM}(\mathbb{R}^d)$ to $\mathcal{M}^\infty(\mathbb{R}^d)$.*

Since this space will play a fundamental role in remainder of the paper, we will characterize it in the following. Let us start with a simple characterization for the positive definite tempered distributions in this space. First, recall that $\omega \in \mathsf{S}'(\mathbb{R}^d)$ is called *positive definite* if for all $f \in \mathsf{S}(\mathbb{R}^d)$, we have

$$\omega(f * \widetilde{f}) \geqslant 0\,.$$

By Bochner–Schwartz's theorem [24, Theorem IX.10], a tempered distribution is positive definite if and only if its Fourier transform is a positive tempered measure.

Recall that given $\omega \in \mathsf{S}'(\mathbb{R}^d)$ and $f \in \mathsf{S}(\mathbb{R}^d)$, the convolution $f * \omega$ is an infinitely many times differentiable function, which is not necessarily bounded. If $f * \omega$ is bounded for all $f \in \mathsf{S}(\mathbb{R}^d)$, we say that ω is a *translation bounded* tempered distribution, see [37] for properties of these tempered distributions. The space of translation bounded tempered distributions is denoted by $\mathsf{S}'_\infty(\mathbb{R}^d)$.

We can now prove the following result.

Proposition 3.2. *Let $\omega \in S'(\mathbb{R}^d)$ be positive definite. Then, the following statements are equivalent.*

(i) $\omega \in \mathcal{DTBM}(\mathbb{R}^d)$.
(ii) $\widehat{\omega} \in S'_\infty(\mathbb{R}^d)$.
(iii) *For all $f \in S(\mathbb{R}^d)$, there exists some constant $C > 0$ such that*

$$\left|\omega(e^{2\pi \mathrm{i} t \cdot} f)\right| \leqslant C$$

for all $t \in \mathbb{R}^d$.
(iv) *There exists some $f \in S(\mathbb{R}^d)$ and $C > 0$ with $f \neq 0$, $\widehat{f} \geqslant 0$ such that*

$$\left|\omega(e^{2\pi \mathrm{i} t \cdot} f)\right| \leqslant C$$

for all $t \in \mathbb{R}^d$.

Proof. The equivalence (i) $\iff$ (ii) follows from [37, Proposition 2.5].

(ii)$\Rightarrow$(iii) Let $f \in \mathsf{S}(\mathbb{R}^d)$, and let $C := \|\widehat{f} * \widehat{\omega}\|_\infty < \infty$. The number C is finite because $\widehat{\omega} \in \mathsf{S}'_\infty(\mathbb{R}^d)$. Then, for all $t \in \mathbb{R}^d$, we have

$$\left|\omega(e^{-2\pi i t\cdot} f)\right| = \left|\widehat{\omega}(\widecheck{e^{-2\pi i t\cdot} f})\right| = \left|(\widehat{f} * \widehat{\omega})(t)\right| \leqslant C\,.$$

(iii)$\Rightarrow$(iv) This is obvious.

(iv)$\Rightarrow$(i) First, $f \neq 0$ implies $\widecheck{f} \neq 0$. Therefore, there exists some $s \in \mathbb{R}^d$ such that $\widecheck{f}(s) \neq 0$. Since $\widecheck{f}(s) = \widehat{f}(-s) \geqslant 0$, we get $\widecheck{f}(s) > 0$. Therefore, there exist some $r > 0$ and $c > 0$ such that

$$\widecheck{f}(x) \geqslant c \qquad \text{for all } x \in B_r(s)\,.$$

Now, let $\mu := \widehat{\omega}$, which is a positive measure since ω is positive definite. Since $\widecheck{f}$ and μ are positive, we have

$$\begin{aligned} c\,\mu(B_r(x)) &= \int_{\mathbb{R}^d} c\,1_{B_r(x)}(y)\,\mathrm{d}\mu(y) = \int_{\mathbb{R}^d} c\,1_{B_r(s)}(y-x+s)\,\mathrm{d}\mu(y) \\ &\leqslant \int_{\mathbb{R}^d} \widecheck{f}(y-x+s)\,\mathrm{d}\mu(y) = \int_{\mathbb{R}^d} \widecheck{e^{2\pi i(s-x)\cdot} f}\,(y)\,\mathrm{d}\widehat{\omega}(y) \\ &= \omega(e^{2\pi i(s-x)\cdot} f) \leqslant C \end{aligned}$$

for all $x \in \mathbb{R}^d$. This gives that

$$\|\mu\|_{B_r(0)} = \sup_{x\in\mathbb{R}^d} \mu(B_r(x)) \leqslant \frac{C}{c}\,,$$

which proves the claim. $\square$

Next, by repeating the arguments of [30], we can give the following characterization of $\mathcal{DTBM}(\mathbb{R}^d)$.

Proposition 3.3. *Let $\omega \in \mathcal{S}'(\mathbb{R}^d)$. Let $X := \{\varphi \in C_c^\infty(\mathbb{R}^d) \,:\, \operatorname{supp}(\varphi) \subseteq [-1,1]^d\}$. Then, the following statements are equivalent.*

(i) $\omega \in \mathcal{DTBM}(\mathbb{R}^d)$.

(ii) $\widehat{\omega} \in \mathcal{S}'_\infty(\mathbb{R}^d)$ *and the operator*

$$T : X \to C_u(\mathbb{R}^d)\,, \qquad \varphi \mapsto \varphi * \widehat{\omega}\,,$$

is a continuous operator.

(iii) There exists a constant $C > 0$ such that, for all $t \in \mathbb{R}^d$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $|\varphi| \leqslant 1_{[-1,1]^d}$, we have

$$\left|\omega(e^{2\pi i t\cdot}\widecheck{\varphi})\right| \leqslant C\,.$$

(iv) There exist positive definite $\omega_1, \omega_2, \omega_3, \omega_4 \in \mathcal{DTBM}(\mathbb{R}^d)$ such that

$$\omega = \omega_1 - \omega_2 + i(\omega_3 - \omega_4)\,.$$

Proof. (i)$\Rightarrow$(iv) This is immediate. Indeed, if $\mu = \widehat{\omega} \in \mathcal{M}^\infty(\mathbb{R}^d)$, then there exist positive measures $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathcal{M}^\infty(\mathbb{R}^d)$ such that

$$\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)\,.$$

As usual, for all $1 \leqslant j \leqslant 4$, there exists some $\omega_j \in \mathcal{DTBM}(\mathbb{R}^d)$ such that $\widehat{\omega_j} = \mu_j$. The claim follows.

(iv)⇒(i) This is also immediate, as a (finite) linear combination of translation bounded measures is translation bounded.

(i)⇒(ii) Let $\mu = \widehat{\omega} \in \mathcal{M}^\infty(\mathbb{R}^d)$. Then, $\widehat{\omega} \in S'_\infty(\mathbb{R}^d)$ by [37, Corollary 2.1]. Moreover, we have

$$\|T(\varphi)\|_\infty = \|\varphi * \mu\|_\infty \leqslant \|\mu\|_{[-1,1]^d} \|\varphi\|_\infty$$

for all $\varphi \in X$.

(ii)⇒(iii) Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $|\varphi| \leqslant 1_{[-1,1]^d}$. Then, $\varphi \in X$ and $\|\varphi\|_\infty \leqslant 1$. Therefore,

$$\left|\omega(e^{2\pi i t \cdot}\check{\varphi})\right| = |(\varphi * \widehat{\omega})(t)| \leqslant \|\varphi * \widehat{\omega}\|_\infty \leqslant \|T\| \, \|\varphi\|_\infty = \|T\| \, .$$

(iii)⇒(ii) Let $\varphi \in X$. Define

$$\phi = \begin{cases} \frac{1}{\|\varphi\|_\infty}\varphi & \text{if } \|\varphi\|_\infty \neq 0 \, , \\ 0 & \text{if } \|\varphi\|_\infty = 0 \, . \end{cases}$$

Then, $\phi \in X$, $|\phi| \leqslant 1_{[-1,1]^d}$, and $\varphi = \|\varphi\|_\infty \phi$. Therefore, we have

$$|(\varphi * \widehat{\omega})(t)| = \|\varphi\|_\infty \left|(\phi * \widehat{\omega})(t)\right| = \|\varphi\|_\infty \left|\omega(e^{2\pi i t \cdot}\check{\phi})\right| \leqslant C\|\varphi\|_\infty$$

for all $t \in \mathbb{R}^d$.

(ii)⇒ (i) Let $N \in \mathbb{N}$ be arbitrary and let $K_N := [-N, N]^d$. Via a standard partition of unity argument, there exist some $\phi_1, \dots, \phi_k \in X$ and $t_1, \dots, t_k \in \mathbb{R}^d$ such that $0 \leqslant \phi_j(x) \leqslant 1$ for all $x \in \mathbb{R}^d$ and

$$\sum_{j=1}^k T_{t_j}\phi_j(x) = 1$$

for all $x \in K_N$. Then, for all $\varphi \in C_c^\infty(\mathbb{R}^d : K_N) := \{\psi \in C_c^\infty(\mathbb{R}^d) \, : \, \operatorname{supp}(\psi) \subseteq K_N\}$, we have

$$\varphi = \sum_{j=1}^k \varphi T_{t_j}\phi_j \, .$$

Therefore, we obtain

$$|\widehat{\omega}(\varphi)| = |(\widehat{\omega} * \varphi^\dagger)(0)| \leqslant \sum_{j=1}^k |(\widehat{\omega} * (\varphi T_{t_j}\phi_j)^\dagger)(0)| = \sum_{j=1}^k \left|(\widehat{\omega} * (\phi_j T_{-t_j}\varphi)^\dagger)(t_j)\right| .$$

Therefore, since $\operatorname{supp}((\phi_j T_{-t_j}\varphi)^\dagger) \subseteq [-1, 1]^d$, (ii) implies

$$\begin{aligned} |\widehat{\omega}(\varphi)| &\leqslant \sum_{j=1}^k \left|(\widehat{\omega} * (\phi_j T_{-t_j}\varphi)^\dagger)(t_j)\right| \leqslant \sum_{j=1}^k \|T\| \, \|(\phi_j T_{-t_j}\varphi)^\dagger\|_\infty \\ &\leqslant \sum_{j=1}^k \|T\| \, \|T_{-t_j}\varphi\|_\infty = C_N \, \|\varphi\|_\infty \, , \end{aligned}$$

where $C_N := k\|T\|$ depends only on N and the choice of $\phi_1, \dots, \phi_k$.

Since $C_c^\infty(\mathbb{R}^d : K_N)$ is dense in $C_c(\mathbb{R}^d : K_N) = \{\psi \in C_c(\mathbb{R}^d) : \operatorname{supp}(\psi) \subseteq K_N\}$, it follows that for all N, $\widehat{\omega}$ can be uniquely extended to a continuous functional on $C_c(\mathbb{R}^d : K_N)$. Therefore, there exists a measure μ_N supported inside

$[-N,N]^d$ such that

$$\mu_N(\varphi) = \widehat{\omega}(\varphi)$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\text{supp}(\varphi) \subseteq [-N,N]^d$. It is easy to see that $\mu_N = \mu_{N+1}|_{[-N,N]^d}$. We can then define $\mu : C_c(\mathbb{R}^d) \to \mathbb{C}$ via

$$\mu(\psi) = \mu_N(\psi) \qquad \text{with } \text{supp}(\psi) \subseteq [-N,N]^d\,,$$

and the definition does not depend on the choice of N. It is easy to see that μ is linear, and

$$|\mu(\psi)| \leqslant C_N \|\psi\|_\infty$$

for all $\psi \in C_c(\mathbb{R}^d)$ with $\text{supp}(\psi) \subseteq [-N,N]^d$. This shows that μ is a measure and

$$\mu(\phi) = \widehat{\omega}(\phi) \qquad \text{for all } \phi \in C_c^\infty(\mathbb{R}^d)\,.$$

Finally, setting $\mathcal{F} := \{\psi \in C_c^\infty(\mathbb{R}^d) \ : \ |\psi| \leqslant 1_{[-1,1]^d}\}$, [30, Corollary 3.4] gives

$$\|\mu\|_{[-1,1]^d} = \sup_{\psi\in\mathcal{F}} \|\psi * \mu\|_\infty = \sup_{\psi\in\mathcal{F}} \|\psi * \widehat{\omega}\|_\infty = \sup_{\psi\in\mathcal{F}} \|T\psi)\|_\infty \leqslant \|T\|\,.$$

This shows that $\mu = \widehat{\omega}$ is a translation bounded measure. □

Next, we give a more explicit description of $\mathcal{DTBM}(\mathbb{R}^d)$ in terms of derivatives of functions in the Fourier–Stieltjes algebra

$$B(\mathbb{R}^d) := \{\widehat{\rho} \ : \ \rho \text{ is a finite measure on } \mathbb{R}^d\}\,.$$

In particular, we will show that each tempered distribution $\omega \in \mathcal{DTBM}(\mathbb{R}^d)$ is a distribution of order (at most) $2d$.

Theorem 3.4. *Let $\omega \in \mathcal{DTBM}(\mathbb{R}^d)$, and let $\nu = \widehat{\omega}$. Then, there exists some function $h \in B(\mathbb{R}^d) \subset C_u(\mathbb{R}^d)$ such that*

(a) $\mu := h\boldsymbol{\lambda}$ is a translation bounded measure and hence a tempered distribution.
(b) For every $\varphi \in C_c(\mathbb{R}^d)$, one has

$$\mu(\varphi) = \int_{\mathbb{R}^d} \frac{\check{\varphi}(x)}{(2\pi \mathrm{i} x_1)^{2d} + \cdots + (2\pi \mathrm{i} x_d)^{2d} + (-1)^d}\, d\nu(x)\,.$$

(c) Let $\Psi_d := (\partial_{x_1})^{2d} + \cdots + (\partial_{x_d})^{2d} + (-1)^d$. Then, $\Psi_d\mu$ is a tempered distribution and

$$\Psi_d \mu = \omega\,.$$

(d) For all $f \in S(\mathbb{R}^d)$, we have

$$|\omega(f)| \leqslant \|h\|_\infty \left(\|f\|_1 + \sum_{j=1}^d \|(\partial_{x_j})^{2d} f\|_1 \right).$$

(e) ω is a distribution of order $2d$.

Proof. Let $\rho := \frac{1}{(2\pi i(\cdot)_1)^{2d}+\cdots+(2\pi i(\cdot)_d)^{2d}+(-1)^d}\nu$. By Lemma 2.3, ρ is a finite measure, and hence $h := \check{\rho} \in B(\mathbb{R}^d) \subset C_{\mathrm{u}}(\mathbb{R}^d)$.

(a) Since $h \in C_{\mathrm{u}}(\mathbb{R}^d)$, we have $\mu \in \mathcal{M}^\infty(\mathbb{R}^d)$. In particular, μ is a tempered distribution.
(b) This follows from Proposition 2.9 with ρ as in the proof of (a).
(c) Let $f \in \mathsf{S}(\mathbb{R}^d)$. Then,

$$\omega(f) = \widehat{\omega}(\check{f}) = \nu(\check{f}) = \int_{\mathbb{R}^d} \frac{((2\pi i x_1)^{2d} + \cdots + (2\pi i x_d)^{2d} + (-1)^d)\,\check{f}(x)}{(2\pi i x_1)^{2d} + \cdots + (2\pi i x_d)^{2d} + (-1)^d}\,\mathrm{d}\nu(x)$$

$$= \int_{\mathbb{R}^d} \frac{\widecheck{\Psi_d f}\,(x)}{(2\pi i x_1)^{2d} + \cdots + (2\pi i x_d)^{2d} + (-1)^d}\,\mathrm{d}\nu(x) = \mu(\Psi_d f) = (\Psi_d \mu)(f)\,.$$

Thus, $\omega = \Psi_d \mu$ as tempered distributions.
(d) Let $f \in \mathsf{S}(\mathbb{R}^d)$. Then,

$$\omega(f) = (\Psi_d\mu) f = \mu\Big(\Big((\partial_{x_1})^{2d} + \cdots + (\partial_{x_d})^{2d} + (-1)^d\Big) f\Big)$$

$$= \int_{R^d} h(x)\Big(\Big((\partial_{x_1})^{2d} + \cdots + (\partial_{x_d})^{2d} + (-1)^d\Big) f\Big)(x)\,\mathrm{d}\boldsymbol{\lambda}(x)\,.$$

Therefore,

$$|\omega(f)| \leqslant \int_{\mathbb{R}^d}\int_{R^d} |h(x)|\Big(|(\partial_{x_1})^{2d} f(x)| + \cdots + |(\partial_{x_d})^{2d} f(x)| + |f(x)|\Big)\mathrm{d}\boldsymbol{\lambda}(x)$$

$$\leqslant \|h\|_\infty \left(\|f\|_1 + \sum_{j=1}^d \|(\partial_{x_j})^{2d} f\|_1 \right).$$

(e) Let K be a compact subset of $\mathbb{R}^d$, and let $\phi \in C_{\mathrm{c}}^\infty(\mathbb{R}^d)$ with $\operatorname{supp}(\phi) \subseteq K$. Now, (d) and Lemma 2.3 imply

$$|\omega(\phi)| \leqslant \|h\|_\infty \left(\|\phi\|_1 + \sum_{j=1}^d \|(\partial_{x_j})^{2d}\phi\|_1 \right) \leqslant \|h\|_\infty\,\boldsymbol{\lambda}(K) \left(\sum_{j=1}^d \|(\partial_{x_j})^{2d}\phi\|_\infty + \|\phi\|_\infty \right).$$

Thus, ω is a distribution of order $2d$. □

Let us now look at a very simple example.

Example 3.5. Let $\omega = \delta_{\mathbb{Z}}$. Then, by Poisson's summation formula $\nu := \widehat{\omega} = \delta_{\mathbb{Z}}$, and hence $\omega \in \mathcal{DTBM}(\mathbb{R}^d)$. In this case, $h(x) = -\frac{1}{2}\sum_{k\in\mathbb{Z}} e^{-|k-x|}$.

To see this, we should first remind the reader that

$$\frac{2}{4\pi^2 x^2 + 1} = \widehat{\phi}(x) \qquad \text{with } \phi(x) = \mathrm{e}^{-|x|}\,.$$

Now, an application of Poisson's summation formula gives

$$h(x) = \int_{\mathbb{R}} \mathrm{e}^{2\pi i x y}\,\frac{1}{-4\pi^2 y^2 - 1}\,\mathrm{d}\delta_{\mathbb{Z}}(y) = -\sum_{k\in\mathbb{Z}} \frac{1}{4\pi^2 k^2 + 1}\,\mathrm{e}^{2\pi i k x}$$

$$= -\frac{1}{2}\sum_{k\in\mathbb{Z}} \widehat{\phi}(k)\,\mathrm{e}^{-2\pi i k x} = -\frac{1}{2}\sum_{k\in\mathbb{Z}} \phi(k-x) = -\frac{1}{2}\sum_{k\in\mathbb{Z}} e^{-|k-x|}\,.$$

Let us note in passing that the proofs in this section can be used to prove the following results about tempered distributions with (not necessarily translation bounded) measure Fourier transform.

Lemma 3.6.

(a) Let $h \in B(\mathbb{R}^d)$ and let

$$\omega = \Psi_d(h\boldsymbol{\lambda}).$$

Then, ω is a tempered distribution and $\nu = \widehat{\omega}$ is a measure satisfying

$$\int_{\mathbb{R}^d} \frac{1}{(2\pi x_1)^{2d} + \cdots + (2\pi x_d)^{2d} + 1} \, d|\nu|(x) < \infty .$$

(b) Let ν be a measure satisfying

$$\int_{\mathbb{R}^d} \frac{1}{(2\pi x_1)^{2d} + \cdots + (2\pi x_d)^{2d} + 1} \, d|\nu|(x) < \infty .$$

Then, there exists some $h \in B(\mathbb{R}^d)$ such that $\omega = \Psi_d(h\boldsymbol{\lambda})$ is a tempered distribution and $\nu = \widehat{\omega}$.

Proof.

(a) Since $h \in B(\mathbb{R}^d)$, there exists a finite measure μ on $\mathbb{R}^d$ such that $h = \widecheck{\mu}$. Let

$$\nu = \big((2\pi \mathrm{i}(\cdot)_1)^{2d} + \cdots + (2\pi \mathrm{i}(\cdot)_d)^{2d} + (-1)^d\big)\mu .$$

Then, ν is a measure.

Now, since $h = \widecheck{\mu}$ and μ is a finite measure, we have

$$\widehat{h\boldsymbol{\lambda}} = \mu \tag{3.1}$$

as measures [22, Lemma 4.9.15]. In particular, (3.1) holds as tempered distributions by Theorem 2.8.

Finally, we obtain

$$\begin{aligned}\widehat{\omega} = \widehat{\Psi_d(h\boldsymbol{\lambda})} &= \big((2\pi \mathrm{i}(\cdot)_1)^{2d} + \cdots + (2\pi \mathrm{i}(\cdot)_d)^{2d} + (-1)^d\big)\widehat{h\boldsymbol{\lambda}} \\ &= \big((2\pi \mathrm{i}(\cdot)_1)^{2d} + \cdots + (2\pi \mathrm{i}(\cdot)_d)^{2d} + (-1)^d\big)\mu = \nu .\end{aligned}$$

Moreover,

$$\int_{\mathbb{R}^d} \frac{1}{(2\pi x_1)^{2d} + \cdots + (2\pi x_d)^{2d} + 1} \, \mathrm{d}|\nu|(x) = |\mu|(\mathbb{R}^d) < \infty ,$$

since μ is a finite measure.

(b) This is similar to the proof of Proposition 3.4. □

Exactly the same way as in Theorem 3.4, we can also prove the following result. Since the proof is identical to the one of Theorem 3.4, we skip it.

Proposition 3.7. *Let $\omega \in \mathcal{DTBM}(\mathbb{R}^d)$, and let $\nu = \widehat{\omega}$. Then, there exists some function $h \in B(\mathbb{R}^d) \subset C_{\mathrm{u}}(\mathbb{R}^d)$ such that*

(a) $\mu := h\boldsymbol{\lambda}$ is a translation bounded measure and hence a tempered distribution.
(b) for every $\varphi \in C_{\mathrm{c}}(\mathbb{R}^d)$, one has

$$\mu(\varphi) = \int_{\mathbb{R}^d} \frac{\widecheck{\varphi}(x)}{((2\pi \mathrm{i} x_1)^2 + \cdots + (2\pi \mathrm{i} x_d)^2)^d + (-1)^d} \, d\nu(x) .$$

(c) let $\Phi_d := \left((\partial_{x_1})^2 + \cdots + (\partial_{x_d})^2\right)^d + (-1)^d$, *then* $\Phi_d \mu$ *is a tempered distribution and*

$$\Phi_d \mu = \omega\,.$$

(d) for all $f \in S(\mathbb{R}^d)$, *we have*

$$|\omega(f)| \leqslant \|h\|_\infty \left(\|f\|_1 + \left\| \left(\sum_{j=1}^{d} (\partial_{x_j})^2 \right)^d f \right\|_1 \right).$$

(e) ω *is a distribution of order* $2d$.

4 | THE EXISTENCE OF A GENERALIZED EBERLEIN DECOMPOSITION

We can now prove the existence of generalized Eberlein decomposition at the level of distributions of order $2d$.

We start with the following simple results.

Proposition 4.1. *Let* $\omega \in \mathcal{DTBM}(\mathbb{R}^d)$. *Then, there exist unique distributions* $\omega_{\mathrm{s}}, \omega_{0\mathrm{a}}$, *and* $\omega_{0\mathrm{s}} \in \mathcal{DTBM}(\mathbb{R}^d)$ *such that* $\omega = \omega_{\mathrm{s}} + \omega_{0\mathrm{a}} + \omega_{0\mathrm{s}}$ *as well as*

$$\widehat{\omega_{\mathrm{s}}} = (\widehat{\omega})_{\mathrm{pp}}\,, \qquad \widehat{\omega_{0\mathrm{a}}} = (\widehat{\omega})_{\mathrm{ac}}, \qquad \textit{and} \qquad \widehat{\omega_{0\mathrm{s}}} = (\widehat{\omega})_{\mathrm{sc}}\,.$$

Moreover, there exist functions h_{s}, $h_{0\mathrm{a}}$, *and* $h_{0\mathrm{s}} \in B(\mathbb{R}^d)$ *such that*

$$\omega_{\mathrm{s}} = \Psi_d(h_{\mathrm{s}}\boldsymbol{\lambda})\,, \qquad \omega_{0\mathrm{a}} = \Psi_d(h_{0\mathrm{a}}\boldsymbol{\lambda}), \qquad \text{and} \qquad \omega_{0\mathrm{s}} = \Psi_d(h_{0\mathrm{s}}\boldsymbol{\lambda})\,,$$

Finally, with $\omega_0 = \omega_{0\mathrm{a}} + \omega_{0\mathrm{s}}$, *the decomposition*

$$\omega = \omega_{\mathrm{s}} + \omega_0$$

is the decomposition of (2.7).

Proof. Since $\omega \in \mathcal{DTBM}(\mathbb{R}^d)$ we have $\nu := \widehat{\omega} \in \mathcal{M}^\infty(\mathbb{R}^d)$, and hence $\nu_{\mathrm{pp}}, \nu_{\mathrm{ac}}, \nu_{\mathrm{sc}} \in \mathcal{M}^\infty(\mathbb{R}^d)$ by [30, Lemma 3.12]. The existence of $\omega_{\mathrm{s}}, \omega_{0\mathrm{a}}$, and $\omega_{0\mathrm{s}} \in \mathcal{DTBM}(\mathbb{R}^d)$ follows now from Fact 3.1. The existence of h_{s}, $h_{0\mathrm{a}}$, and $h_{0\mathrm{s}}$ follows from Theorem 3.4. The last claim follows from Theorem 2.19. □

Remark 4.2. Let $\omega \in \mathcal{DTBM}(\mathbb{R}^d)$ be positive definite and let $\nu = \widehat{\omega}$. We can ask wether or not the distributions ω_{s}, $\omega_{0\mathrm{a}}$, and $\omega_{0\mathrm{s}}$ can be chosen as measures. However, for the distributions to be measures, it is necessary that ν_{pp}, ν_{ac}, and ν_{sc} are weakly almost periodic [22, Theorem 4.11.12]. So, in general the answer is "no."

As an example, let $\nu = \boldsymbol{\lambda}|_{[0,\infty)}$ and let $\omega = \check{\nu}$. Then, $\omega \in \mathcal{DTBM}(\mathbb{R}^d)$ cannot be a measure, as this would imply that ν is weakly almost periodic, and hence, by [22, Lemma 4.10.7] the limit

$$\lim_{n\to\infty} \frac{1}{2n} \nu([t-n, t+n])$$

would exist uniformly in t. It is easy to see that this is not the case, as

$$\frac{1}{2n}\nu([t-n, t+n]) = \begin{cases} 1 & \text{if } t \geqslant n \\ \frac{1}{2} & \text{if } t = 0 \\ 0 & \text{if } t \leqslant -n \end{cases}.$$

Similarly, with $\omega = \widecheck{\delta_{\mathbb{N}}}$, the distribution $\omega = \omega_{\mathrm{s}}$ is not a measure.

In the case of Fourier transformable measures, Proposition 4.1 yields the following consequence.

Theorem 4.3. *Let γ be measure on $\mathbb{R}^d$, which is Fourier transformable as a measure. Then, there exist a unique Fourier transformable measure γ_{s}, tempered distributions $\omega_{0\mathrm{a}}, \omega_{0\mathrm{s}} \in \mathcal{DTBM}(\mathbb{R}^d)$ of order $2d$ and functions h_{s}, $h_{0\mathrm{a}}$, and $h_{0\mathrm{s}} \in B(\mathbb{R}^d)$ such that:*

(a) $\gamma_0 := \omega_{0\mathrm{a}} + \omega_{0\mathrm{s}}$ is a Fourier transformable measure.
(b) $\gamma = \gamma_{\mathrm{s}} + \omega_{0\mathrm{a}} + \omega_{0\mathrm{s}}$.
(c) $\gamma = \gamma_{\mathrm{s}} + \gamma_0$ is the Eberlein decomposition of Proposition 2.17.
(d) $\widehat{\gamma_{\mathrm{s}}} = (\widehat{\gamma})_{\mathrm{pp}}$, $\widehat{\omega_{0\mathrm{a}}} = (\widehat{\gamma})_{\mathrm{ac}}$, and $\widehat{\omega_{0\mathrm{s}}} = (\widehat{\gamma})_{\mathrm{sc}}$.
(e) $\gamma_{\mathrm{s}} = \Psi_d\,(h_{\mathrm{s}}\boldsymbol{\lambda})$, $\omega_{0\mathrm{a}} = \Psi_d\,(h_{0\mathrm{a}}\boldsymbol{\lambda})$, and $\omega_{0\mathrm{s}} = \Psi_d\,(h_{0\mathrm{s}}\boldsymbol{\lambda})$.
(f) If γ is positive definite, then $\gamma_{\mathrm{s}}, \gamma_0$ are positive definite measures, and $\omega_{0\mathrm{a}}, \omega_{0\mathrm{s}}$ are positive definite tempered distributions.

Proof. Since γ is a Fourier transformable measure, we have $\gamma \in \mathcal{DTBM}(\mathbb{R}^d)$ by Theorem 2.8. Now, let

$$\gamma = \gamma_{\mathrm{s}} + \gamma_0$$

be the usual Eberlein decomposition of γ from Theorem 2.18, and let

$$\gamma = \omega_{\mathrm{s}} + \omega_{0\mathrm{a}} + \omega_{0\mathrm{s}}$$

be the decomposition of Proposition 4.1. Then, we have

$$\widehat{\gamma_{\mathrm{s}}} = (\widehat{\gamma})_{\mathrm{pp}} = \widehat{\omega_{\mathrm{s}}} \qquad \text{and} \qquad \widehat{\gamma_0} = (\widehat{\gamma})_{\mathrm{c}} = (\widehat{\gamma})_{\mathrm{ac}} + (\widehat{\gamma})_{\mathrm{sc}} = \widehat{\omega_{0\mathrm{s}}} + \widehat{\omega_{0\mathrm{a}}}$$

in the sense of tempered distributions. The injectivity of the Fourier transform then gives

$$\gamma_{\mathrm{s}} = \omega_{\mathrm{s}} \qquad \text{and} \qquad \gamma_0 = \omega_{0\mathrm{s}} + \omega_{0\mathrm{a}}.$$

Note here that $\omega_{0\mathrm{a}}, \omega_{0\mathrm{s}} \in \mathcal{DTBM}(\mathbb{R}^d)$ by Proposition 4.1. Also, by Theorem 2.8, we have $\gamma_{\mathrm{s}} \in \mathcal{DTBM}(\mathbb{R}^d)$. In particular, Theorem 3.4 gives the existence of h_{s}, $h_{0\mathrm{a}}$, and $h_{0\mathrm{s}} \in B(\mathbb{R}^d)$ such that (e) holds.

The injectivity of the Fourier transform for tempered distributions and uniqueness of the Lebesgue decomposition for $\widehat{\gamma}$ gives the uniqueness of $\gamma_{\mathrm{s}}, \gamma_0, \omega_{0\mathrm{a}}, \omega_{0\mathrm{s}}$.

Finally, if γ is a positive definite measure, then $\widehat{\gamma}$ is a positive measure and hence so are $(\widehat{\gamma})_{\mathrm{pp}}, (\widehat{\gamma})_{\mathrm{c}}, (\widehat{\gamma})_{\mathrm{ac}}, (\widehat{\gamma})_{\mathrm{sc}}$. This gives (f) and completes the proof. □

Now, Proposition 4.1 makes the following definitions natural.

Definition 4.4.

$$\begin{aligned}
\mathcal{DTBM}_{\mathrm{s}}(\mathbb{R}^d) &:= \{\omega \in \mathcal{DTBM}(\mathbb{R}^d) \ : \ \widehat{\omega} \text{ is pure point}\}.\\
\mathcal{DTBM}_{0\mathrm{s}}(\mathbb{R}^d) &:= \{\omega \in \mathcal{DTBM}(\mathbb{R}^d) \ : \ \widehat{\omega} \text{ is singular continuous}\}.\\
\mathcal{DTBM}_{0\mathrm{a}}(\mathbb{R}^d) &:= \{\omega \in \mathcal{DTBM}(\mathbb{R}^d) \ : \ \widehat{\omega} \text{ is absolutely continuous}\}.\\
\mathcal{DTBM}_{0}(\mathbb{R}^d) &:= \{\omega \in \mathcal{DTBM}(\mathbb{R}^d) \ : \ \widehat{\omega} \text{ is continuous}\}.
\end{aligned}$$

As usual, we denote by $\mathcal{M}^\infty_{\mathrm{pp}}(\mathbb{R}^d)$, $\mathcal{M}^\infty_{\mathrm{c}}(\mathbb{R}^d)$, $\mathcal{M}^\infty_{\mathrm{ac}}(\mathbb{R}^d)$, $\mathcal{M}^\infty_{\mathrm{sc}}(\mathbb{R}^d)$ the spaces of translation bounded pure point measures, translation bounded continuous measures, translation bounded absolutely continuous measures, and translation bounded singular continuous measures, respectively.

Proposition 4.5. *These spaces just defined have the following properties.*

(a) $\mathcal{DTBM}(\mathbb{R}^d) \subseteq \mathsf{WAP}(\mathbb{R}^d)$, $\mathcal{DTBM}_{\mathrm{s}}(\mathbb{R}^d) \subseteq \mathsf{SAP}(\mathbb{R}^d)$, $\mathcal{DTBM}_0(\mathbb{R}^d) \subseteq \mathsf{WAP}_0(\mathbb{R}^d)$.
(b) $\mathcal{DTBM}(\mathbb{R}^d) = \mathcal{DTBM}_{\mathrm{s}}(\mathbb{R}^d) \oplus \mathcal{DTBM}_0(\mathbb{R}^d)$.
(c) $\mathcal{DTBM}_0(\mathbb{R}^d) = \mathcal{DTBM}_{0\mathrm{s}}(\mathbb{R}^d) \oplus \mathcal{DTBM}_{0\mathrm{a}}(\mathbb{R}^d)$.
(d) $\mathcal{DTBM}_{\mathrm{s}}(\mathbb{R}^d) = \mathcal{DTBM}(\mathbb{R}^d) \cap \mathsf{SAP}(\mathbb{R}^d)$.
(e) $\mathcal{DTBM}_0(\mathbb{R}^d) = \mathcal{DTBM}(\mathbb{R}^d) \cap \mathsf{WAP}_0(\mathbb{R}^d)$.
(f) *The Fourier transform induces the following bijections:*

$$\begin{array}{cccc}
\mathcal{DTBM}(\mathbb{R}^d) & = \mathcal{DTBM}_{\mathrm{s}}(\mathbb{R}^d) & \oplus \mathcal{DTBM}_{0\mathrm{s}}(\mathbb{R}^d) & \oplus \mathcal{DTBM}_{0\mathrm{a}}(\mathbb{R}^d) \\
\mathcal{F}\updownarrow & \mathcal{F}\updownarrow & \mathcal{F}\updownarrow & \mathcal{F}\updownarrow \\
\mathcal{M}^\infty(\mathbb{R}^d) & = \mathcal{M}^\infty_{\mathrm{pp}}(\mathbb{R}^d) & \oplus \mathcal{M}^\infty_{\mathrm{sc}}(\mathbb{R}^d) & \oplus \mathcal{M}^\infty_{\mathrm{ac}}(\mathbb{R}^d)
\end{array}$$

Proof. (a) is a consequence of Theorem 2.19, [37, Proposition 6.1], and [37, Proposition 6.2]. (b) and (c) are an immediate consequence of Proposition 4.1, Fact 3.1, and [30, Lemma 3.12]. (d) and (e) follow from [37, Proposition 6.1] and [37, Proposition 6.2], respectively. (f) is a consequence of Fact 3.1. □

5 | ON THE COMPONENTS OF THE GENERALIZED EBERLEIN DECOMPOSITION

Let γ be the autocorrelation of $\omega \in \mathcal{M}^\infty(\mathbb{R}^d)$. Then, $\gamma \in \mathcal{M}^\infty(\mathbb{R}^d) \cap \mathcal{DTBM}(\mathbb{R}^d)$, since γ is positive definite. In order to better understand the pure point, continuous, absolute continuous, and singular continuous spectrum, respectively, we need a better understanding of the components of the generalized Eberlein decomposition of γ. In particular, we need to understand the spaces $\mathcal{DTBM}_{\mathrm{s}}(\mathbb{R}^d)$, $\mathcal{DTBM}_0(\mathbb{R}^d)$, $\mathcal{DTBM}_{0\mathrm{s}}(\mathbb{R}^d)$, and $\mathcal{DTBM}_{0\mathrm{a}}(\mathbb{R}^d)$.

The subspaces $\mathcal{DTBM}_{\mathrm{s}}(\mathbb{R}^d)$, $\mathcal{DTBM}_0(\mathbb{R}^d)$ are characterized by Proposition 4.5. Indeed, for a distribution $\omega \in \mathcal{DTBM}(\mathbb{R}^d)$, one has the following:

(1) $\omega \in \mathcal{DTBM}_{\mathrm{s}}(\mathbb{R}^d)$ if and only if $f * \omega \in SAP(\mathbb{R}^d)$ for all $f \in \mathsf{S}(\mathbb{R}^d)$.
(2) $\omega \in \mathcal{DTBM}_0(\mathbb{R}^d)$ if and only if $f * \omega \in WAP_0(\mathbb{R}^d)$ for all $f \in \mathsf{S}(\mathbb{R}^d)$.

While we cannot constructively obtain the (restricted) Eberlein decomposition this way, it can be used to check if a given decomposition is the Eberlein decomposition. Indeed, we have the following result, compare [33, Proposition 5.8.7.].

Lemma 5.1. *Let $\omega, \omega_1, \omega_2 \in \mathcal{DTBM}(\mathbb{R}^d)$ be such that $\omega = \omega_1 + \omega_2$. Then, $\omega_1 = \omega_{\mathrm{s}}$ and $\omega_2 = \omega_0$ if and only if, for all $f \in S(\mathbb{R}^d)$, the following two conditions hold:*

(1) $f * \omega_1 \in SAP(\mathbb{R}^d)$.
(2) $\lim_{n\to\infty} \frac{1}{(2n)^d} \int_{[-n,n]^d} |(f * \omega_2)(x)|\, dx = 0$.

Proof. $\Longrightarrow$: One has $f * \omega_1 = f * \omega_{\mathrm{s}} \in SAP(\mathbb{R}^d)$ and $f * \omega_2 = f * \omega_0 \in WAP_0(\mathbb{R}^d)$ for all $f \in \mathsf{S}(\mathbb{R}^d)$. Therefore, we have

$$\lim_{n\to\infty} \frac{1}{(2n)^d} \int_{[-n,n]^d} |(f * \omega_2)(x)|\, \mathrm{d}x = 0.$$

$\Longleftarrow$: Let $f \in \mathsf{S}(\mathbb{R}^d)$. Then, $f * \omega_1 \in SAP(\mathbb{R}^d)$. Moreover, by Proposition 4.5 (a), we have $f * \omega_2 \in WAP(\mathbb{R}^d)$ and hence $f * \omega_2 \in WAP_0(\mathbb{R}^d)$ by assumption. Thus, $f * \omega = f * \omega_1 + f * \omega_2$ is the Eberlein decomposition of $f * \omega \in WAP(\mathbb{R}^d)$. Therefore, by [37, Theorem 5.2], we have

$$f * \omega_1 = (f * \omega)_{\mathrm{s}} = f * (\omega_{\mathrm{s}}) \qquad \text{and} \qquad f * \omega_2 = (f * \omega)_0 = f * (\omega_0)\,.$$

As this holds for all $f \in \mathsf{S}(\mathbb{R}^d)$, we get $\omega_1 = \omega_{\mathrm{s}}, \omega_2 = \omega_0$. □

Consequently, we obtain the following simpler characterization of $\omega \in \mathcal{DTBM}_0(\mathbb{R}^d)$.

Corollary 5.2. *Let $\omega \in \mathcal{DTBM}(\mathbb{R}^d)$. Then, $\omega \in \mathcal{DTBM}_0(\mathbb{R}^d)$ if and only if*

$$\lim_{n\to\infty} \frac{1}{(2n)^d} \int_{[-n,n]^d} |(f * \omega)(x)| \, dx = 0$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

Going further into the generalized Eberlein decomposition, the problem becomes much more complicated. The main issue is that the space $\mathcal{DTBM}_{0s}(\mathbb{R}^d)$ is very mysterious, and there is not much we can say about it right now.

Instead, we will take a closer look at $\mathcal{DTBM}_{0a}(\mathbb{R}^d)$. Via an application of the Riemann–Lebesgue lemma and Plancherel's theorem, we find necessary conditions and sufficient conditions for a tempered distribution $\omega \in \mathcal{DTBM}(\mathbb{R}^d)$ to belong in this space. Finding a necessary and sufficient condition seems to be a difficult problem, which is likely related to finding an intrinsic characterization of the *Fourier algebra*

$$A(\widehat{\mathbb{R}^d}) = \{\widehat{f} \,:\, f \in L^1(\mathbb{R}^d)\}.$$

Let us start with necessary conditions. Here, we follow the approach of [29, Theorem 27].

Proposition 5.3. *Let $\omega \in \mathcal{DTBM}_{0a}(\mathbb{R}^d)$. Then, for all $f \in \mathcal{S}(\mathbb{R}^d)$, we have $f * \omega \in C_0(\mathbb{R}^d)$.*

Proof. Let $g \in L^1_{\text{loc}}(\mathbb{R}^d)$ be such that $\widehat{\omega} = g\boldsymbol{\lambda}$, and let $f \in \mathsf{S}(\mathbb{R}^d)$. Since $\nu := g\boldsymbol{\lambda} \in \mathcal{M}^\infty(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \left|\widehat{f}(x)\, g(x)\right| \mathrm{d}x \leqslant \left\| \left(\sum_{j=1}^d (2\pi(\cdot)_j)^{2d} + 1 \right) \widehat{f} \right\|_\infty \int_{\mathbb{R}^d} \frac{1}{\sum_{j=1}^d (2\pi x_j)^{2d} + 1} \, \mathrm{d}|\nu|(x) < \infty$$

by Lemma 2.3. Therefore, $\widehat{f}\, g \in L^1(\mathbb{R}^d)$. The Riemann–Lebesgue lemma now gives

$$f * \omega = \widecheck{(\widehat{f} g)} \in C_0(\mathbb{R}^d). \qquad \square$$

Remark 5.4.

(a) The converse of Proposition 5.3 is not true [29, Remark 28].
(b) Let $\omega \in \mathsf{S}'(\mathbb{R}^d)$ be such that $\widehat{\omega}$ is absolutely continuous (but not necessarily translation bounded). Then, exactly as in Proposition 5.3, one can show that $\widecheck{\varphi} * \omega \in C_0(\mathbb{R}^d)$, for all $\varphi \in C_c^\infty(\mathbb{R}^d)$.

Next, we give a sufficient condition for the Fourier transform of a tempered distribution to be a continuous measure with $L^2_{\text{loc}}(\mathbb{R}^d)$-density function.

Proposition 5.5. *Let $\omega \in \mathcal{S}'(\mathbb{R}^d)$. Then, there exists some $f \in L^2_{\text{loc}}(\mathbb{R}^d)$ such that $\widehat{\omega} = f\boldsymbol{\lambda}$ if and only if $\widecheck{\varphi} * \omega \in L^2(\mathbb{R}^d)$ for all $\varphi \in C_c^\infty(\mathbb{R}^d)$.*

Proof. $\Longrightarrow$ Let $\varphi \in C_c^\infty(\mathbb{R}^d)$, and let $K \subset \mathbb{R}^d$ be the support of φ. Let $g := f\varphi$. Then,

$$\int_{\mathbb{R}^d} |g(x)|^2 \, \mathrm{d}x = \int_K |f(x)|^2 \, |\varphi(x)|^2 \, \mathrm{d}x \leqslant \|\varphi\|_\infty^2 \int_K |f(x)|^2 \, \mathrm{d}x < \infty.$$

Therefore, $g \in L^2(\mathbb{R}^d)$, and hence, by Plancherel's theorem, there exists some $h \in L^2(\mathbb{R}^d)$ such that $\widehat{h} = g$.

Now, by [1, Theorem 2.2], $h\boldsymbol{\lambda}$ is Fourier transformable as a measure and $\widehat{h\boldsymbol{\lambda}} = g\boldsymbol{\lambda}$. In particular, $h\boldsymbol{\lambda}$ is a tempered distribution by Theorem 2.8, and as tempered distributions, we have

$$\widehat{h\boldsymbol{\lambda}} = g\boldsymbol{\lambda} = f\varphi\boldsymbol{\lambda} = \widehat{(\check{\varphi} * \omega)\boldsymbol{\lambda}}.$$

Therefore, the functions h and $\omega * \check{\varphi}$ agree as tempered distributions, and hence agree almost everywhere.

Finally, $h \in L^2(\mathbb{R}^d)$ implies $\omega * \check{\varphi} \in L^2(\mathbb{R}^d)$ as claimed.

$\Longleftarrow$ For each $\varphi \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$, we have $\omega * \check{\varphi} \in L^2(\mathbb{R}^d)$. Let $f_{\varphi} \in L^2(\mathbb{R}^d)$ be the L^2-Fourier transform of $\omega * \check{\varphi}$. Then, exactly as above, as tempered distributions, we have

$$\varphi\widehat{\omega} = \widehat{(\omega * \check{\varphi})\boldsymbol{\lambda}} = f_{\varphi}\boldsymbol{\lambda}. \tag{5.1}$$

Next, for each $n \in \mathbb{Z}^d$, fix some $\varphi_n \in C_{\mathrm{c}}(\mathbb{R}^d)$ such that $\varphi_n(x) = 1$ for all $x \in n + [0,1)^d$. Now, define

$$f(x) := f_{\varphi_n}(x) \qquad \text{if } x \in n + [0,1)^d\,. \tag{5.2}$$

Then, $\varphi_n \in L^2(\mathbb{R}^d)$ implies $f|_{n+[0,1)^d} \in L^2(\mathbb{R}^d)$. As every compact set be covered by finitely many sets of the form $n + [0,1)^d$, we obtain $f \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$.

Finally, if $\varphi \in C_{\mathrm{c}}(\mathbb{R}^d)$ such that $\mathrm{supp}(\varphi) \subseteq n + [0,1)^d$ for some n, we have $\varphi = \varphi\varphi_n$, and hence by (5.1) and (5.2),

$$\widehat{\omega}(\varphi) = (\varphi_n\widehat{\omega})(\varphi) = (f_{\varphi_n}\boldsymbol{\lambda})(\varphi) = \int_{n+[0,1)^d} f_{\varphi_n}(x)\,\varphi(x)\,\mathrm{d}x = (f\boldsymbol{\lambda})(\varphi)\,.$$

Now, via a partition of unity argument, we can write each $\phi \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$ as a linear combination of $\varphi \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$ with $\mathrm{supp}(\varphi) \subseteq n + [0,1)^d$ and hence $\omega = f\boldsymbol{\lambda}$ on $C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$. Therefore, by the density of $C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$ in $\mathcal{S}(\mathbb{R}^d)$, the two tempered distributions agree. □

Cauchy–Schwarz's or Hölder's inequality imply that $L^2_{\mathrm{loc}}(\mathbb{R}^d) \subseteq L^1_{\mathrm{loc}}(\mathbb{R}^d)$, which immediately leads to the following sufficient conditions.

Corollary 5.6. *Let $\omega \in \mathcal{DTBM}(\mathbb{R}^d)$. If $\widehat{\varphi} * \omega \in L^2(\mathbb{R}^d)$ for all $\varphi \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$, then $\omega \in \mathcal{DTBM}_{0\mathrm{a}}(\mathbb{R}^d)$.*

Corollary 5.7. *Let $\omega \in \mathcal{DTBM}(\mathbb{R}^d)$. If $g * \omega \in L^2(\mathbb{R}^d)$ for all $g \in \mathcal{S}(\mathbb{R}^d)$, then $\omega \in \mathcal{DTBM}_{0\mathrm{a}}(\mathbb{R}^d)$.*

Consider now a Fourier transformable measure μ with uniformly discrete support. If we could strengthen Corollary 5.7 by replacing "for all $g \in \mathcal{S}(\mathbb{R}^d)$" by "for all $g \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$," then a simple computation allows us to replace this condition by a much simpler restriction on the coefficients of μ. This will in turn have interesting consequences for Fourier transformable measures with Meyer set support.

In order to do this, let us introduce the following definition.

Definition 5.8. A measurable function $f : \mathbb{R}^d \to \mathbb{C}$ is called *Schwartz to L^2 compatible* if $gf \in L^2(\mathbb{R}^d)$ for all $g \in \mathcal{S}(\mathbb{R}^d)$. We denote the space of Schwartz to L^2 compatible functions by[1] $\mathcal{LS}_2(\mathbb{R}^d)$.

A measurable function $f : \mathbb{R}^d \to \mathbb{C}$ is called *weakly Schwartz to L^2 compatible* if $\widehat{\varphi}f \in L^2(\mathbb{R}^d)$ for all $\varphi \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$. We denote the space of weakly Schwartz to L^2 compatible functions by $\mathcal{WLS}_2(\mathbb{R}^d)$.

Let us start with some straightforward consequences of the definition.

Lemma 5.9.

(a) $\mathcal{LS}_2(\mathbb{R}^d) \subseteq \mathcal{WLS}_2(\mathbb{R}^d) \subseteq L^2_{\mathrm{loc}}(\mathbb{R}^d) \subseteq L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

(b) *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. If there exists some $k \in \mathbb{N}$ and some $2 \leqslant p \leqslant \infty$ such that $\frac{1}{(1+|\cdot|^2)^k} f \in L^p(\mathbb{R}^d)$, then $f \in \mathcal{LS}_2(\mathbb{R}^d)$.*

(c) $L^{\infty}(\mathbb{R}^d) \subseteq \mathcal{LS}_2(\mathbb{R}^d)$.

Proof.

(a) $\mathcal{LS}_2(\mathbb{R}^d) \subseteq \mathcal{WLS}_2(\mathbb{R}^d)$ follows from the fact that $\widehat{\varphi} \in \mathsf{S}(\mathbb{R}^d)$ for all $\varphi \in C_c^\infty(\mathbb{R}^d)$.

Next, let $f \in \mathcal{WLS}_2(\mathbb{R}^d)$. Let $K \subseteq \mathbb{R}^d$ be any compact set. It is easy to see that there is some $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\widehat{\varphi} \geqslant 1_K$. Then,

$$\int_K |f(x)|^2 \,\mathrm{d}x \leqslant \int_{\mathbb{R}^d} |f(x)|^2 \,|\widehat{\varphi}(x)|^2 \,\mathrm{d}x < \infty\,.$$

This shows that $\mathcal{WLS}_2(\mathbb{R}^d) \subseteq L^2_{\mathrm{loc}}(\mathbb{R}^d)$. As mentioned above, $L^2_{\mathrm{loc}}(\mathbb{R}^d) \subseteq L^1_{\mathrm{loc}}(\mathbb{R}^d)$ follows from Cauchy–Schwarz's or Hölder's inequality.

(b) Let $g \in \mathsf{S}(\mathbb{R}^d)$, and let $2 \leqslant q \leqslant \infty$ be so that $\frac{2}{p} + \frac{2}{q} = 1$. Such a q exists, since $p \geqslant 2$. Then, an application of Hölder's inequality gives

$$\begin{aligned}
\|gf\|_2^2 &= \left\| \left| \frac{1}{(1+|\cdot|^2)^k} f \right|^2 \left| (1+|\cdot|^2)^k g \right|^2 \right\|_1 \\
&\leqslant \left\| \left| \frac{1}{(1+|\cdot|^2)^k} f \right|^2 \right\|_{\frac{p}{2}} \left\| \left| (1+|\cdot|^2)^k g \right|^2 \right\|_{\frac{q}{2}} \\
&= \left\| \frac{1}{(1+|\cdot|^2)^k} f \right\|_p \left\| (1+|\cdot|^2)^k g \right\|_q < \infty
\end{aligned}$$

because of the assumption on f and $g \in \mathsf{S}(\mathbb{R}^d)$.

(c) This follows from (b). □

Remark 5.10.

(a) Let

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{if } 0 < x \leqslant 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then, $f \in L^1(\mathbb{R}) \subseteq L^1_{loc}(\mathbb{R})$ and $f \in L^p(\mathbb{R})$ for all $1 \leqslant p < 2$.

Next, if $g \in \mathsf{S}(\mathbb{R})$ is any function such that $g \geq 1$ on $[0,1]$, then $fg \notin L^2(\mathbb{R})$. This shows that $f \notin \mathcal{LS}_2(\mathbb{R}^d)$ and $f \notin \mathcal{WLS}_2(\mathbb{R}^d)$. In particular, the condition $2 \leqslant p \leqslant \infty$ in Lemma 5.9 (b) is sharp.

(b) Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ be such that $f\boldsymbol{\lambda}$ is a tempered measure in strong sense. Then, there exists some $n \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^d} \frac{1}{(1+|x|^2)^n} |f(x)| \,\mathrm{d}x < \infty,$$

which is the condition from Lemma 5.9 (b) for $p = 1$. Unfortunately, as pointed above, this is not enough to conclude that $f \in \mathcal{LS}_2(\mathbb{R}^d)$.

The next result is an immediate consequence of Lemma 5.9.

Corollary 5.11. *We have the following equality of spaces.*

$$\begin{aligned}
L^\infty(\mathbb{R}^d) &= L^1_{\mathrm{loc}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) = L^2_{\mathrm{loc}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \\
&= \mathcal{WLS}_2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) = \mathcal{LS}_2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\,.
\end{aligned}$$

Proof. Lemma 5.9 (c) implies

$$L^\infty(\mathbb{R}^d) \subseteq \mathcal{LS}_2(\mathbb{R}^d) \subseteq \mathcal{WLS}_2(\mathbb{R}^d) \subseteq L^2_{\mathrm{loc}}(\mathbb{R}^d) \subseteq L^1_{\mathrm{loc}}(\mathbb{R}^d)\,.$$

The claim follows. ☐

Now, we can prove the following result. Note here that the only difference between this result and Proposition 5.5, is that the definition of $\mathcal{LS}_2(\mathbb{R}^d)$ and $\mathcal{WLS}_2(\mathbb{R}^d)$, respectively, allows us replace in the proof of Proposition 5.5 the function $\check{\check{g}}$ for $g \in C_\mathrm{c}^\infty(\mathbb{R}^d)$ by $h \in \mathsf{S}(\mathbb{R}^d)$ and $\varphi \in C_\mathrm{c}^\infty(\mathbb{R}^d)$, respectively. Repeating the proof with these changes, we get the following result, which strengthens Corollary 5.7. As the conversion of the proof is straightforward, we skip it.

Proposition 5.12. *Let $\omega \in \mathcal{S}'(\mathbb{R}^d)$.*

(a) There exists some $f \in \mathcal{LS}_2(\mathbb{R}^d)$ such that $\widehat{\omega} = f\boldsymbol{\lambda}$ if and only if $h * \omega \in L^2(\mathbb{R}^d)$ for all $h \in \mathcal{S}(\mathbb{R}^d)$.
(b) There exists some $f \in \mathcal{WLS}_2(\mathbb{R}^d)$ such that $\widehat{\omega} = f\boldsymbol{\lambda}$ if and only if $\varphi * \omega \in L^2(\mathbb{R}^d)$ for all $\varphi \in C_\mathrm{c}^\infty(\mathbb{R}^d)$.

5.1 | Measures with uniformly discrete support

We can now prove the following result, which has interesting consequences for measures with Meyer sets support.

Theorem 5.13. *Let $\Lambda \subset \mathbb{R}^d$ be a uniformly discrete set, and let*

$$\mu = \sum_{x\in\Lambda} c_x \delta_x$$

be a tempered measure. Then, the following statements are equivalent.

(i) There exists some $f \in \mathcal{WLS}_2(\mathbb{R}^d)$ such that $\widehat{\mu} = f\boldsymbol{\lambda}$.
(ii) $\varphi * \mu \in L^2(\mathbb{R}^d)$ for all $\varphi \in C_\mathrm{c}^\infty(\mathbb{R}^d)$.
(iii) $\sum_{x\in\Lambda} |c_x|^2 < \infty$.

Proof. (i) $\iff$ (ii): This is a consequence of Proposition 5.12.

(ii)⇒ (iii): Let $U \subseteq \mathbb{R}^d$ be such that Λ is U-uniformly discrete. Pick $\varphi \in C_\mathrm{c}^\infty(\mathbb{R}^d)$ such that $\mathrm{supp}(\varphi) \subseteq U$ and $\varphi \geqslant 1$ on some compact set $K \subseteq U$ of positive Lebesgue measure. Then, by [33, Lemma 5.8.3] and its proof, we have

$$|\mu * \varphi|(y) = \sum_{x\in\Lambda} |c_x|\,|\varphi(y-x)|\,,$$

for all $y \in \mathbb{R}^d$, and the sum $\sum_{x\in\Lambda} |c_x|\,|\varphi(y-x)|$ contains at most one nonzero term. Then,

$$|(\mu * \varphi)(y)|^2 = \sum_{x\in\Lambda} |c_x|^2\,|\varphi(y-x)|^2 \geqslant \sum_{x\in\Lambda} |c_x|^2\,|1_K(y-x)| \qquad \text{for all } y \in \Lambda\,.$$

Note here that if Λ is finite, then

$$\lambda(K) \sum_{x\in\Lambda} |c_x|^2 = \sum_{x\in\Lambda} |c_x|^2 \int_{\mathbb{R}^d} |1_K(y-x)|\,\mathrm{d}y = \int_{\mathbb{R}^d} \sum_{x\in\Lambda} |c_x|^2\,|1_K(y-x)|\,\mathrm{d}y\,.$$

On the other hand, if Λ is infinite, it is countable by uniform discreteness. Let $\{x_n\}$ be an enumeration of Λ. Then, the sequence of functions $f_n := \sum_{k=1}^n |c_{x_k}|^2 |1_K(\cdot - x_k)| \in L^1(\mathbb{R}^d)$ is increasing, and hence

$$\lambda(K) \sum_{x\in\Lambda} |c_x|^2 = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} \lim_{n\to\infty} f_n(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} \left(\sum_{x\in\Lambda} |c_x|^2\,|1_K(y-x)| \right) \mathrm{d}y,$$

where we applied the monotone convergence theorem. Therefore, in both cases, we obtain

$$\lambda(K)\sum_{x\in\Lambda}|c_x|^2=\int_{\mathbb{R}^d}\left(\sum_{x\in\Lambda}|c_x|^2|1_K(y-x)|\right)\mathrm{d}y\leqslant\int_{\mathbb{R}^d}|(\mu*\varphi)(y)|^2\,\mathrm{d}y<\infty.$$

(iii)⇒(ii): Note first that (iii) implies that

$$\nu:=\sum_{x\in\Lambda}|c_x|^2\delta_x$$

is a finite positive measure on $\mathbb{R}^d$.

Let again $U\subseteq\mathbb{R}^d$ be such that Λ is U-uniformly discrete. Let $\varphi\in C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$. By a standard partition of unity argument, there exist $\varphi_1,\dots,\varphi_k\in C_{\mathrm{c}}^{\infty}(\mathbb{R}^d)$ and $y_1,\dots,y_k\in\mathbb{R}^d$ such that

$$\varphi=\sum_{j=1}^{k}\varphi_j\qquad\text{with}\qquad\operatorname{supp}(\varphi_j)\subseteq y_j+U\ \text{ for all } 1\leqslant j\leqslant k\,.$$

By [33, Lemma 5.8.3] and its proof, we have

$$|(\mu*\varphi_j)(y)|=(|\mu|*|\varphi_j|)(y)=\sum_{x\in\Lambda}|c_x|\,|\varphi_j(y-x)|$$

with $|c_x||\varphi_j(y-x)|=0$, for all $1\leqslant j\leqslant k$ and all $y\in\mathbb{R}^d$, for all but at most one $x\in\Lambda$. In particular, this implies

$$|(\mu*\varphi_j)(y)|^2=\sum_{x\in\Lambda}|c_x|^2\,|\varphi_j(y-x)|^2=\int_{\mathbb{R}^d}|\varphi_j(y-x)|^2\,\mathrm{d}\nu(x)$$

for all $1\leqslant j\leqslant k$ and $y\in\mathbb{R}^d$. Therefore, an application of Fubini's theorem gives

$$\begin{aligned}\int_{\mathbb{R}^d}|(\mu*\varphi_j)(y)|^2\mathrm{d}y&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\varphi_j(y-x)|^2\,\mathrm{d}\nu(x)\,\mathrm{d}y\\&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\varphi_j(y-x)|^2\,\mathrm{d}y\,\mathrm{d}\nu(x)=\|\varphi_j\|_2^2\int_{\mathbb{R}^d}1\,\mathrm{d}\nu(x)\\&=\|\varphi_j\|_2^2\sum_{x\in\Lambda}|c_x|^2<\infty.\end{aligned}$$

This completes the proof. □

The next corollary is an immediate consequence.

Corollary 5.14. *Let $\Lambda\subset\mathbb{R}^d$ be a uniformly discrete set, and let*

$$\mu=\sum_{x\in\Lambda}c_x\delta_x$$

be a tempered measure such that $\sum_{x\in\Lambda}|c_x|^2<\infty$. Then, its Fourier transform in the tempered distribution sense is an absolutely continuous measure.

In particular, if μ is Fourier transformable as a measure, $\widehat{\mu}$ is absolutely continuous.

Fourier transformable measures with Meyer set support are of special interest in some branches of mathematics like aperiodic order. Therefore, we are going to list some consequences for this special class of measures.

Theorem 5.15. *Let μ be a Fourier transformable measure supported inside a Meyer set $\Gamma \subset \mathbb{R}^d$. Then, there exists a Meyer set $\Lambda \subset \mathbb{R}^d$ and a bounded function $c : \Lambda \to \mathbb{C}$ such that*

(a) $\mu_{0\mathrm{a}} = \sum_{x\in\Lambda} c(x)\,\delta_x$.
(b) $\lim_{\|x\|\to\infty} |c(x)| = 0$.
(c) if there exists some $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *such that* $(\widehat{\mu})_{\mathrm{ac}} = f\boldsymbol{\lambda}$, *then*

$$\sum_{x\in\Lambda} |c(x)|^2 < \infty\,.$$

Proof.

(a) This follows from [34, Theorem 4.1] or [36, Theorem 5.7].
(b) This is a consequence of [29, Corollary 35].
(c) By Corollary 5.11, we have $f \in \mathcal{WLS}_2(\mathbb{R}^d)$. The claim follows now from Theorem 5.13. □

Remark 5.16. Under the assumptions of Theorem 5.15, if $(\mathbb{R}^d, H, \mathcal{L})$ is any cut-and-project scheme and $W \subseteq H$ any compact set such that $\Lambda \subseteq \curlywedge(W)$, we can choose $\Gamma = \curlywedge(W)$ [36, Theorem 5.7].

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ENDNOTE

[1] We reversed the order of letters here since SL_2 is used for the special linear group.

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