Capacities And Embeddings Of Besov Spaces Via General Convolution Kernels

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The final publication is available at Springer via
https://doi.org/10.1007/s44007-024-00091-4

CAPACITIES AND EMBEDDINGS OF BESOV SPACES VIA GENERAL CONVOLUTION KERNELS

PENGTAO LI, RUI HU, AND ZHICHUN ZHAI

Abstract. This note is denoted to establish equivalent characterizations of Carleson embeddings of fractional Besov spaces $\dot{\Lambda}^{p,q}_{\beta}(\mathbb{R}^n)$ into the Lorentz spaces $L^{q_0,p}_{\mu}(\mathbb{R}^{1+n}_+)$ induced by general convolution kernels $\Phi_t(\cdot)$. When $(p,q) \in (1, n/\beta) \times (1,\infty)$, the embeddings will be characterized in terms of capacitary type inequalities for open subsets of $\mathbb{R}^n$. When $p = q \in (0,1]$, the embeddings will be characterized in terms of fractional Besov capacities or the associated variational functional of a nonnegative Radon measure μ. Especially, when $p = q = 1$ and $\beta \in (0,1)$, the characterization can be also established in terms of fractional perimeters.

Contents

1. Introduction

1.1. **Motivations.** Due to the profound theoretical background in physics, the concept of capacities has been widely applied in the research of analysis and partial differential equations. Let μ be a positive charge distribution on a compact subset K of $\mathbb{R}^3$. In mathematics, the charge μ can be considered as a Radon measure and the total charge on K is defined as $\mu(K)$. For a unit test charge at $x \in \mathbb{R}^3 \setminus K$, the Newton potential of μ at x is

$$U^{\mu}(x) = \int_K \frac{d\mu(y)}{|x-y|}.$$

2020 *Mathematics Subject Classification.* Primary 42B35; 46E30, 30H25.

Key words and phrases. Carleson embeddings; Fractional capacities; Besov spaces, fractional perimeters.

Project supported: The research was partly supported by National Natural Science Foundation of China (No. 11871293, No. 12071272) and Shandong Natural Science Foundation of China (No. ZR2020MA004).

The capacity of K, denoted by $Cap(K)$, is interpreted as the charge which makes the equilibrium potential of μ equal to one. Denote by $\mathcal{M}_+(K)$ the set of all positive Radon measures on K. It can be proved that

$$Cap(K) = \inf\left\{\int_K\int_K \frac{d\mu(y)}{|x-y|}d\mu(x) : \mu \in \mathcal{M}_+(K),\ U^\mu \geq 1 \text{ on } K\right\}.$$

Moreover, the capacity $Cap(K)$ can also be represented as

$$Cap(K) = \inf\left\{\frac{1}{4\pi}\int_{\mathbb{R}^3} |\nabla\varphi(x)|^2dx : \varphi \in C_0^\infty(\mathbb{R}^3),\ \varphi \geq 1 \text{ on } K\right\},$$

which is related with the Sobolev space $W^{1,2}(\mathbb{R}^n)$.

The above observation inspired many researchers to investigate capacities associated to function spaces, see [1], [2], [3], [16] and the references therein. As an ideal substitute of measures, capacities have been applied to investigate the continuity of the embeddings: $L^p(\mathbb{R}^n) \to L^q_\mu(\mathbb{R}^n)$, where μ denotes a positive Radon measure. Let g be a rapidly decreasing convolution kernel on $\mathbb{R}^n$. The classical L^p-capacity is defined as follows. Let $1 \leq p < \infty$ and $E \subset \mathbb{R}^n$. Define

$$C_{g,p}(E) := \inf\left\{\int_{\mathbb{R}^n} |f(x)|^p dx :\ f \in L^p_+(\mathbb{R}^n),\ g * f \geq 1 \text{ for all } x \in E\right\}.$$

Here $f \in L^p_+(\mathbb{R}^n)$ contains all nonnegative elements of $L^p(\mathbb{R}^n)$. If $g * f < 1$ on E for all $f \in L^p_+(\mathbb{R}^n)$, we set $C_{g,p}(E) = \infty$. The following capacitary weak type inequality holds:

$$C_{g,p}(\{x \in \mathbb{R}^n :\ g * f \geq \lambda\}) \leq \lambda^{-p}\int_{\mathbb{R}^n} |f(x)|^p dx, \tag{1.1}$$

which implies that, for $E \subset \mathbb{R}^n$, $C_{g,p}(E) = 0$ if and only if there exists an $f \in L^p_+(\mathbb{R}^n)$ such that $E \subset \{x :\ g * f = \infty\}$. In fact, it can be proved that the following capacitary strong type inequality holds. If g is a radially decreasing convolution kernel on $\mathbb{R}^n$, then there holds

$$\int_0^\infty C_{g,p}(\{x :\ g * f(x) \geq \lambda\})d\lambda^p \lesssim \|f\|_p^p$$

for $p \in (1, \infty)$ and all $f \in L^p_+(\mathbb{R}^n)$, see [3, Theorem 7.1.1]. Here $A \lesssim B$ means that there exists a positive constant C such that $A \leq CB$.

As an application of the strong type inequality, there holds the following characterization of the embedding

$$\mathcal{G} :\ f \in L^p(\mathbb{R}^n) \to g * f \in L^q_\mu(\mathbb{R}^n).$$

Theorem 1.1. ([3, Theorem 7.2.1]) *Let g be a radially decreasing convolution kernel, and let $\mu \in \mathcal{M}_+(\mathbb{R}^n)$. Then for $1 < p \leq q < \infty$, the following properties of μ are equivalent:*
(i) *There holds*

$$\left(\int_{\mathbb{R}^n} |g * f(x)|^q d\mu(x)\right)^{1/q} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \quad \forall\ f \in L^p(\mathbb{R}^n).$$

(ii) *There holds*

$$\|g * \mu_K\|_{L^{p'}(\mathbb{R}^n)} \lesssim \mu(K)^{1/q'}$$

for all compact sets $K \subset \mathbb{R}^n$.
(iii) *There holds*

$$\sup_{t>0} t(\mu(\{x :\ |g * f(x)| \geq t\}))^{1/q} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \quad \forall\ f \in L^p(\mathbb{R}^n).$$

(iv) *There holds*

$$(\mu(K))^{1/q} \lesssim C_{g,p}(K)^{1/p}$$

for all compact sets $K \subset \mathbb{R}^n$.

The first version of Theorem 1.1 was proved by Maz'ya in 1960s when studying the Schrödinger operator $L = -\triangle - \mu$, for a positive measure μ on a domain $\Omega \subset \mathbb{R}^n, n \geq 3$. The idea of Theorem 1.1 motivated many researchers to study the embeddings of functions induced by partial differential equations. Adams and Xiao [4] studied the embddings of Besov spaces via the classical Poisson kernel. Xiao [29, 30] established characterizations of Carleson embeddings of fractional Sobolev/Besov spaces induced by the classical heat equation. The authors of [29, 30, 33, 10, 15, 32, 27] studied the characterization of the embeddings of fractional Sobolev spaces, Besov spaces and Lebesgue spaces via the spatial fractional heat type equations. Following the ideas of these former progress, Li, Zhai and their collaborators investigated the equivalent characterizations of Carleson embeddings of function spaces via the Caffarelli-Silvestre extension and space-time fractional equations, see [19, 17, 20, 18, 21].

Motivated by the idea of Theorem 1.1 and the above-mentioned works, in this article, we aim to characterize the Carleson type embeddings from the function spaces on $\mathbb{R}^n$ to the ones defined on the upper-half Euclidean spaces $\mathbb{R}^{1+n}_+$ and associated to $u(t,x) = \Phi_t * f(x)$, where $\{\Phi_t(\cdot)\}_{t>0}$ is a family of convolution kernels satisfying some regularity condition and canceling condition. More specifically, in this article, we make the following two assumptions on $\Phi_t(\cdot)$.

- Assumption 1. $\Phi_t(x) = t^{-n}\Phi(x/t)$ for all $t > 0$ with Φ being nonnegative, lower semicontinuous, radially decreasing and integral on $\mathbb{R}^n$ satisfying $\int_{\mathbb{R}^n} \Phi(x)dx = 1$.
- Assumption 2. For any $\delta > 0$, there exists a constant C such that

$$\inf \{\Phi_t(x) : |x| \leq \delta t\} \geq C t^{-n}.$$

It is easy to see that these assumptions can be satisfied when Φ_t is an approximate identity generalized by a Schwartz function. Based on these assumptions, we will characterize a nonnegative measure μ on $\mathbb{R}^{1+n}_+$ such that

$$\|u(\cdot,\cdot)\|_{L^{q_0}_\mu(\mathbb{R}^{1+n}_+)} \lesssim \|f\|_{\dot{\Lambda}^{p,q}_\beta(\mathbb{R}^n)} \quad \forall\, f \in C_0^\infty(\mathbb{R}^n).$$

Note that $\Phi_t(x)$ in the assumption 1 can be replaced by $\Phi_t(x) = t^{-n\alpha}\Phi(x/t^\alpha)$ for $\alpha > 0$, which covers the spatial/spatial-time fractional heat kernel. Similar embedding problems have been studied in [10, 15, 27, 29, 30, 32, 33].

In the rest of this article, we will use the following function spaces. Let $C_0^\infty(\mathbb{R}^n)$ be the class of all infinitely smooth functions with compact support in $\mathbb{R}^n$. For $0 < p, q < \infty$ and a nonnegative Radon measure μ on $\mathbb{R}^{n+1}_+$, $L^{q,p}_\mu(\mathbb{R}^{1+n}_+)$ and $L^q_\mu(\mathbb{R}^{1+n}_+)$ denote the Lorentz space and the Lebesgue space of all functions $g(\cdot,\cdot)$ on $\mathbb{R}^{n+1}_+$, respectively. Here,

$$\|g(\cdot,\cdot)\|_{L^{q,p}_\mu(\mathbb{R}^{1+n}_+)} := \left(\int_0^\infty \left(\mu\left(\{(t,x) \in \mathbb{R}^{n+1}_+ : \ |g(t,x)| > \lambda\}\right)\right)^{p/q} d\lambda^p\right)^{1/p} < \infty$$

and

$$\|g(\cdot,\cdot)\|_{L^q_\mu(\mathbb{R}^{1+n}_+)} := \left(\int_{\mathbb{R}^{n+1}_+} |g(t,x)|^q d\mu\right)^{1/q} < \infty.$$

Moreover, we denote by $L^{q,\infty}_\mu(\mathbb{R}^{1+n}_+)$ the set of all $\mu-$measurable functions $g(\cdot,\cdot)$ on $\mathbb{R}^{n+1}_+$ with

$$\|g(\cdot,\cdot)\|_{L^{q,\infty}_\mu(\mathbb{R}^{1+n}_+)} := \sup_{s>0} s\left(\mu\left(\left\{(t,x) \in \mathbb{R}^{n+1}_+ : \ |g(t,x)| > s\right\}\right)\right)^{1/q} < \infty.$$

In this paper, $\dot{\Lambda}^{p,q}_{\beta}(\mathbb{R}^n)$ is the homogeneous Besov space defined as the completion of all $C_0^\infty(\mathbb{R}^n)$ functions with $\|f\|_{\dot{\Lambda}^{p,q}_{\beta}(\mathbb{R}^n)} < \infty$, where the norm $\|f\|_{\dot{\Lambda}^{p,q}_{\beta}(\mathbb{R}^n)}$ is defined as follows. For $(\beta, p, q) \in (0,\infty) \times (0,\infty) \times (0,\infty)$,

$$\|f\|_{\dot{\Lambda}^{p,q}_{\beta}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \|\Delta_h^k f\|^q_{L^p(\mathbb{R}^n)} |h|^{-(n+\beta q)} dh\right)^{1/q}.$$

Here $k = 1 + [\beta], \beta = [\beta] + \{\beta\}$ with $[\beta] \in \mathbb{Z}_+$, $\{\beta\} \in (0,1)$ and

$$\Delta_h^k f = \Delta_h^1 \Delta_h^{k-1} f, \quad \Delta_h^1 = f(x+h) - f(x) \quad \forall\, x \in \mathbb{R}^n.$$

The Besov capacities $C^{p,q}_{\beta}(\cdot)$ have been studied in [2, 4].

Definition 1.2. Let $(\beta, p, q) \in (0,\infty)\times(0,\infty)\times(0,\infty)$. For a compact set $K \subset \mathbb{R}^n$, its Besov capacity $C^{p,q}_{\beta}(K)$ is defined as

$$C^{p,q}_{\beta}(K) := \inf\left\{\|f\|^p_{\dot{\Lambda}^{p,q}_{\beta}(\mathbb{R}^n)} : \ f \in C_0^\infty(\mathbb{R}^n) \text{ and } f \geq 1_K\right\}$$

and for any set $E \subset \mathbb{R}^n$, its Besov capacity

$$C^{p,q}_{\beta}(E) := \inf_{\text{open } O \supseteq E} \ \sup_{\text{compact } K \subseteq O} \left\{C^{p,q}_{\beta}(K)\right\},$$

where 1_A stands for the characteristic function of the set A.

The Besov capacity $C^{p,q}_{\beta}(\cdot)$ is equivalent to the classical Hausdorff capacity $H_d^{(\infty)}$ defined as, for $d \in (0,\infty)$,

$$H_d^{(\infty)}(K) := \inf\Big\{\sum_{i=0}^{\infty} m_i 2^{-di}\Big\},$$

where the infimum is taken over all countable coverings of $K \subset \mathbb{R}^n$ by balls whose radii $r_j < \infty$, while m_i is the number of balls from this covering whose radii r_j belong to the interval $(2^{-i-1}, 2^{-i}], i = 0, 1, 2, \ldots$. Then, it follows from [24, Theorem 2] that $C^{p,p}_{\beta}(\cdot) \approx H^{(\infty)}_{n-\beta p}(\cdot)$ when $\beta \in (0,n)$, $p \in (0,\infty)$ and $0 < p\beta < n$.

For any open set $O \subset \mathbb{R}^n$, the tent space $T(O) \subset \mathbb{R}^{1+n}_+$ based on O is defined as

$$T(O) = \left\{(t,x) \in \mathbb{R}^{1+n}_+ : \ B(x,t) \subset O\right\}.$$

For a nonnegative measure μ on $\mathbb{R}^{n+1}_+$ and $\lambda > 0$,

$$c^{\beta}_{p,q}(\mu;\lambda) := \inf\left\{C^{p,q}_{\beta}(O) : \ \text{open } O \subseteq \mathbb{R}^n, \mu(T(O)) > \lambda\right\} \tag{1.2}$$

is the variational capacity minimizing function associated with $C^{p,q}_{\beta}(\cdot)$.

When $(p,q) \in (1, n/\beta)\times(1,\infty)$, the embeddings will be characterized in terms of capacitary type inequalities for open subsets of $\mathbb{R}^n$. When $p = q \in (0,1]$, the embeddings will be characterized in terms of fractional Besov capacities $C^{p,q}_{\beta}(\cdot)$ or the associated variational functional $c^{\beta}_{p,q}(\mu;\lambda)$ of a nonnegative Radon measure μ. Especially, when $p = q = 1$, the characterization can be established in terms of the fractional perimeter. Let us review some preliminaries on the fractional perimeter. See [31] for more details.

When $\beta \in (0,1)$, the fractional $\beta-$perimeter of E is half of the fractional Besov norm $\|\cdot\|_{\dot{\Lambda}^{1,1}_{\beta}(\mathbb{R}^n)}$ of 1_E, that is,

$$P_\beta(E) = \int_E \int_{E^c} \frac{1}{|x-y|^{n+\beta}} dxdy = \frac{1}{2}\|1_E\|_{\dot{\Lambda}^{1,1}_{\beta}(\mathbb{R}^n)}$$

which is also closely related to the $C^{1,1}_\beta(\cdot)$. For more significant roles of the fractional perimeter $P_\beta(E)$ in modern geometric analysis, see [5, 6, 7, 8, 9, 12, 13, 14, 25, 26].

Now, we are ready to state our main results.

1.2. Statements of main results.

The first result characterizes the Carleson embeddings in terms of several geometric inequalities when $p \in (n/(n+\beta), 1]$.

Proposition 1.3. *Let $\beta \in (0,n)$ when $p = 1$, or $\beta \in (0,1)$ when $p \in (n/(n+\beta), 1)$. Let $q_0 \in [p,\infty)$ and μ be a nonnegative Radon measure on $\mathbb{R}^{1+n}_+$. Denote $u(t,x) = \Phi_t * f(x)$. Then the following statements are equivalent.*

(i) *There holds*

$$\|u(\cdot,\cdot)\|_{L^{q_0,p}_\mu(\mathbb{R}^{1+n}_+)} \lesssim \|f\|_{\dot\Lambda^{p,p}_\beta(\mathbb{R}^n)} \quad \forall\, f \in C^\infty_0(\mathbb{R}^n).$$

(ii) *There holds*

$$\|u(\cdot,\cdot)\|_{L^{q_0}_\mu(\mathbb{R}^{1+n}_+)} \lesssim \|f\|_{\dot\Lambda^{p,p}_\beta(\mathbb{R}^n)} \quad \forall\, f \in C^\infty_0(\mathbb{R}^n).$$

(iii) *There holds*

$$\|u(\cdot,\cdot)\|_{L^{q_0,\infty}_\mu(\mathbb{R}^{1+n}_+)} \lesssim \|f\|_{\dot\Lambda^{p,p}_\beta(\mathbb{R}^n)} \quad \forall\, f \in C^\infty_0(\mathbb{R}^n).$$

(iv) *There holds*

$$\sup_{t>0} \frac{t^{p/q_0}}{c^\beta_{p,p}(\mu;t)} < \infty.$$

(v) *There holds*

$$\sup\left\{ \frac{(\mu(T(O)))^{p/q_0}}{C^{p,p}_\beta(O)} : \textit{open set } O \subset \mathbb{R}^n \right\} < \infty.$$

(vi) *There holds*

$$\sup_{(r,x)\in\mathbb{R}^{1+n}_+} \frac{(\mu(T(B(x,r))))^{p/q_0}}{r^{n-p\beta}} < \infty.$$

Denote

$$p \vee q = \max\{p,q\} \quad \text{and} \quad p \wedge q = \min\{p,q\}.$$

When $(p,q) \in (1,n/\beta) \times (1,\infty)$, we can establish the following result.

Proposition 1.4. *Let μ be a nonnegative Borel measure on $\mathbb{R}^{1+n}_+$. Denote $u(t,x) = \Phi_t * f(x)$. If $\beta \in (0,n)$ and $(p,q) \in (1,n/\beta) \times (1,\infty)$, then*

$$\|u(\cdot,\cdot)\|_{L^{p\wedge q,p\vee q}_\mu(\mathbb{R}^{1+n}_+)} \lesssim \|f\|_{\dot\Lambda^{p,q}_\beta(\mathbb{R}^n)} \quad \forall\, f \in C^\infty_0(\mathbb{R}^n) \tag{1.3}$$

if and only if

$$\mu(T(O)) \lesssim (C^{p,q}_\beta(O))^{(p\wedge q)/p} \quad \forall\ \textit{open set } O \subset \mathbb{R}^n. \tag{1.4}$$

The embeddings from $\dot\Lambda^{1,1}_\beta(\mathbb{R}^n)$ to $L^q_\mu(\mathbb{R}^{1+n}_+)$ can be characterized in terms of the following geometric inequality of the fractional perimeter.

Proposition 1.5. *Let μ be a nonnegative Radon measure on $\mathbb{R}^{1+n}_+$ and $1 \le q < \infty$. The following two statements are equivalent.*

(i) *There holds the analytic inequality:*

$$\|u(\cdot,\cdot)\|_{L^q_\mu(\mathbb{R}^{1+n}_+)} \lesssim \|f\|_{\dot\Lambda^{1,1}_\beta(\mathbb{R}^n)} \quad \forall f \in C^\infty_0.$$

(ii) *There holds the geometric inequality:*

$$\mu(T(O))^{1/q} \lesssim P_\beta(O)$$

for all bounded domain $O \subset \mathbb{R}^n$.

Our last result extends Theorem 1.1, i.e., [3, Theorem 7.2.1] to the endpoint case $p = 1$.

Proposition 1.6. *Let g be a radially decreasing convolution kernel, and let $\mu \in \mathcal{M}_+(\mathbb{R}^n)$. Then for $1 = p \leq q < \infty$, the following properties of μ are equivalent:*
(iii) *There holds*

$$\sup_{t>0} t(\mu(\{x : |g * f(x)| \geq t\}))^{1/q} \lesssim \|f\|_{L^1(\mathbb{R}^n)} \quad \forall f \in L^1(\mathbb{R}^n).$$

(iv) *There holds*

$$(\mu(K))^{1/q} \lesssim C_{g,1}(K)$$

for all compact sets $K \subset \mathbb{R}^n$.

In next section, we will provide the proofs of our main results.

2. Proof of main results

Before proving our main results, we need to establish the following basic lemmas.

2.1. **Basic lemmas.**

Lemma 2.1. *For $f \in C_0^\infty(\mathbb{R}^n)$, $\lambda > 0$, and a nonnegative measure μ on $\mathbb{R}^{1+n}_+$, let $u(t,x) = \Phi_t * f(x)$,*

$$E_\lambda(f) = \left\{(t,x) \in \mathbb{R}^{n+1}_+ : |u(t,x)| > \lambda\right\}$$

and

$$R_\lambda(f) = \left\{y \in \mathbb{R}^n : \sup_{|y-x|<t} |u(x,t)| > \lambda\right\}.$$

Then the following four statements are true.
(i) *For any natural number k*

$$\mu\left(E_\lambda(f) \cap T(B(0,k))\right) \leq \mu\left(T\left(R_\lambda(f) \cap B(0,k)\right)\right).$$

(ii) *For any natural number k,*

$$Cap^{\beta,p}_{\mathbb{R}^n}\left(R_\lambda(f) \cap B(0,k)\right) \geq c^\beta_{p,p}(\mu, \mu\left(T\left(R_\lambda(f) \cap B(0,k)\right)\right).$$

(iii) *There exists a constant $\theta_n > 0$ such that*

$$\sup_{|y-x|<t} |u(t,y)| \leq \theta_n \mathcal{M}f(x), \quad x \in \mathbb{R}^n,$$

where $\mathcal{M}$ denotes the Hardy-Littlewood maximal operator:

$$\mathcal{M}f(x) = \sup_{r>0} r^{-n} \int_{B(x,r)} |f(y)|dy, \ x \in \mathbb{R}^n.$$

Proof. (i) The assumption 1 implies that $\sup_{|y-x|<t} |u(x,t)|$ is lower semicontinuous on $\mathbb{R}^n$. So, $R_\lambda(f)$ is an open subset of $\mathbb{R}^n$ and thus

$$\begin{cases} E_\lambda(f) \subseteq T(R_\lambda(f)); \\ \mu(E_\lambda(f)) \leq \mu(T(R_\lambda(f))). \end{cases}$$

Therefore, we get

$$\mu(E_\lambda(f) \cap T(B(0,k))) \quad \leq \quad \mu(T(R_\lambda(f) \cap T(B(0,k)))) = \mu(T(R_\lambda(f) \cap B(0,k))).$$

(ii) It follows from the definition of $c^\beta_{p,p}(\mu; t)$.

(iii) It follows from the assumption 1 and [28, page 57, Proposition] that

$$\sup_{|y-x|<t} |\Phi_t * f(t,y)| \leq \theta_n \mathcal{M}f(x).$$

□

We need the following strong type estimate for Besov capacities $C^{p,q}_\beta(\cdot)$.

Lemma 2.2. *When* $\beta \in (0,1)$ & $p = q \in (n/(n+\beta), 1)$, *or* $\beta \in (0,n)$ & $p = q = 1$, *or* $(\beta, p, q) \in (0,n) \times (1, n/\beta) \times (1,\infty)$,

$$\int_0^\infty \left(C^{p,q}_\beta(\{x \in \mathbb{R}^n : |\mathcal{M}f(x)| > \lambda\})\right)^{1\vee(q/p)} d\lambda^{p\vee q} \lesssim \|f\|^{p\vee q}_{\dot{\Lambda}^{p,q}_\beta(\mathbb{R}^n)} \quad \forall\, f \in C^\infty_0(\mathbb{R}^n).$$

Proof. When $p = q > 1$, this statement was proved by Maz'ya [22]. When $1 \le p \le q < \infty$ and $0 < \beta < 1$, it was obtained by Wu [34]. Adams and Xiao [4] established the case $0 < \beta < \infty$ and $(p,q) \in (1, n/\beta) \times (1,\infty)$. When $\beta \in (0,n)$ and $p = q = 1$, it was proved by Xiao in [30]. Xiao and Zhai [32] showed this statement when $0 < \beta < 1$ and $n/(n+\beta) < p = q < 1$. □

The following lemma was proved by Dafni and Xiao in [11].

Lemma 2.3. ([11, Lemma 4.1]) *Let* $d \in (0,n]$ *and* $\{I_j\}$ *be a sequence of dyadic cubes in* $\mathbb{R}^n$ *such that* $\sum_j |I_j|^{d/n} < \infty$. *Then there exists a sequence of dyadic cubes* $\{J_k\}$ *with mutually disjoint interiors,* $\cup_k J_k = \cup_j I_j$, *and*

$$\sum_k |J_k|^{d/n} \le \sum_j |I_j|^{d/n}.$$

Moreover, if a set $O \subset \cup_j I_j$, *then the tent*

$$T(O) \subset \cup_k T(J^*_k),$$

where J^*_k *is the cube with the same center as* J_k *but* $5\sqrt{n}$ *times the sidelength.*

The following result is due to Xiao in [31].

Lemma 2.4. ([31, Theorem 2]) *If* $\beta \in (0,1)$ *and* K *is a compact subset of* $\mathbb{R}^n$, *then*

$$C^{1,1}_\beta(K) = 2 \inf_{O \in O^\infty(K)} P_\beta(O),$$

where $O^\infty(K)$ *denotes the class of all open sets with* C^∞ *boundary that contain* K.

2.2. **Proof of Proposition 1.3.**

Proof. We divide the proof into three parts.

Part 1: (i)⟹(ii)⟹(iii)⟹(v)⟹(i). (i)⟹(ii)⟹(iii) is a direct result of the monotone increasing with respect to the second index of Lorentz spaces $L^{p,q}_\mu(\mathbb{R}^{1+n}_+)$.

In the following, we show (iii)⟹(v)⟹(i). We first show (iii)⟹(v). Assume that (iii) holds. Let $O \subset \mathbb{R}^n$ be an open set. Let K be a compact subset of O. If a nonnegative function $f \in C^\infty_0(\mathbb{R}^n)$ satisfies $f \ge 1$ on K, since $B(x,t) \subset K$ for $(x,t) \in T(K)$, we can derive from the assumption 2 that, for $|x-y| < t$,

$$\Phi_t(x-y) \ge Ct^{-n}.$$

So, we get

$$\begin{aligned} |\Phi_t * f(x)| &= \int_{\mathbb{R}^n} \Phi_t(x-y) f(y)\,dy \\ &\ge C \int_K \Phi_t(x-y)\,dy \\ &\ge C \int_{B(x,t)} t^{-n}\,dy \\ &\ge C, \end{aligned}$$

which yields

$$T(K)\subset\Big\{(t,x)\in\mathbb{R}^{1+n}_+:\ |\Phi_t*f(x)|>C\Big\}.$$

Hence, (iii) implies

$$\mu(T(K))\quad\leq\quad\mu\Big(\Big\{(t,x)\in\mathbb{R}^{1+n}_+:\ |\Phi_t*f(x)|>C\Big\}\Big)\lesssim\|f\|^{q_0}_{\dot{\Lambda}^{p,p}_\beta(\mathbb{R}^n)}.$$

Taking the supermum on f gives

$$\mu(T(K))\leq(C^{p,p}_\beta(K))^{q_0/p}.$$

The definition of $C^{p,p}_\beta(\cdot)$ implies

$$\begin{aligned}\mu(T(O))&=\sup_{K\subset O}\mu(T(K))\\&\leq\sup_{K\subset O}(C^{p,p}_\beta(K))^{q_0/p}\\&\leq(C^{p,p}_\beta(O))^{q_0/p}.\end{aligned}$$

This proves (v).

Below we prove (v) $\Longrightarrow$ (i). Given $f\in C^\infty_0(\mathbb{R}^n)$. (i)-(iv) and (v) of Lemma 2.1 give us

$$\mu(E_\lambda)\leq\mu\Big(T\Big(\Big\{x\in\mathbb{R}^n:\ \mathcal{M}f(x)>\lambda/C\Big\}\Big)\Big)$$

with a constant C.

If (v) holds for every open set $O\subset\mathbb{R}^n$, then we get

$$\mu(E_\lambda(f))\lesssim\Big(C^{p,p}_\beta\Big(\Big\{x\in\mathbb{R}^n:\ \mathcal{M}f(x)>\lambda/C\Big\}\Big)\Big)^{q_0/p}.$$

So, Lemma 2.2 implies

$$\begin{aligned}\int_0^\infty\Big(\mu(E_\lambda(f))\Big)^{p/q_0}d\lambda^p&\lesssim\int_0^\infty C^{p,p}_\beta\Big(\Big\{x\in\mathbb{R}^n:\ \mathcal{M}f(x)>\lambda/C\Big\}\Big)d\lambda^p\\&\lesssim\|f\|^p_{\dot{\Lambda}^{p,p}_\beta(\mathbb{R}^n)}.\end{aligned}$$

This proves (i).

Part 2: (iii)$\Longrightarrow$(iv)$\Longrightarrow$(i). Assume that (iii) holds. Define

$$T_p(\mu)=\sup_{f\in C^\infty_0(\mathbb{R}^n)\&\|f\|_{\dot{\Lambda}^{p,p}_\beta(\mathbb{R}^n}>0}\frac{\|u(\cdot,\cdot)\|_{L^{q_0,\infty}_\mu(\mathbb{R}^{1+n}_+)}}{\|f\|_{\dot{\Lambda}^{p,p}_\beta(\mathbb{R}^n)}}.$$

Then we get $T_p(\mu)<\infty$. For any bounded open set $O\subset\mathbb{R}^n$, let $f\in C^\infty_0(\mathbb{R}^n)$ with $f\geq1$ on O. For $(t,x)\in T(O)$, we get $\Phi_t*|f|(x)>\eta_n$. Then

$$\begin{aligned}(\mu(T(O)))^{1/q_0}&\leq\Big(\mu\Big(\Big\{(x,t)\in\mathbb{R}^{1+n}_+:\ \Phi_t*f(x)>\eta_n\Big\}\Big)\Big)^{1/q_0}\\&\leq T_p(\mu)\|f\|_{\dot{\Lambda}^{p,p}_\beta(\mathbb{R}^n)}.\end{aligned}$$

For any $0<t<\mu(T(O))$, we get $t\leq(T_p(\mu)\|f\|_{\dot{\Lambda}^{p,p}_\beta(\mathbb{R}^n)})^{q_0}$. Taking the infimum on $\|f\|_{\dot{\Lambda}^{p,p}_\beta(\mathbb{R}^n)}$ gives $t\leq(T_p(\mu))^{q_0}(C^{p,p}_\beta(O))^{q_0/p}$. Thus

$$\sup_{t>0}\frac{t^{p/q_0}}{c^\beta_{p,p}(\mu;t)}\leq(T_p(\mu))^p.$$

On the other hand, if (iv) is true, then we have

$$\begin{aligned}
&\int_0^\infty \mu(E_\lambda(f)\cap T(B(0,k)))^{p/q_0}d\lambda^p\\
&\leq \int_0^\infty \left\{\frac{(\mu(E_\lambda(f)\cap T(B(0,k))))^{p/q_0}}{c^\beta_{p,p}(\mu;\mu(E_\lambda(f)\cap T(B(0,k))))}\right\}\times C^{p,p}_\beta(R_\lambda(f)\cap B(0,k))d\lambda^p\\
&\leq \left(\sup_{t>0}\frac{t^{p/q_0}}{c^\beta_{p,p}(\mu;t)}\right)\int_0^\infty C^{p,p}_\beta\Big(\Big\{x\in\mathbb{R}^n:\ \theta_n\mathcal{M}f(x)>\lambda\Big\}\cap B(0,k)\Big)d\lambda^p.
\end{aligned}$$

Then Lemma 2.2 implies

$$\int_0^\infty \mu(E_\lambda(f)\cap T(B(0,k)))^{p/q_0}d\lambda^p \lesssim \left(\sup_{t>0}\frac{t^{p/q_0}}{c^\beta_{p,p}(\mu;t)}\right)\|f\|^p_{\dot{\Lambda}^{p,p}_\beta(\mathbb{R}^n)}.$$

By letting $k\to\infty$, we can derive (i).

Part 3: (v)$\Longleftrightarrow$(vi). We assume that (v) is true. Then let $O=B(x,r)$. (v) implies

$$(\mu(T(B(x,r))))^{p/q_0}\lesssim C^{p,p}_\beta(B(x,r)).$$

Since $C^{p,p}_\beta(B(x,r))\approx H^{(\infty)}_{n-p\beta}(B(x,r))\approx r^{n-p\beta}$, we can get

$$\begin{aligned}
\sup_{(r,x)\in\mathbb{R}^{1+n}_+}\frac{(\mu(T(B(x,r))))^{p/q_0}}{r^{n-p\beta}} &\approx \sup_{(r,x)\in\mathbb{R}^{1+n}_+}\frac{(\mu(T(B(x,r))))^{p/q_0}}{C^{p,p}_\beta(B(x,r))}\\
&\lesssim \sup\left\{\frac{(\mu(T(O)))^{p/q_0}}{C^{p,p}_\beta(O)}:\ \text{open set } O\subset\mathbb{R}^n\right\}<\infty.
\end{aligned}$$

Thus (vi) is true.

Conversely, if (vi) holds, then we have

$$S_{q_0,n,\beta,p}(\mu):=\sup_{(r,x)\in\mathbb{R}^{1+n}_+}\frac{\mu\left(T(B(x,r))\right)}{r^{q_0(n-p\beta)/p}}<\infty.$$

For an given open set $O\subset\mathbb{R}^n$, let $\{B(x_j,r_j)\}_{j\in\mathbb{Z}_+}$ be a sequence of covering balls of O, i.e., $O\subseteq\cup_j B(x_j,r_j)$.

We only need to verify that (vi) holds for bounded sets. In fact, if (v) holds for bounded sets, for any open set $O\subset\mathbb{R}^n$, let $\Omega_0=B(0,1)$ and $\Omega_k=B(0,2^k)\setminus B(0,2^{k-1})$, $k\geq 1$. Then $\mathbb{R}^n=\cup_{k=0}^\infty\Omega_k$ and

$$O=O\cap\Big(\cup_{k=0}^\infty\Omega_k\Big)=\cup_{k=1}^\infty\Big(O\cap\Omega_k\Big).$$

So, we can derive

$$\begin{aligned}
(\mu(T(O)))^{p/q_0} &= \left(\mu(T(\cup_{k=1}^{\infty}(O\cap E_k)))\right)^{p/q_0}\\
&\leq \left(\mu(\cup_{k=1}^{\infty}T((O\cap\Omega_k)))\right)^{p/q_0}\\
&\leq \left(\sum_{k=0}^{\infty}\mu(T(O\cap\Omega_k))\right)^{p/q_0}\\
&\lesssim \left(\sum_{k=0}^{\infty}C_{\beta}^{p,p}(O\cap\Omega_k)^{q_0/p}\right)^{p/q_0}\\
&\lesssim \sum_{k=0}^{\infty}C_{\beta}^{p,p}(O\cap\Omega_k)\\
&\lesssim C_{\beta}^{p,p}(\cup_{k=0}^{\infty}(O\cap\Omega_k))\\
&\lesssim C_{\beta}^{p,p}(O).
\end{aligned}$$

In the following, we assume that O is a bounded open subset of $\mathbb{R}^n$. Then, there exists a sequence of dyadic cubes $\{I_j^{(1)}\}$ in $\mathbb{R}^n$ such that $O\subseteq\cup_j I_j^{(1)}$ and $|B(x_j,r_j)|\approx|I_j^{(1)}|$. Applying Lemma 2.3 with $d=n-p\beta$, we can find a sequence of dyadic cubes $\{I_j^{(2)}\}$ and $\{I_j^{(3)}\}$ its $5\sqrt{n}$ expansion such that

$$\cup_j I_j^{(2)}=\cup_j I_j^{(1)};\quad \sum_j\left|I_j^{(2)}\right|^{1-p\beta/n}\leq\sum_j\left|I_j^{(1)}\right|^{1-p\beta/n};\quad T(O)\subseteq\cup_j T(I_j^{(3)}).$$

The fact $|B(x_j,r_j)|\approx|I_j^{(1)}|$ implies $r_j\approx\ell(I_j^{(1)})\approx\ell(I_j^{(3)})$. Denote by x_{I_j} the center of $I_j^{(1)}$. Then we can get $I_j^{(3)}\subseteq B(x_{I_j},\ell(I_j^{(3)}))$. We derive

$$\begin{aligned}
\mu(T(O)) &\leq \sum_j\mu(T(I_j^{(3)}))\\
&\leq \sum_j\mu\left(T(B(x_{I_j},\ell(I_j^{(3)})))\right)\\
&\leq \sum_j\left(\ell(I_j^{(3)})\right)^{q_0(p\beta-n)/p}\mu\left(T(B(x_{I_j},\ell(I_j^{(3)})))\right)\left(\ell(I_j^{(3)})\right)^{q_0(n-p\beta)/p}\\
&\lesssim S_{q_0,n,\beta,p}(\mu)\sum_j\left(\ell\left(I_j^{(3)}\right)\right)^{q_0(n-p\beta)/p}\\
&\lesssim S_{q_0,n,\beta,p}(\mu)\sum_j\left|I_j^{(3)}\right|^{q_0(1-p\beta/n)/p}.
\end{aligned}\tag{2.1}$$

On the other hand, we have

$$\sum_j\left|I_j^{(3)}\right|^{q_0(1-p\beta/n)/p}\lesssim\Big(\sum_j\left|I_j^{(1)}\right|^{1-p\beta/n}\Big)^{q_0/p}\lesssim\Big(\sum_j\left|B(x_j,r_j)\right|^{1-p\beta/n}\Big)^{q_0/p},$$

which implies

$$\sum_j\left|I_j^{(3)}\right|^{q_0(1-p\beta/n)/p}\lesssim\left(H_{n-p\beta}^{(\infty)}(O)\right)^{q_0/p}.\tag{2.2}$$

Then (2.1) and (2.2) imply

$$\mu(T(O))\lesssim S_{q_0,n,\beta,p}(\mu)\left(H_{n-\beta p}^{(\infty)}(O)\right)^{q_0/p}.$$

Finally, the fact $C^{p,p}_{\beta}(\cdot) \approx H^{(\infty)}_{n-\beta p}(\cdot)$ (see [2, Theorems 3.6 and 3.8] and [2, page 32]) implies that (v) holds. □

2.3. **Proof of Proposition 1.4.**

Proof. The proof of (1.3)$\Longrightarrow$ (1.4) can be done in a similar way as (iii)$\Longrightarrow$ (v) of Theorem 1.3. Assume that (1.4) holds. According to (i) of Lemma 2.1,

$$E_{\lambda}(f) \subset T(R_{\lambda}), \quad \mu(E_{\lambda}(f)) \le \mu(T(R_{\lambda}(f))).$$

Then, (1.4) gives

$$\mu(E_{\lambda}(f)) \le \mu(T(R_{\lambda}(f))) \lesssim (C^{p,q}_{\beta}(R_{\lambda}(f)))^{(p\wedge q)/p},$$

which yields

$$\begin{aligned}\int_0^{\infty} (\mu(E_{\lambda}(f)))^{(p\vee q)/(p\wedge q)} d\lambda^{p\vee q} &\le \int_0^{\infty} (\mu(T(R_{\lambda}(f))))^{(p\vee q)/(p\wedge q)} d\lambda^{p\vee q}\\ &\lesssim \int_0^{\infty} (C^{p,q}_{\beta}(R_{\lambda}(f)))^{1\vee(q/p)} d\lambda^{p\vee q}.\end{aligned}$$

Then (iii) of Lemma 2.1 implies $R_{\lambda} \subset \left\{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda/C\right\}$ with a constant C. So, it reaches to

$$C^{p,q}_{\beta}(R_{\lambda}) \le C^{p,q}_{\beta}\Big(\left\{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda/C\right\}\Big).$$

Thus, Lemma 2.2 implies

$$\begin{aligned}\int_0^{\infty} (\mu(E_{\lambda}(f)))^{(p\vee q)/(p\wedge q)} d\lambda^{p\vee q} &\lesssim \int_0^{\infty} C^{p,q}_{\beta}(\{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda/C\})^{1\vee(q/p)} d\lambda^{p\vee q}\\ &\lesssim \|f\|^{p\vee q}_{\dot{\Lambda}^{p,q}_{\beta}(\mathbb{R}^n)}\end{aligned}$$

which reaches (1.3). □

2.4. **Proof of Proposition 1.5.** It follows from Theorem 1.3 that (i) is equivalent to

$$(\mu(T(O)))^{1/q} \lesssim C^{1,1}_{\beta}(O) \tag{2.3}$$

for every bounded open set $O \subseteq \mathbb{R}^n$. Thus it suffices to show that (2.3) $\Longrightarrow$ (ii) $\Longrightarrow$ (i).

Firstly, we show that (2.3) $\Longrightarrow$ (ii). Note that $C^{1,1}_{\beta}(\cdot) \approx H^{(\infty)}_{n-\beta}(\cdot)$. On the other hand, Ponce-Spector in [26] proved that

$$H^{(\infty)}_{n-\beta}(O) \lesssim P_{\beta}(O)$$

holds for every bounded open set $O \subseteq \mathbb{R}^n$. Thus, (ii) follows from

$$(\mu(T(O)))^{1//q} \lesssim C^{1,1}_{\beta}(O) \lesssim H^{(\infty)}_{n-\beta}(O) \lesssim P_{\beta}(O).$$

For (ii) $\Longrightarrow$ (i), denote

$$S_q(\mu) := \sup \frac{(\mu(T(O)))^{1/q}}{P_{\beta}(O)} < \infty,$$

where the supermum is taken over all bounded open sets $O \subseteq \mathbb{R}^n$. For any $f \in C^{\infty}_0(\mathbb{R}^n)$ and any positive integer k, denote

$$\Omega = \{x \in \mathbb{R}^n : \theta_n \mathcal{M}(f)(x) > s\} \cap B(0,k).$$

It follows from Lemma 2.4 that

$$C^{1,1}_{\beta}(\overline{\Omega}) = 2 \inf_{O \in O^{\infty}(\overline{\Omega})} P_{\beta}(\overline{O}),$$

where $O^\infty(\overline{\Omega})$ is the class of all open sets with C^∞ boundary that contain $\overline{\Omega}$. Thus, we have

$$\begin{aligned}\mu(T(\Omega))^{1/q} &\leq \mu(T(\overline{\Omega}))^{1/q}\\ &\leq \inf_{O\in O^\infty(\overline{\Omega})} \mu(T(O))^{1/q}\\ &\leq S_q(\mu) \inf_{O\in O^\infty(\overline{\Omega})} P_\beta(O)\\ &= \frac{S_q(\mu)}{2} C_\beta^{1,1}(\overline{\Omega})\\ &\lesssim C_\beta^{1,1}(\{x\in\mathbb{R}^n : \theta_\alpha \mathcal{M}(f)(x)\geq s\}).\end{aligned}$$

Then, applying Lemma 2.1, we get, for any $f\in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned}&\int_0^\infty (\mu(E_s(f)\cap T(B(0,k))))^{1/q}\,ds\\ &\leq \int_0^\infty (\mu(T(R_s(f)\cap B(0,k))))^{1/q} ds\\ &\leq \int_0^\infty (\mu(T(\{x\in\mathbb{R}^n : \theta_n\mathcal{M}(f)(x)> s\}\cap B(0,k)))^{1/q} ds\\ &\lesssim \int_0^\infty C_\beta^{1,1}(\{x\in\mathbb{R}^n : \theta_n\mathcal{M}(f)(x)\geq s\}) ds.\end{aligned}$$

Then, Lemma 2.2 implies

$$\int_0^\infty (\mu(E_s(f)\cap T(B(0,k)))^{1/q}\,ds \lesssim \|f\|_{\dot{\Lambda}_\beta^{1,1}(\mathbb{R}^n)},$$

which derives (i) by letting $k\longrightarrow\infty$.

2.5. **Proof of Proposition 1.6.** Assume that (i) holds. Let K be a compact subset of $\mathbb{R}^n$ and f be a nonnegative function satisfying $g*f\geq 1_K$. Then (i) implies

$$\mu(K)^{1/q}\leq \sup_{t>0} t(\mu(\{x\in\mathbb{R}^n : |g*f|\geq t\}))^{1/q}\lesssim \|f\|_{L^1(\mathbb{R}^n)},$$

which implies

$$\mu(K)^{1/q}\lesssim C_{g,1}(K)$$

by the definition of $C_{g,1}(K)$. So, (ii) holds.

Now, assume (ii) is true. For any compact $K\subset\mathbb{R}^n$, choose a nonnegative function $f\in C_0^\infty(\mathbb{R}^n)$ and $\lambda>0$. Define $E_\lambda(f)=\{x\in\mathbb{R}^n : |g*f|>\lambda\}$. Then (ii) implies

$$\mu(E_\lambda(f))\lesssim C_{g,1}(E_\lambda(f))^q\lesssim \|f/\lambda\|^q_{L^1(\mathbb{R}^n)}.$$

So, we have

$$\lambda\mu(\{x\in\mathbb{R}^n : |g*f|>\lambda\})^{1/q}\lesssim \|f\|_{L^1(\mathbb{R}^n)},$$

which implies (i) by taking supremum over $\lambda>0$.

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