Leptin Densities in Amenable Groups

Felix Pogorzelski[1] [ID] · **Christoph Richard[2]** [ID] · **Nicolae Strungaru[3,4]** [ID]

Abstract

Consider a positive Borel measure on a locally compact group. We define a notion of uniform density for such a measure, which is based on a group invariant introduced by Leptin in 1966. We then restrict to unimodular amenable groups and to translation bounded measures. In that case our density notion coincides with the well-known Beurling density from Fourier analysis, also known as Banach density from dynamical systems theory. We use Leptin densities for a geometric proof of the model set density formula, which expresses the density of a uniform regular model set in terms of the volume of its window, and for a proof of uniform mean almost periodicity of such model sets.

Communicated by Hartmut Führ.

✉ Christoph Richard
  christoph.richard@fau.de

  Felix Pogorzelski
  felix.pogorzelski@math.uni-leipzig.de

  Nicolae Strungaru
  strungarun@macewan.ca

1 Institut für Mathematik, Universität Leipzig, Augustusplatz 10, 04109 Leipzig, Germany

2 Department für Mathematik, Friedrich-Alexander-Universität Erlangen-Nürnberg, Cauerstrasse 11, 91058 Erlangen, Germany

3 Department of Mathematical Sciences, MacEwan University, 10700–104 Avenue, Edmonton, AB T5J 4S2, Canada

4 Institute of Mathematics "Simon Stoilow", Bucharest, Romania

1 Introduction

This article addresses asymptotic frequencies of point sets in locally compact groups G that are amenable [50].[1] We introduce a canonical notion of uniform density, which is intimately related to the group invariant $I(G)$ introduced by Horst Leptin in [38], and we thus call it the *Leptin density* of a locally finite point set or more generally of a positive Borel measure on G. Let us restrict to amenable locally compact groups that are unimodular. This setting encompasses locally compact groups that are exponentially bounded [50, Proposition 6.8], such as locally compact abelian groups. Let us further restrict to translation bounded positive Borel measures, as defined below. In that case, a main result of our article asserts that our notion of Leptin density coincides with the classical notion of Banach density, also known as Beurling density or Beurling-Landau density. The latter densities may be evaluated on any so-called strong Følner or van Hove net. We will show that such nets can be obtained from any Følner net by a simple thickening procedure.

Our result might be useful for sampling and interpolation problems on such groups, as necessary conditions for sampling and interpolation are usually formulated in terms of the Beurling density of the underlying point set. Our definition of Leptin density is in fact inspired by work of Gröchenig, Kutyniok and Seip [24] in that direction, compare [52, Lemma 9.3]. In Euclidean space, it was already noted by Landau that the Beurling density can be defined on averaging sets beyond boxes [34, Lemma 4], and Gabardo [21] gave a definition of Beurling density which does not resort to any averaging sequence. Due to its connection to amenability, our observation clarifies why the Beurling density is independent of the chosen averaging sets. Making proper use of Leptin densities might streamline existing proofs, as one does not have to resort to particular averaging sets. The notion of Leptin density might also provide insight as how to extend the notion of Beurling density beyond the group setting, compare [20, 44].

A further focus of this article is on applications to mathematical diffraction theory [4, 28, 54]. In that field, it is common to work with van Hove (strong Følner) sequences, which restricts the analysis to σ-compact amenable groups. However one might drop the assumption of σ-compactness and use van Hove (strong Følner) nets, at least if one does not use dynamical arguments such as the pointwise ergodic theorem. In fact one might avoid nets at all, using Leptin densities. As a prominent example, consider the class of model sets in σ-compact locally compact abelian groups. Such point sets are obtained from a cut-and-project construction, originating in Meyer's ground breaking work during the 70s [41–43, 57, 58]. Further investigation had been advocated by Moody in the 90s [45, 46], motivated by the experimental discovery of quasicrystals. In fact the cut-and-project construction had been re-developed independently in that context, see e.g. the historical discussion in [53, Section 2]. By now the class of model sets constitutes the most important example of pure point diffractive structures, see e.g., the monograph [2] and references therein. Moreover, the last couple of years has seen an emerging theory of aperiodic order beyond the abelian situation, see [11–13] for regular model sets in general locally compact second countable groups

---

[1] We will implicitly assume that such groups always satisfy the Hausdorff property.

and homogeneous spaces, as well as [9, 10, 40] for foundations of a new theory on approximate lattices.

The so-called density formula expresses the density of a regular model set in terms of the volume of its window. A first proof for Euclidean space, which relies on the Poisson summation formula, goes back to Meyer [42, 43], compare [54] for a recent account. In varying degrees of generality, alternative proofs have been given by geometric methods [55], dynamical methods [47], and by methods based on almost periodicity, see e.g., [4, 22]. We refer to [29, Section 3] for a detailed account of the history of the density formula until 2015. For non-abelian regular model sets as defined in [11], a variant of density formula can be deduced using dynamical methods, see Remark 7.3 below. This subsumes results for the non-abelian setting from [29, Remark 3.3] and [30, Section 9.3]. In the present paper we use Leptin densities in order to give a geometric proof of the density formula for uniform regular model sets in amenable groups. The method is via tilings with disjoint copies of translates of a relatively compact fundamental domain arising from a uniform lattice. In Euclidean space, a similar combinatorial proof was given by Schlottmann in [55]. We emphasize at this point that the lattice in the cut-and-project scheme being cocompact is a crucial ingredient in the line of argumentation. In particular, the density formula presented here is not the most general version. We will also use Leptin densities in order to analyse almost periodicity of a regular model set. This complements previous results in the σ-compact abelian case [4, 25, 26, 36, 37].

The article is structured as follows. In the following section, we fix the class of point sets, using a measure-theoretic description. Our setting is that of right-translation bounded measures. Section 3 is devoted to the notion of Leptin density and discusses some of its properties. In Section. 4 we prove that Leptin densities coincide with Beurling densities, when defined without resorting to an averaging net. In Section. 5, we discuss various notions of boundary and how they characterise averaging nets. Moreover, we describe Beurling and Leptin densities via approximation by certain Følner nets. In Section. 6, we compute the Leptin density of a cocompact lattice in a locally compact amenable group via a tiling argument for fundamental domains. The same strategy is used in Section. 7, where we give a geometric proof of the density formula for (regular) model sets in amenable groups. In Section. 8, we discuss uniform versions of almost periodicity with focus on a description avoiding nets. We prove uniform mean almost periodicity of regular model sets using the notion of Leptin density. For the convenience of the reader, Appendix A collects basic facts about nets, Appendix B explains box decomposition arguments that are useful for monotilable groups, instead of our more general approach. Appendix C describes the relation between Leptin densities and the density notion in [24], which has inspired our work.

2 Delone Measures

Instead of counting the number of points in locally finite sets, we use the more general setting of (positive) Borel measures. For some of our results we restrict to translation bounded measures, a notion which encompasses uniformly discrete point sets. See the discussion below.

2.1 Left Translations Versus Right Translations

It is somewhat common in works on non-abelian aperiodic order to define uniform discreteness and relative denseness for point sets via left translations, cf. e.g., [7, 9, 11, 12, 29, 48]. The same remark holds for works on sampling and interpolation, see e.g., [20, 44], where uniform discreteness is called separatedness.

On the other hand, central aspects of classical dynamical systems theory around amenable groups build on the Ornstein–Weiss quasi-tiling theory [49]. The latter is based on left-invariant Følner sequences and almost tessellations by right translates of Følner sets. Here, in the presence of a metric, tiles are isometric when dealing with a right-invariant metric. Moreover, counting quantities defined via approximation by left-asymptotically invariant (left-Følner) sets show desirable uniformity properties with respect to translations from the right. This phenomenon has also been observed in the theory of aperiodic order, see e.g., [6, 48, 51].

We thus have to make a choice whether we work with asymptotic invariance from the left and uniformity of translations from the right, or with asymptotic invariance from the right and uniformity of translations from the left. We decided to go with the former since in classical dynamical systems theory, densities are often defined via left Følner sequences. Moreover the classical theory about Følner sets and amenability, which we use in our article, is traditionally formulated for left-Følner sets, see e.g., [50].

However, we point out that this choice is somewhat arbitrary, and we could likewise have worked with asymptotic invariance from the right and uniformity properties from the left. In fact, our results can be formulated and proven in the other realm as well. To do so, point translations (e.g., in the definitions of Delone measures, Beurling densities and Banach densities) have to be changed from right to left, while thickening by compact sets must be changed from left to right (e.g., in the definitions of Leptin densities, boundaries and Følner sets).

The alternate versions of many statements in the manuscript can be obtained by passing from the measure ν to its inverse measure $\nu^\dagger$ defined by $\nu^\dagger(A) = \nu(A^{-1})$, using unimodularity. This will be indicated throughout the manuscript, see Remark 2.2 (iii), Remark 3.4 (iii) and Remark 5.7. For example, our main result on comparison of densities (cf. Proposition 5.14) shows that the uniform densities of a (right) Delone measure ν, see Definition 2.1, coincide with the corresponding uniform densities of its inverse measure $\nu^\dagger$, which is a left Delone measure, see Remark 2.2 (iii).

2.2 Definition of Delone Measures

Throughout the paper, we denote by G a locally compact topological group. When we speak of a locally compact group G in this work, we will implicitly assume that G also satisfies the Hausdorff property, i.e., any two points in G can be separated by disjoint open neighborhoods. For simplicity we will call a positive Borel measure a measure in the following. Recall that we decided to use right translations in our definitions.

Definition 2.1 (*Delone measure*) Let ν be a measure on a locally compact group G.

(a) We call ν upper translation bounded if there exist $C_u < \infty$ and a compact symmetric unit neighborhood $B_u \subseteq G$ such that $\sup_{x \in G} \nu(B_u x) \leq C_u$.

(b) We call ν lower translation bounded if there exist $C_l > 0$ and a compact symmetric unit neighborhood $B_l \subseteq G$ such that $\inf_{x \in G} \nu(B_l x) \geq C_l$.

(c) We call ν a Delone measure if ν is both upper and lower translation bounded.

Remark 2.2 (i) In the above definition, by convenience we restricted to compact symmetric unit neighborhoods. Using covering arguments, it is readily seen that ν is upper translation bounded iff $\sup_{x \in G} \nu(Kx) < \infty$ for every compact $K \subseteq G$. In particular, upper translation boundedness implies local finiteness, i.e., $\nu(A)$ is finite for every compact A. For a complex measure, upper and lower translation boundedness can be defined via its (positive) variation measure.

(ii) Upper translation boundedness is traditionally called translation boundedness [1]. The name Delone measure is motivated from weak Delone sets [33], i.e., from point sets that are weakly uniformly discrete and relatively dense. For point sets Λ, weak uniform discreteness is equivalent to upper translation boundedness of the associated point measure defined via $\nu(A) = \text{card}(\Lambda \cap A)$. Relative denseness is equivalent to lower translation boundedness of the associated point measure. For point sets, uniform discreteness is also called separatedness, and weak uniform discreteness is also called relative separatedness, see e.g., [20, 44].

(iii) As discussed in Section. 2.1, one might also consider versions of upper/lower translation boundedness based on left translation. Then ν is left upper/lower translation bounded if and only if $\nu^\dagger$ is (right) upper/lower translation bounded, where $\nu^\dagger(A) = \nu(A^{-1})$. In our applications in Section. 6, 7 and 8 we consider Delone measures built from point sets $\Lambda \subseteq G$, i.e., $\nu(A) = \text{card}(\Lambda \cap A)$, such that ν is both left and right upper/lower translation bounded.

In this article, we will analyse various notions of density of a measure. Upper translation boundedness will result in finite upper density. Lower translation boundedness will result in positive lower density. Examples of Delone measures are any (left or right) Haar measure on a unimodular locally compact group G and, for a uniform lattice L in G, the Haar measure on L viewed as a measure on G.

2.3 Standard Estimates on Unimodular Groups

For later use, we provide standard estimates for Delone measures on unimodular groups. These are based on the following observation from [49]. In the sequel, we denote by $\mathcal{K} = \mathcal{K}(G)$ the collection of nonempty compact subsets of G. We denote by AB the Minkowski product of $A, B \subseteq G$, i.e., we have $AB = \{ab : a \in A, b \in B\}$.

Lemma 2.3 *Let G be a unimodular locally compact group with Haar measure m. Consider any compact $A \in \mathcal{K}$ and any unit neighborhood B. Take $\{a_1, \ldots, a_n\} \subseteq A$ maximal such that $Ba_i \cap Ba_j = \varnothing$ for $i \neq j$. Then*

$$\bigcup_{i=1}^n Ba_i \subseteq BA , \qquad A \subseteq \bigcup_{i=1}^n B^{-1} Ba_i .$$

where the union on the lhs is disjoint. In particular we have

$$n \cdot m(B) \leq m(BA) , \qquad m(A) \leq n \cdot m(B^{-1}B) .$$

Proof The first inclusion is obvious. For the second inclusion, assume $a \in A$ satisfies $a \notin \bigcup_{i=1}^{n} B^{-1}Ba_i$. Then for arbitrary fixed $i \in \{1, \dots, n\}$ we have $a \notin B^{-1}Ba_i$. Hence $ba \neq b'a_i$ for all $b, b' \in B$, which implies $Ba \cap Ba_i = \varnothing$. As $i \in \{1, \dots, n\}$ was arbitrary, this contradicts maximality. The inequalities for the Haar measure follow from unimodularity.   □

For an upper translation bounded measure we have the following upper bound.

Lemma 2.4 *Let G be a unimodular locally compact group with Haar measure m. Let ν be an upper translation bounded measure on G. Take a compact symmetric unit neighborhood $B_u \subseteq G$ and a finite number C_u such that $\nu(B_u^2 x) \leq C_u$ for all $x \in G$. We then have for any $A \in \mathcal{K}$ the estimate*

$$\nu(A) \leq \frac{C_u}{m(B_u)} \cdot m(B_u A) .$$

Proof Fix any $A \in \mathcal{K}$ and consider the setting of Lemma 2.3 with $B = B_u$. We then have

$$\nu(A) \leq \nu \left(\bigcup_{i=1}^{n} B_u^2 a_i \right) \leq n \cdot \max_{1 \leq i \leq n} \nu(B_u^2 a_i) \leq \frac{C_u}{m(B_u)} \cdot m(B_u A) .$$

□

For a measure that is lower translation bounded, we have the following lower bound.

Lemma 2.5 *Let G be a unimodular locally compact group with Haar measure m. Let ν be a lower translation bounded measure on G. Take a compact symmetric unit neighborhood $B_l \subset G$ and a positive number C_l such that $\nu(B_l x) \geq C_l$ for all $x \in G$. We then have for any compact set $A \in \mathcal{K}$ the estimate*

$$\nu(B_l A) \geq \frac{C_l}{m(B_l^2)} \cdot m(A) .$$

Proof Fix any $A \in \mathcal{K}$ and consider the setting of Lemma 2.3 with $B = B_l$. We then have

$$\nu(B_l A) \geq \nu \left(\bigcup_{i=1}^{n} B_l a_i \right) = \sum_{i=1}^{n} \nu(B_l a_i) \geq n \cdot C_l \geq \frac{C_l}{m(B_l^2)} \cdot m(A) .$$

□

3 Leptin Density of a Measure

We will define a certain uniform density that is based on the group invariant $I(G)$ introduced by Leptin [38], see also [50, Cor. 4.14]. Our notion is inspired by the work of Gröchenig, Kutyniok and Seip in [24], see Appendix C for a discussion of the connection.

3.1 Definition

Let G be a locally compact group. Take any left Haar measure m on G. Let $\mathcal{K}$ denote the collection of nonempty compact subsets of G, and let $\mathcal{K}_p \subseteq \mathcal{K}$ denote the collection of compact subsets of G of positive left Haar measure. Consider

$$ I(G) = \sup_{K \in \mathcal{K}} \inf_{A \in \mathcal{K}_p} \frac{m(KA)}{m(A)} \, . $$

Note that in the definition of $I(G)$ we may restrict to K being a compact symmetric unit neighborhood without loss of generality. The quantity $I(G)$ is a group invariant as it does not depend on the choice of the left Haar measure. It exhibits the following dichotomy.

Lemma 3.1 *Let G be a locally compact group with left Haar measure m. Then either $I(G) = 1$ or $I(G) = \infty$.*

Proof Note that $I(G) \geq 1$ by definition. For any compact unit neighborhood $K \in \mathcal{K}_p$ consider

$$ C(K) = \inf_{A \in \mathcal{K}_p} \frac{m(KA)}{m(A)} $$

and note that $C(K^2) \geq C(K)^2$. Assume now that $I(G) > 1$. Then there exist a compact unit neighborhood $K_0 \in \mathcal{K}_p$ and $\varepsilon_0 > 0$ such that $C(K_0) > 1 + \varepsilon_0 > 1$. This implies $I(G) \geq C(K_0^n) \geq C(K_0)^n \to \infty$. Thus $I(G) = \infty$. □

In the following, we use the convention $1/I(G) = I(G)^{-1} = 0$ for $I(G) = \infty$.

Recall that a locally compact group G is *amenable* if it admits a left-invariant mean on $L^\infty(G)$. The following important result is well known, see e.g., [17, 23] and the monograph [50].

Theorem 3.2 [50, Cor. 4.14] *Let G be a locally compact group. Then G is amenable if and only if $I(G) = 1$.* □

We now define the notion of Leptin density of a measure. Although our definition is given in a general setting, the notion of Leptin density seems most useful in amenable unimodular locally compact groups. Moreover it will be infinite on measures that are not upper translation bounded. This will be discussed below.

Definition 3.3 (*Leptin densities*) Let G be a locally compact group. Let m be a left Haar measure on G. For a measure ν on G consider

$$L_\nu^- = \sup_{K \in \mathcal{K}} \inf_{A \in \mathcal{K}_p} \frac{\nu(KA)}{m(A)}, \qquad L_\nu^+ = \inf_{K \in \mathcal{K}} \sup_{A \in \mathcal{K}_p} \frac{\nu(A)}{m(KA)}.$$

We call L_ν^- and L_ν^+ the lower Leptin density respectively the upper Leptin density of the measure ν. If $L_\nu^- = L_\nu^+ = L_\nu$, we call L_ν the Leptin density of ν.

> **Remark 3.4** (i) In the above definition we may assume $e \in K$ without loss of generality, due to left invariance of the Haar measure. The attribution to Leptin is motivated by $L_m^- = I(G)$ and $L_m^+ = I(G)^{-1}$. Note that the left Haar measure m has Leptin density 1 if and only if G is amenable, as a consequence of Theorem 3.2.
> (ii) If $L_\nu^-, L_\nu^+ \in (0, \infty)$, then the lower and upper Leptin densities provide uniform estimates which relate the measure ν to the Haar measure m. Namely, for every $\varepsilon > 0$ there exists $K \in \mathcal{K}$ such that for all $A \in \mathcal{K}_p$ we have $\nu(KA) \geq (L_\nu^- - \varepsilon) \cdot m(A)$. Also, for every $\varepsilon > 0$ there exists $K \in \mathcal{K}$ such that for all $A \in \mathcal{K}_p$ we have $\nu(A) \leq (L_\nu^+ + \varepsilon) \cdot m(KA)$.
> (iii) As discussed in Section 2.1, one might alternatively use thickening from the right instead of thickening from the left. Let us denote the corresponding densities by R_ν^- and R_ν^+. If G is unimodular, then we have $R_\nu^- = L_{\nu^\dagger}^-$ and $R_\nu^+ = L_{\nu^\dagger}^+$. In particular if ν satisfies $\nu = \nu^\dagger$, compare Remark 2.2 (iii), then both notions coincide.

3.2 Some Properties of Leptin Densities

Let us analyse how Leptin densities behave with respect to translation. For a given measure ν on G we consider its left translation $\nu \mapsto \delta_t * \nu$ and its right translation $\nu \mapsto \nu * \delta_t$ for $t \in G$, where $(\delta_t * \nu)(A) = \nu(t^{-1}A)$ and $(\nu * \delta_t)(A) = \nu(At^{-1})$.

Lemma 3.5 *Let ν be a measure on a locally compact group G. Then the Leptin densities of ν are invariant under left translation, i.e., we have for all $t \in G$ that*

$$L_{\delta_t * \nu}^- = L_\nu^-, \qquad L_{\delta_t * \nu}^+ = L_\nu^+.$$

If G is assumed to be unimodular, then the Leptin densities of ν are also invariant under right translation, i.e., we have for all $t \in G$ that

$$L_{\nu * \delta_t}^- = L_\nu^-, \qquad L_{\nu * \delta_t}^+ = L_\nu^+. \qquad \square$$

Proof This is immediate from the definition of Leptin density. For illustration, observe

$$L_{\nu * \delta_t}^- = \sup_{K \in \mathcal{K}} \inf_{A \in \mathcal{K}_p} \frac{(\nu * \delta_t)(KA)}{m(A)} = \sup_{K \in \mathcal{K}} \inf_{A \in \mathcal{K}_p} \frac{\nu(KAt^{-1})}{m(At^{-1})}$$

$$= \sup_{K \in \mathcal{K}} \inf_{A \in \mathcal{K}_p} \frac{\nu(KA)}{m(A)} = L_\nu^-,$$

where we assumed G to be unimodular for the second equation. $\square$

Leptin densities reflect amenability of the underlying group. Whereas they are degenerate for Delone measures in non-amenable groups, they are well behaved in the amenable case.

Lemma 3.6 *Let v be a measure on a locally compact group G. If G is amenable, then $L_v^- \le L_v^+$.*

Proof Assume $L_v^+ < L_v^-$ and pick $a < b$ such that $L_v^+ < a < b < L_v^-$. Since $b < L_v^-$, there exists $K_b \in \mathcal{K}$ such that for all $A \in \mathcal{K}_p$ we have $b \cdot m(A) \le v(K_b A)$. Since $a > L_v^+$, there exists $K_a \in \mathcal{K}$ such that for all $A \in \mathcal{K}_p$ we have $a \cdot m(K_a A) \ge v(A)$. Now fix arbitrary $A \in \mathcal{K}_p$. We thus have $b \cdot m(A) \le v(K_b A) \le a \cdot m(K_a K_b A)$. Therefore, for all $A \in \mathcal{K}_p$ we have

$$\frac{m(K_a K_b A)}{m(A)} \ge \frac{b}{a} > 1 .$$

Using the notation of the proof of Lemma 3.1, this implies $C(K_a K_b) > 1$. Hence $I(G) = \infty$, and G is not amenable by Theorem 3.2. $\qquad\square$

For a converse of the above statement in the unimodular case, see Lemma 3.9.

3.3 Leptin Densities of Delone Measures in Unimodular Groups

In this subsection, we restrict to unimodular locally compact groups. The standard estimates of Section. 2.3 have the following consequence.

Lemma 3.7 (Standard estimates) *Let G be a unimodular locally compact group. Let v be any Delone measure on G. Take compact symmetric unit neighborhoods $B_l, B_u \subseteq G$ and positive finite constants C_l, C_u such that*

$$C_l \le v(B_l x) , \qquad v(B_u^2 x) \le C_u$$

for all $x \in G$. Then the following estimates hold.

$$\frac{C_l}{m(B_l^2)} \cdot I(G) \le L_v^- \le \frac{C_u}{m(B_u)} \cdot I(G) ,$$

$$\frac{C_l}{m(B_l^2)} \cdot I(G)^{-1} \le L_v^+ \le \frac{C_u}{m(B_u)} \cdot I(G)^{-1} .$$

Here the upper estimates rely on upper translation boundedness, and the lower estimates rely on lower translation boundedness.

Proof We consider L_v^- first and use the estimate in Lemma 2.5 to obtain

$$L_v^- \ge \sup_{K \in \mathcal{K}} \inf_{A \in \mathcal{K}_p} \frac{v(B_l K A)}{m(A)} \ge \frac{C_l}{m(B_l^2)} \cdot \sup_{K \in \mathcal{K}} \inf_{A \in \mathcal{K}_p} \frac{m(K A)}{m(A)} = \frac{C_l}{m(B_l^2)} \cdot I(G) .$$

For an upper estimate, we use the estimate in Lemma 2.4 to obtain

$$L_\nu^- = \sup_{K \in \mathcal{K}} \inf_{A \in \mathcal{K}_p} \frac{\nu(KA)}{m(A)} \le \frac{C_u}{m(B_u)} \cdot \sup_{K \in \mathcal{K}} \inf_{A \in \mathcal{K}_p} \frac{m(B_u K A)}{m(A)} \le \frac{C_u}{m(B_u)} \cdot I(G) \,.$$

The proofs of the assertions on L_ν^+ are analogous and use Lemma 2.4.   □

The following lemma relates the notions of translation boundedness and Leptin density. This also motivates the notion of Delone measure. Part (a) of the lemma is analogous to [25, Cor. 2].

Lemma 3.8 *Let G be a unimodular locally compact group, and let ν be any measure on G. Then the following hold.*

  (a) *ν is upper translation bounded if and only if L_ν^+ is finite.*
  (b) *ν is lower translation bounded if and only if L_ν^- is positive.*
  (c) *ν is a Delone measure if and only if L_ν^- is positive and L_ν^+ is finite.*

Proof (c) follows from (a) and (b). The claims (a $\Rightarrow$) and (b $\Rightarrow$) follow from Lemma 3.7. In order to prove the remaining claims, fix a Haar measure m on G.

(a $\Leftarrow$) Assume $L_\nu^+ < \infty$. Then there exist a compact set K and a finite constant C such that $\nu(A) \le Cm(KA)$ for all compact sets A of positive measure. Choosing $A = Bx$ for some compact symmetric unit neighbourhood B, the claim follows from unimodularity of G.
(b $\Leftarrow$) Assume that $L_\nu^- > 0$. Then there exist a compact set K and a positive constant C such that $\nu(KA) \ge Cm(A)$ for all compact sets A of positive measure. Choosing $A = Bx$ for some compact symmetric unit neighbourhood B, the claim follows from unimodularity of G.   □

Due to the above standard estimates, Leptin densities of Delone measures characterise amenability of unimodular locally compact groups.

Lemma 3.9 *Let G be a unimodular locally compact group. Let ν be any Delone measure on G. Then the following assertions are equivalent.*

  (i) *G is amenable.*
  (ii) *$L_\nu^- \le L_\nu^+$.*
  (iii) *L_ν^- is finite.*
  (iv) *L_ν^+ is positive.*

Proof The implication (i) $\Rightarrow$ (ii) is Lemma 3.6. For the implication (ii) $\Rightarrow$ (iii) assume $L_\nu^- = \infty$. Then (ii) implies $L_\nu^+ = \infty$. Hence by Lemma 3.7 we have $I(G) = 0$ and thus $L_\nu^- = 0$, which is a contradiction. For the implication (iii) $\Rightarrow$ (iv), note that (iii) excludes the case $I(G) = \infty$ by Lemma 3.7. In view of Lemma 3.1, we have $I(G) = 1$. But then $L_\nu^+ > 0$ by Lemma 3.7. For the implication (iv) $\Rightarrow$ (i), note that (iv) excludes the case $I(G) = \infty$ by Lemma 3.7, hence $I(G) = 1$ by Lemma 3.1. Thus, G is amenable by Theorem 3.2.   □

4 Leptin Densities and Beurling Densities

4.1 Leptin Densities and Beurling Densities

Often uniform densities are used, which are invariant under shifting the averaging sequence or the measure. These densities are sometimes called Beurling densities or Beurling-Landau densities, see [8, p. 346] and [34, p. 47] for $G = \mathbb{R}^d$. In ergodic theory they are known by the name Banach densities. In fact these densities can be defined without resorting to an averaging sequence. For discrete groups, this is studied in [16]. The case $G = \mathbb{R}^d$ is considered in [21] by a linear functional approach.

Definition 4.1 (*Beurling densities*) Assume that G is a unimodular locally compact group with left Haar measure m. Let ν be a measure on G and consider

$$B_\nu^- = \sup_{A \in \mathcal{K}_p} \inf_{s \in G} \frac{\nu(As)}{m(A)}, \qquad B_\nu^+ = \inf_{A \in \mathcal{K}_p} \sup_{s \in G} \frac{\nu(As)}{m(A)}.$$

We call B_ν^- and B_ν^+ the lower Beurling density respectively the upper Beurling density of the measure ν. If $B_\nu^- = B_\nu^+ = B_\nu$, we call B_ν the Beurling density of ν.

In the following, we will suppress the subscript ν if the reference to the measure is clear.

Remark 4.2 Note that $B_\nu^+ < \infty$ if ν is upper translation bounded, as a consequence of Lemma 2.4. The attribution to Beurling is motivated by Proposition 5.14 below, which states that, for unimodular groups, the above densities coincide with a definition based on averaging.

Let us discuss the relation between Beurling densities and Leptin densities. Beyond the amenable situation, these notions differ. Indeed, for any left Haar measure m on a unimodular non-amenable group G, we have $B_m^+ = B_m^- = 1$, but $L_m^- = \infty$ and $L_m^+ = 0$. It follows from the next theorem that, for a locally finite measure ν on a unimodular amenable group, the apriori different notions of Beurling density and Leptin density coincide. Precisely, the lower Beurling density of ν coincides with its lower Leptin density and the upper Beurling density of ν coincides with its upper Leptin density.

Theorem 4.3 *Assume that G is a unimodular locally compact group. Consider any locally finite measure ν on G. For the densities associated to ν we then have*

$$B_\nu^- \leq L_\nu^- \leq B_\nu^- \cdot I(G), \qquad I(G)^{-1} \cdot B_\nu^+ \leq L_\nu^+ \leq B_\nu^+.$$

In particular if G is additionally amenable, then we have $B_\nu^- = L_\nu^- \leq L_\nu^+ = B_\nu^+$.

4.2 Proof of Theorem 4.3

We will treat the four inequalities separately. The inequalities $L^- \leq B^- \cdot I(G)$ and $B^+ \leq L^+ \cdot I(G)$ follow from elementary estimates.

Proof of $L^- \leq B^- \cdot I(G)$ Consider any $A \in \mathcal{K}_p$, any nonempty $K \in \mathcal{K}$ and any $s \in G$. We have

$$\frac{\nu(KAs)}{m(A)} = \frac{m(KA)}{m(A)} \cdot \frac{\nu(KAs)}{m(KA)} .$$

Taking the infimum over $A \in \mathcal{K}$ on the lhs and using unimodularity of G, we see that the lhs is independent of $s \in G$. Hence we may take the infimum over s on the rhs and arrive at

$$\inf_{A \in \mathcal{K}_p} \frac{\nu(KA)}{m(A)} \leq \frac{m(KA)}{m(A)} \cdot \inf_{s \in G} \frac{\nu(KAs)}{m(KA)} \leq \frac{m(KA)}{m(A)} \cdot B^- .$$

As the lhs is independent of $A \in \mathcal{K}_p$, we may take the infimum over A on the rhs and the supremum over K on both sides to conclude

$$L^- = \sup_{K \in \mathcal{K}} \inf_{A \in \mathcal{K}_p} \frac{\nu(KA)}{m(A)} \leq \sup_{K \in \mathcal{K}} \inf_{A \in \mathcal{K}_p} \frac{m(KA)}{m(A)} \cdot B^- \leq I(G) \cdot B^- .$$

□

Proof of $I(G)^{-1} \cdot B^+ \leq L^+$ Consider any $A \in \mathcal{K}_p$, nonempty $K \in \mathcal{K}$ and any $s \in G$ and note that by unimodularity of G we have

$$\frac{m(A)}{m(KA)} \cdot \frac{\nu(As)}{m(A)} = \frac{\nu(As)}{m(KA)} \leq \sup_{A' \in \mathcal{K}_p} \frac{\nu(A')}{m(KA')} ,$$

where the rhs is independent of $s \in G$. Hence we may take the supremum over s on the lhs and arrive at

$$\frac{m(A)}{m(KA)} \cdot B^+ \leq \frac{m(A)}{m(KA)} \cdot \sup_{s \in G} \frac{\nu(As)}{m(A)} \leq \sup_{A' \in \mathcal{K}_p} \frac{\nu(A')}{m(KA')} .$$

As the rhs is independent of $A \in \mathcal{K}_p$, we may take the supremum over A on the lhs and the infimum over K on both sides to conclude $I(G)^{-1} \cdot B^+ \leq L^+$.   □

The proofs of $B^- \leq L^-$ and of $L^+ \leq B^+$ are more delicate. We will follow ideas from [16, Lemma 2.9]. Note the following two elementary results.

Lemma 4.4 *Let G be a unimodular locally compact group. Fix a left Haar measure m on G and let ν be a locally finite measure on G. Then we have for any compact sets $A, B \subseteq G$ that*

$$\int_A \nu(aB) \, dm(a) = \int_B \nu(Ab) \, dm(b) . \tag{4.1}$$

Proof We evaluate the lhs and use Tonelli's theorem, which is applicable as ν is assumed to be locally finite, and A and B are compact sets. We obtain

$$
\int_A \nu(aB)\,dm(a) = \int_A \int_{aB} d\nu(x)\,dm(a) = \int_G \int_{A \cap xB^{-1}} dm(a)\,d\nu(x)
$$
$$
= \int_G m(A \cap xB^{-1})\,d\nu(x)\,,
$$

where we used $x \in aB$ iff $a \in xB^{-1}$. Evaluating the rhs and using Tonelli's theorem yield

$$
\int_B \nu(Ab)\,dm(b) = \int_B \int_{Ab} d\nu(x)\,dm(b) = \int_G \int_{B \cap A^{-1}x} dm(b)\,d\nu(x)
$$
$$
= \int_G m(B \cap A^{-1}x)\,d\nu(x)\,,
$$

where we used $x \in Ab$ iff $b \in A^{-1}x$. Now equality follows from unimodularity and left invariance of the Haar measure. Indeed we have

$$
m(B \cap A^{-1}x) = m(B^{-1} \cap x^{-1}A) = m(xB^{-1} \cap A)\,.
$$

$\square$

Lemma 4.5 *Let G be a unimodular locally compact group. Fix a left Haar measure m on G and let ν be a locally finite measure on G. Let $A, B \subseteq G$ be compact sets. Assume that there is a positive finite constant $C = C(A, B)$ such that for all $b \in B$ we have*

$$
\nu(Ab) \geq C \cdot m(A)\,.
$$

Then there exists $a \in A$ such that

$$
\nu(aB) \geq C \cdot m(B)\,.
$$

Remark 4.6 In the two inequalities of the above lemma, replace "$\geq$" by "$\leq$". Then the resulting statement remains true, as the following proof shows.

Proof Let A, B be compact sets and assume that the assertion of the lemma is wrong. Then we have for all $a \in A$ that $\nu(aB) < C \cdot m(B)$. Integrate this inequality to obtain

$$
\int_A \nu(aB)\,dm(a) < C \cdot m(A) \cdot m(B)\,. \tag{4.2}
$$

On the other hand, the inequality in the assumption of the lemma can be integrated to yield

$$
\int_B \nu(Ab)\,dm(b) \geq C \cdot m(A) \cdot m(B)\,. \tag{4.3}
$$

But the left hand sides of the above two inequalities are equal due to Lemma 4.4. This is contradictory.   □

Proof of $B^- \leq L^-$ Recall from Definition 4.1 that

$$B^- = \sup_{A \in \mathcal{K}_p} \inf_{s \in G} \frac{\nu(As)}{m(A)} \,, \qquad L^- = \sup_{K \in \mathcal{K}} \inf_{A \in \mathcal{K}_p} \frac{\nu(KA)}{m(A)} \,.$$

Fix arbitrary $\varepsilon > 0$ and take $A \in \mathcal{K}_p$ such that for all $s \in G$ we have

$$B^- - \varepsilon \leq \frac{\nu(As)}{m(A)} \,.$$

Let now $S \in \mathcal{K}_p$ be arbitrary. Recall that $m(A) \cdot (B^- - \varepsilon) \leq \nu(As)$ for all $s \in S$. Hence by Lemma 4.5 there exists $a \in A$ such that $m(S) \cdot (B^- - \varepsilon) \leq \nu(aS) \leq \nu(AS)$. Hence we have

$$B^- - \varepsilon \leq \frac{\nu(AS)}{m(S)} \,.$$

As $S \in \mathcal{K}_p$ and $\varepsilon > 0$ were arbitrary, the claim follows.   □

Proof of $L^+ \leq B^+$ Recall from Definition 4.1 that

$$L^+ = \inf_{K \in \mathcal{K}} \sup_{A \in \mathcal{K}_p} \frac{\nu(A)}{m(KA)} \,, \qquad B^+ = \inf_{A \in \mathcal{K}_p} \sup_{s \in G} \frac{\nu(As)}{m(A)} \,.$$

Assume without loss of generality that $B^+ < \infty$. Fix arbitrary $\varepsilon > 0$ and take $A \in \mathcal{K}_p$ such that for all $s \in G$ we have

$$\frac{\nu(As)}{m(A)} \leq B^+ + \varepsilon \,.$$

Let now $S \in \mathcal{K}_p$ be arbitrary. Since $\nu(At) \leq (B^+ + \varepsilon) \cdot m(A)$ for all $t \in A^{-1}S$, by Remark 4.6 there exists $a \in A$ such that $\nu(aA^{-1}S) \leq (B^+ + \varepsilon) \cdot m(A^{-1}S)$. Since $S \subseteq aA^{-1}S$, we have

$$\frac{\nu(S)}{m(A^{-1}S)} \leq B^+ + \varepsilon \,.$$

As $S \in \mathcal{K}_p$ and $\varepsilon > 0$ were arbitrary, the claim follows.   □

5 Følner Conditions and Følner Nets

As for Følner sequences or nets in locally compact groups there are various definitions in the literature depending on the definition of boundary. In this section we clarify some subtle connections, most of which can be found in the literature.

5.1 Various Boundaries

The geometric intuition behind amenability is that, for suitable sets A, the measure of the boundary of A is small when compared to the measure of A. Traditionally this is formalised using the so-called Følner boundaries

$$\delta^K A := K A \triangle A = \left(K A \cap A^c\right) \cup \left((K A)^c \cap A\right) .$$

Here and also in the sequel of this text, $\triangle$ denotes the symmetric set difference, i.e., $A \triangle B = (A \cap B^c) \cup (A^c \cap B)$. We write $A^c := G \setminus A$ for the complement set of A.

As we discuss in the following sections, it is often useful to consider different notions of boundaries such as the *van Hove boundary*, defined as

$$\partial^K A := (K A \cap \overline{A^c}) \cup (K^{-1}\overline{A^c} \cap A)$$

or the *strong Følner boundary*, defined as

$$\partial_K A := K^{-1}A \cap K^{-1}A^c = \{g \in G : Kg \cap A \neq \varnothing \wedge Kg \cap A^c \neq \varnothing\}$$

for sets $K, A \subseteq G$.

Remark 5.1 The attribution to van Hove is due to Schlottmann [56]. Van Hove boundaries are often used in model set analysis, see [3] and [48, Sec. 2] for further background. Strong Følner boundaries have been introduced and used by Ornstein and Weiss in [49]. The name strong Følner boundary is coined in [51]. Also note the slightly different definition in [27], which is more in line with the notion of van Hove boundary, compare [27, Remark 2.2(ii)]. Such boundaries naturally arise in tiling problems leading to sub- or almost additive convergence results beyond Euclidean space, cf. e.g., [14, 27, 32, 39, 51].

Note that the van Hove boundary and the strong Følner boundary are both monotonic in K, in contrast to the classical Følner boundary. Strong Følner boundaries additionally satisfy the simple relation $L\partial_K A \subseteq \partial_{KL^{-1}}A$, which is very convenient in calculations. Let us collect some relations between the three types of boundary for later use.

Lemma 5.2 (Boundary comparison) *Let G be a locally compact group. Let $K \subseteq G$ be any symmetric unit neighbourhood and $A \subseteq G$ be arbitrary. Then*

$$\partial_K A \subseteq \partial^K A \subseteq \partial_{K^2} A , \qquad \delta^K A \subseteq \partial_K A \subseteq K\delta^K A , \qquad \partial_K(K A) \subseteq \delta^{K^2} A .$$

Remark 5.3 The statements comparing the boundaries ∂_K and δ^K can be found in [32, Lemma 2.2]. The inclusion $\partial_K(K A) \subseteq \delta^{K^2} A$ has been observed in [49] and leads to the uniform Følner condition (UFC) in [49, p. 19]. We provide the proofs for the sake of self-containment.

Proof Let $K = K^{-1}$ be any symmetric unit neighborhood. We have to show five inclusions which we enumerate by (i)–(v) in its order of appearance in the statement.
(i) To show $\partial_K A \subseteq \partial^K A$, let $x \in KA \cap KA^c$ and assume $x \notin KA \cap \overline{A^c}$. Then $x \in A$, which implies $x \in KA \cap KA^c \cap A = A \cap KA^c \subseteq A \cap K\overline{A^c} \subseteq \partial^K A$.
(ii) Noting $\overline{A^c} \subseteq KA^c$ we have $KA \cap \overline{A^c} \subseteq KA \cap KA^c \subseteq K^2 A \cap K^2 A^c$. Likewise we have $K\overline{A^c} \cap A \subseteq KKA^c \cap KA \subseteq K^2 A^c \cap K^2 A$. We thus have $\partial^K A \subseteq \partial_{K^2} A$.
(iii) As $(KA)^c \subseteq KA^c$, we have $\delta^K A = (KA \cap A^c) \cup ((KA)^c \cap A) \subseteq (KA \cap A^c) \cup (KA^c \cap A) \subseteq \partial_K A$.
(iv) We have $\partial_K A = KA \cap KA^c \cap (A \cup A^c) = (A \cap KA^c) \cup (KA \cap A^c)$. As we have $A \cap KA^c \subseteq K(KA \cap A^c)$, this implies $\partial_K A \subseteq K \delta^K A$.
(v) We have $\partial_K(KA) = K(KA) \cap K(KA)^c = K^2 A \cap K(KA)^c \subseteq K^2 A \cap A^c \subseteq \delta^{K^2} A$. $\qquad \square$

Let us denote by $\mathcal{K}_{s0} \subseteq \mathcal{K}$ the collection of compact sets which are symmetric and contain the identity. The following proposition characterizes amenability. It is well-known, we give a proof for the convenience of the reader.

Proposition 5.4 *Let G be a locally compact group with left Haar measure m. Then the following are equivalent.*

(i) *For all $K \in \mathcal{K}$ we have $\inf_{A \in \mathcal{K}_p} m(KA)/m(A) = 1$.*
(ii) *For all $K \in \mathcal{K}$ we have $\inf_{A \in \mathcal{K}_p} m(\delta^K A)/m(A) = 0$.*
(iii) *For all $K \in \mathcal{K}$ we have $\inf_{A \in \mathcal{K}_p} m(\partial_K A)/m(A) = 0$.*
(iv) *For all $K \in \mathcal{K}$ we have $\inf_{A \in \mathcal{K}_p} m(\partial^K A)/m(A) = 0$.*

In fact the claimed equivalences continue to hold if in any of the four statements (i) to (iv) the condition "$K \in \mathcal{K}$" is replaced by "$K \in \mathcal{K}_{s0}$".

Proof We will first prove the claimed equivalences for $K \in \mathcal{K}_{s0}$. The equivalence of (i) and (ii) is a direct consequence of $A \,\dot\cup\, \delta^K A = KA$, which holds as $e \in K$. For (ii) $\Rightarrow$ (iii), let $\varepsilon > 0$ be arbitrary and choose $A' \in \mathcal{K}_p$ such that $m(\delta^{K^2} A')/m(A') < \varepsilon$. Letting $A = KA'$, we then estimate $\partial_K A = \partial_K(KA') \subseteq \delta^{K^2} A'$. As we have $m(A) = m(KA') \geq m(A')$, we obtain $m(\partial_K A)/m(A) \leq m(\delta^{K^2} A')/m(A') < \varepsilon$. As $\varepsilon > 0$ was arbitrary, then claim follows.

The implications (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (ii) follow from the comparison lemma 5.2. Indeed that lemma states $\partial^K A \subseteq \partial_{K^2} A$ and $\delta^K A \subseteq \partial^K A$.

Consider now the general case $K \in \mathcal{K}$ in the above statements. Then statements (i), (iii), (iv) hold if these statements hold for $K \in \mathcal{K}_{0s}$, as the corresponding measure ratios are monotonic in K. With respect to (ii), note $\delta^K A \subseteq \partial_{K^{-1} \cup \{e\}} A$. Hence (ii) follows from (iii), which is already established. $\qquad \square$

5.2 Følner Nets

Consider a Delone measure ν on an amenable locally compact group G. In that case, one might ask whether there exists a sequence $(A_n)_{n \in \mathbb{N}}$ of compact sets in G such that the Leptin density L_ν^- is realised on that sequence, instead of taking the infimum over all $A \in \mathcal{K}_p$. Natural candidates are Følner sequences, which exist in any σ-compact

amenable group. For general locally compact groups, amenability is characterized by the existence of Følner nets. We refer to [50, Chapter 4] for the construction of Følner nets. For the convenience of the reader, background about nets is collected in Appendix A.

Definition 5.5 (*Følner net*) Let G be an amenable locally compact group with left Haar measure m. Let $(\mathbb{I}, \prec)$ be a directed poset and let $(A_i)_{i \in \mathbb{I}} \in (\mathcal{K}_p)^{\mathbb{I}}$. Then $(A_i)_{i \in \mathbb{I}}$ is called a *Følner net* if

$$\sup_{K \in \mathcal{K}} \lim_{i \in \mathbb{I}} \frac{m(\delta^K A_i)}{m(A_i)} = 0 \, .$$

Remark 5.6 Note that any subnet of a Følner net is a Følner net. If G is unimodular and $(A_i)_{i \in \mathbb{I}}$ is a Følner net, then $(A_i x_i)_{i \in \mathbb{I}}$ is a Følner net for every net $(x_i)_{i \in \mathbb{I}} \in G^{\mathbb{I}}$. This is a direct consequence of right-invariance of the Haar measure in that case. For σ-compact amenable G, there always exists a Følner sequence. See the proof of [50, Theorem 4.16].

Clearly, one can define alternative Følner nets or sequences using ∂^K or ∂_K as a notion of boundary. This has lead to the terminology of *van Hove sequences* or *strong Følner sequences* in the literature. Let us introduce these notions in the general context of nets. Namely, $(A_i)_{i \in \mathbb{I}} \in (\mathcal{K}_p)^{\mathbb{I}}$ is called *strong Følner net*, respectively a *van Hove net* if

$$\sup_{K \in \mathcal{K}} \lim_{i \in \mathbb{I}} \frac{m(\partial_K A_i)}{m(A_i)} = 0, \quad \text{respectively} \quad \sup_{K \in \mathcal{K}} \lim_{i \in \mathbb{I}} \frac{m(\partial^K A_i)}{m(A_i)} = 0.$$

Remark 5.7 Concerning the discussion in Section. 2.1, note that for a unimodular amenable group with left (strong) Følner net (F_i), a right (strong) Følner net is given by (F_i^{-1}). Note that for many groups of interest there are symmetric Følner sequences, which are invariant from the left and from the right simultaneously. This is for instance the case for homogeneous Lie groups (for a beautiful exposition see [19, Chapter 3]) endowed with a left-invariant (homogeneous) metric. For those groups one can consider the sequence (B_n), with the B_n being the closed balls of radius n around the identity, cf. [6, Proposition 4.4].

It is obvious from the boundary comparison Lemma 5.2 that the notions of strong Følner net and van Hove net coincide, a fact that has already been remarked in [27, Remark 2.2(ii)]. In the remainder, we will mostly work with strong Følner boundaries. It is easy to see that every strong Følner net is a Følner net, since $\delta^K A \subseteq \partial_{K^{-1} \cup \{e\}} A$ for all $K \in \mathcal{K}$ and each $A \in \mathcal{K}_p$. An example of a Følner sequence $(A_n)_{n \in \mathbb{N}}$ in $G = \mathbb{R}$ that is not a strong Følner sequence can be provided by compact nowhere dense sets $A_n \subset [0, n]$ of Lebesgue measure $n - 1/n$, see [62]. An example in $G = \mathbb{R}^d$ has been given in [60, Appendix, Ex. 3.4]. The next proposition characterizes strong Følner nets. Whereas parts of it are known, we give a proof for the convenience of the reader.

Proposition 5.8 (Characterization of strong Følner nets) *Let G be an amenable locally compact group with left Haar measure m. Let $(A_i)_{i\in\mathbb{I}} \in \mathcal{K}_p^{\mathbb{I}}$. Then the following are equivalent.*

   (i) *$(A_i)_{i\in\mathbb{I}}$ is a strong Følner net.*
  (ii) *$(A_i)_{i\in\mathbb{I}}$ is a Følner net and there is an open unit neighborhood O such that*
     $\lim_{i\in\mathbb{I}} \frac{m\left(\bigcap_{o\in O} oA_i\right)}{m(A_i)} = 1.$
 (iii) *$(A_i)_{i\in\mathbb{I}}$ is a Følner net and for every compact symmetric unit neighborhood K*
     $\lim_{i\in\mathbb{I}} \frac{m\left(\bigcap_{k\in K} kA_i\right)}{m(A_i)} = 1.$
 (iv) *For all $K, L \in \mathcal{K}$ we have $\lim_{i\in\mathbb{I}} \frac{m(L\delta^K A_i)}{m(A_i)} = 0.$*
  (v) *For all $K, L \in \mathcal{K}$ we have $\lim_{i\in\mathbb{I}} \frac{m(L\partial^K A_i)}{m(A_i)} = 0.$*
 (vi) *For all $K, L \in \mathcal{K}$ we have $\lim_{i\in\mathbb{I}} \frac{m(L\partial_K A_i)}{m(A_i)} = 0.$*

In fact the same equivalences (iv), (v) and (vi) hold if "$K, L \in \mathcal{K}$" is replaced by "$K, L \in \mathcal{K}_{s0}$" in any of the above statements. In particular, $(A_i)_{i\in\mathbb{I}}$ is a van Hove net (sequence) if and only if it is a strong Følner net (sequence). If G is additionally assumed to be discrete, then every Følner net (sequence) is also a strong Følner net (sequence).

Remark 5.9 The equivalence (i)⇔(ii) already appears in [60, Appendix (3.K)]. It shows that Følner nets are strong Følner nets if the they display some asymptotic invariance with respect to small topological pertubations. In this situation, one gets with (iii) asymptotic invariance for the "inner part" $A_i \setminus \mathrm{Int}_K(A_i)$, where $\mathrm{Int}_K(A) = \{g \in A : Kg \subseteq A\}$ for a compact symmetric unit neighborhood K and a set A. The observation that for discrete groups, the notions of a Følner net (or sequence) and a strong Følner net (or sequence) are equivalent has also been observed in [32, Proposition 2.3] and [14, Proposition 2.4]. Note also that the authors deal in [14] with semigroups and with boundary defined as the inner part of the strong Følner boundary, i.e., $\tilde{\partial}_K A := A \setminus \mathrm{Int}_K(A)$.

Proof We have indicated above that every strong Følner net is a Følner net. In order to complete the proof of the implication (i)⇒(ii), let O be an open relatively compact, symmetric unit neighborhood. Since $A_i \setminus \bigcap_{o\in O} oA_i \subseteq \partial_O(A_i)$, and since (A_i) is a strong Følner net by (i), we obtain

$$\liminf_{i\in\mathbb{I}} \frac{m\left(\bigcap_{o\in O} oA_i\right)}{m(A_i)} \geq \left(1 - \limsup_{i\in\mathbb{I}} \frac{m\left(\partial_O A_i\right)}{m(A_i)}\right) = 1.$$

We turn to the proof of (ii)$\Rightarrow$(iii). Let K be any compact symmetric unit neighborhood. We find a finite cover of K by left-translates $s_j O$. Then

$$
\begin{aligned}
\limsup_{i \in \mathbb{I}} \frac{m\big(A_i \setminus \bigcap_{k \in k} kA_i\big)}{m(A_i)} &\leq \limsup_{i \in \mathbb{I}} \frac{m\big(A_i \setminus \bigcap_j \bigcap_{o \in O} s_j o A_i\big)}{m(A_i)} \\
&\leq \limsup_{i \in \mathbb{I}} \sum_j \frac{m\big(s_j^{-1} A_i \setminus \bigcap_{o \in O} o A_i\big)}{m(A_i)} \\
&\leq \sum_j \left(\limsup_{i \in \mathbb{I}} \frac{m\big(s_j^{-1} A_i \triangle A_i\big)}{m(A_i)} + \limsup_{i \in \mathbb{I}} \frac{m\big(A_i \setminus \bigcap_{o \in O} o A_i\big)}{m(A_i)} \right) \\
&= 0.
\end{aligned}
$$

Note that we used for the last equality that (A_i) is a Følner net and the assumption on O. This shows the assertion (iii). We turn to the proof of the implication (iii)$\Rightarrow$(i). By Lemma 5.2 and the fact that (A_i) is a Følner net,

$$
\begin{aligned}
\limsup_{i \in \mathbb{I}} \frac{m\big(\partial_K A_i\big)}{m(A_i)} &\leq \limsup_{i \in \mathbb{I}} \frac{m\big(K(K A_i \setminus A_i)\big)}{m(A_i)} \leq \limsup_{i \in \mathbb{I}} \frac{m\big(K^2 A_i \setminus \bigcap_{k \in K} kA_i\big)}{m(A_i)} \\
&\leq \limsup_{i \in \mathbb{I}} \frac{m(K^2 A_i \setminus A_i)}{m(A_i)} + \limsup_{i \in \mathbb{I}} \frac{m(A_i \setminus \bigcap_{k \in K} kA_i)}{m(A_i)} \\
&= 0.
\end{aligned}
$$

Due to the monotonicity of the strong Følner boundary in K, the same is true for general $K \in \mathcal{K}$. Hence (A_i) is a strong Følner net. The equivalence of (i) and (vi) is due to the relation $L\partial_K A \subseteq \partial_{KL^{-1}} A$ for general compact sets K, L, A. If restricting oneself to compact symmetric unit neighborhoods K and L, the equivalences (iv)$\Leftrightarrow$(v)$\Leftrightarrow$(vi) follow from the boundary comparison lemma, Lemma 5.2. Due to monotonicity, these assertions are equivalent when formulated for general $K, L \in \mathcal{K}$. Assuming in addition that G is discrete, then the implication (ii) $\Rightarrow$ (i) shows that every Følner net is a strong Følner net. □

It is well known that existence of a Følner sequence implies existence of a strong Følner sequence, cf. [51, Lemma 2.6] or [62, Lemma 2.2]. Here we describe how a strong Følner net can be constructed from any Følner net by a simple "thickening procedure".

Proposition 5.10 (Construction of strong Følner nets) *Let G be an amenable locally compact group with left Haar measure m. If (A_i) is a Følner net and L is a compact symmetric unit neighborhood, then the net (A_i') with $A_i' := L A_i$ is a strong Følner net.*

Proof We first note that (A_i') is a Følner net which follows from the Følner property of (A_i) and the inclusion

$$
A_i' \triangle K A_i' \subseteq \big(L A_i \triangle A_i\big) \cup \big(A_i \setminus K A_i\big) \cup \big(K L A_i \setminus A_i\big)
$$

for arbitrary $K \in \mathcal{K}$. Further, take an open symmetric unit neighborhood $O \subseteq L$. We claim that $A_i \subseteq \bigcap_{o \in O} o A_i'$. Indeed, for any $a \in A_i$ and each $o \in O$, we have $a = o o^{-1} a \subseteq o A_i'$. Using that (A_i) is a Følner net, we obtain

$$\liminf_{i \in \mathbb{I}} \frac{m\left(\bigcap_{o \in O} o A_i'\right)}{m(A_i')} \geq \liminf_{i \in \mathbb{I}} \frac{m(A_i)}{m(L A_i)} = 1.$$

Now the claim follows from the previous proposition, equivalence (i)⇔(ii).   □

The following lemma shows that (left)-asymptotic invariance observed via strong Følner nets carries over to upper translation bounded measures. For σ-compact LCA groups, weaker versions of it are well known, see e.g., [36, Proposition 9.1] and [48, Lemma 2.2, Proposition 5.1].

Lemma 5.11 *Let G be a unimodular amenable locally compact group with Haar measure m. Let $(\mathbb{I}, \prec)$ be a directed poset and let $(A_i)_{i \in \mathbb{I}}$ be a strong Følner net. Let ν be a measure on G that is upper translation bounded. We then have*

$$\sup_{K,L \in \mathcal{K}} \limsup_{i \in \mathbb{I}} \sup_{s \in G} \frac{\nu(L \delta^K A_i s)}{m(A_i)} = \sup_{K,L \in \mathcal{K}} \limsup_{i \in \mathbb{I}} \sup_{s \in G} \frac{\nu(L \partial_K A_i s)}{m(A_i)}$$

$$= \sup_{K,L \in \mathcal{K}} \limsup_{i \in \mathbb{I}} \sup_{s \in G} \frac{\nu(L \partial^K A_i s)}{m(A_i)} = 0 .$$

Proof We use Lemma 2.4 to transfer the properties in Proposition 5.8 to the measure ν. For instance, for the strong Følner boundary this yields

$$\limsup_{i \in \mathbb{I}} \sup_{s \in G} \frac{\nu(L \partial_K A_i s)}{m(A_i)} \leq \lim_{i \in \mathbb{I}} \frac{C_u}{m(B_u)} \frac{m(B_u L \partial_K A_i)}{m(A_i)} = 0 ,$$

where we used right invariance of the Haar measure in the estimate.   □

5.3 Densities on Følner Nets in Unimodular Groups

In this section, we restrict ourselves to amenable locally compact groups that are unimodular. We define densities of a measure via averaging with respect to a Følner net.

Definition 5.12 ($\mathcal{A}$-densities) Let G be a unimodular amenable locally compact group with left Haar measure m. Let ν be a measure on G, and let $\mathcal{A} = (A_i)_{i \in \mathbb{I}}$ be a strong Følner net in G. We define the upper and lower densities of ν with respect to $\mathcal{A}$ by

$$D_{\mathcal{A}}^- = \liminf_{i \in \mathbb{I}} \frac{1}{m(A_i)} \nu(A_i) , \qquad D_{\mathcal{A}}^+ = \limsup_{i \in \mathbb{I}} \frac{1}{m(A_i)} \nu(A_i)$$

$$B_{\mathcal{A}}^- = \liminf_{i \in \mathbb{I}} \frac{1}{m(A_i)} \inf_{s \in G} \nu(A_i s) , \qquad B_{\mathcal{A}}^+ = \limsup_{i \in \mathbb{I}} \frac{1}{m(A_i)} \sup_{s \in G} \nu(A_i s) .$$

If $D_{\mathcal{A}}^- = D_{\mathcal{A}}^+ = D_{\mathcal{A}}$, we say that ν has $\mathcal{A}$-density $D_{\mathcal{A}}$.

Remark 5.13 We have $0 \leq B_{\mathcal{A}}^- \leq D_{\mathcal{A}}^- \leq D_{\mathcal{A}}^+ \leq B_{\mathcal{A}}^+ \leq \infty$ by definition. If ν is upper translation bounded, we have $B_{\mathcal{A}}^+ < \infty$. If ν is lower translation bounded, we have $B_{\mathcal{A}}^- > 0$.

Proposition 5.14 *Assume that G is a unimodular amenable locally compact group with left Haar measure m. Let ν be an upper translation bounded measure on G. Take any strong Følner net $\mathcal{A} = (A_i)_{i \in \mathbb{I}}$ in G. We then have*

$$B_{\mathcal{A}}^- = \lim_{i \in \mathbb{I}} \frac{1}{m(A_i)} \inf_{s \in G} \nu(A_i s), \qquad B_{\mathcal{A}}^+ = \lim_{i \in \mathbb{I}} \frac{1}{m(A_i)} \sup_{s \in G} \nu(A_i s).$$

Moreover we have

$$0 \leq L^- = B^- = B_{\mathcal{A}}^- \leq D_{\mathcal{A}}^- \leq D_{\mathcal{A}}^+ \leq B_{\mathcal{A}}^+ = B^+ = L^+ < \infty,$$

where L^-, L^+ are the Leptin densities from Definition 3.3, and where B^-, B^+ are the Beurling densities from Definition 4.1. In particular, the above $\mathcal{A}$-densities do not depend on the choice of the strong Følner sequence.

Remark 5.15 In the following proof, note that upper translation boundedness and the strong version of the Følner property only enter in showing that $B_{\mathcal{A}}^- = B_-$. Assuming only local finiteness of ν we still have $B_{\mathcal{A}}^- \leq B^- = L^- \leq L^+ = B^+ \leq B_{\mathcal{A}}^+$.

Proof Consider

$$C_{\mathcal{A}}^- = \limsup_{i \in \mathbb{I}} \frac{1}{m(A_i)} \inf_{s \in G} \nu(A_i s), \qquad C_{\mathcal{A}}^+ = \liminf_{i \in \mathbb{I}} \frac{1}{m(A_i)} \sup_{s \in G} \nu(A_i s).$$

(i) Note that for each $i \in \mathbb{I}$ we have

$$\frac{1}{m(A_i)} \inf_{s \in G} \nu(A_i s) \leq \sup_{A \in \mathcal{K}_p} \frac{1}{m(A)} \inf_{s \in G} \nu(A s) = B^-,$$

which gives $C_{\mathcal{A}}^- \leq B^-$. If $L^- \leq C_{\mathcal{A}}^-$, then $C_{\mathcal{A}}^- = B^- = L^-$ by Theorem 4.3. To show $L^- \leq C_{\mathcal{A}}^-$, we take arbitrary $d < L^-$. Take $K \in \mathcal{K}$ such that for all $A \in \mathcal{K}_p$ we have $d \leq \nu(KA)/m(A)$. Choosing $A = A_i s$ for $i \in \mathbb{I}$ we obtain the estimate

$$d \leq \frac{\nu(K A_i s)}{m(A_i)} \leq \frac{\nu(A_i s)}{m(A_i)} + \frac{\nu((\delta^K A_i)s)}{m(A_i)},$$

where we used the estimate $K As \subseteq As \cup (\delta^K A)s$, and the unimodularity of G. Now upper translation boundedness of ν, right invariance of the Haar measure and the strong Følner property of $\mathcal{A}$ yield $d \leq C_{\mathcal{A}}^-$, cf. Lemma 5.11. As $d < L^-$ was arbitrary, we infer $L^- \leq C_{\mathcal{A}}^-$. Therefore, we have $C_{\mathcal{A}}^- = B^- = L^-$.

(ii) Since for all $i \in \mathbb{I}$ we have

$$\frac{1}{m(A_i)} \sup_{s \in G} \nu(A_i s) \geq \inf_{A \in \mathcal{K}_p} \frac{1}{m(A)} \sup_{s \in G} \nu(As) = B^+ ,$$

we get $B^+ \leq C_{\mathcal{A}}^+$. If $C_{\mathcal{A}}^+ \leq L^+$, then $C_{\mathcal{A}}^+ = B^+ = L^+$ by Theorem 4.3, which uses local finiteness of ν. Let us thus show $L^+ \geq C_{\mathcal{A}}^+$. Assume without loss of generality $L^+ < \infty$. Assume $d > L^+$ for some finite d. Take $K \in \mathcal{K}$ such that for all $A \in \mathcal{K}_p$ we have $d \geq \nu(A)/m(KA)$. Choosing $A = A_i s$ for $i \in \mathbb{I}$ we obtain with the unimodularity of G the estimate

$$d \geq \frac{\nu(A_i s)}{m(KA_i)} .$$

By the Følner property of $\mathcal{A}$ this implies $d \geq C_{\mathcal{A}}^+$. As $d > L^+$ was arbitrary, we infer $L^+ \geq C_{\mathcal{A}}^+$. Therefore, we have $C_{\mathcal{A}}^+ = B^+ = L^+$.

(iii) By (i) and (ii), the values $C_{\mathcal{A}}^-$ and $C_{\mathcal{A}}^+$ are independent of the choice of the strong Følner net. This means that $C_{\mathcal{A}}^-$ and $C_{\mathcal{A}}^+$ are limits. In particular this implies $C_{\mathcal{A}}^- = B_{\mathcal{A}}^- \leq D_{\mathcal{A}}^- \leq D_{\mathcal{A}}^+ \leq B_{\mathcal{A}}^+ = C_{\mathcal{A}}^+$. Also note $B^+ < \infty$ by upper translation boundedness of ν, see Remark 4.2. $\qquad \square$

6 Leptin Densities for Lattice Point Counting

In the sequel, we consider groups G that admit a uniform lattice Γ, i.e., a discrete subgroup $\Gamma \subseteq G$ that is also co-compact. Beyond the abelian situation, lattices in locally compact groups are defined as discrete subgroups permitting a finite G-invariant measure on the quotient space. These subgroups are not necessarily uniform (co-compact). However, there are large classes of amenable groups, such as nilpotent locally compact groups, where this is the case. For an in-depth study of this and related phenomena via the so-called property (M), we refer to [5]. Recall that existence of a uniform lattice in G implies that G is unimodular, see e.g., [15, Theorem 9.1.6]. A uniform lattice is both left-uniformly discrete and right-uniformly discrete. Analogously, a uniform lattice Γ in G is both left-relatively dense and right-relatively dense. Hence it admits a measurable, relatively compact left-fundamental domain and also a measurable, relatively compact right-fundamental domain. For a proof, see Theorem 1 and Remark 6 in [18]. Since we are only interested in the Haar measure of a fundamental domain, we can make this choice freely and fix a relatively compact measurable left-fundamental domain F, i.e., we have $\Gamma F = G$ and $F^{-1}\gamma_1 \cap F^{-1}\gamma_2 \neq \varnothing$ for some $\gamma_1, \gamma_2 \in \Gamma$ implies $\gamma_1 = \gamma_2$. Moreover, for any $\gamma \in \Gamma$, we observe that

$$\operatorname{card}(\Gamma \cap F^{-1}\gamma) = \operatorname{card}(\Gamma\gamma^{-1} \cap F^{-1}) = \operatorname{card}(\Gamma \cap F) = 1. \qquad (6.1)$$

This fact will be used below in the proofs of the density formulas.

When identified with the Delone measure $\delta_\Gamma = \sum_{\gamma \in \Gamma} \delta_\gamma$, uniform lattices Γ have a positive finite Leptin density, which equals the reciprocal of its covolume $\operatorname{covol}(\Gamma)$.

Recall that the covolume is defined as the Haar measure of any measurable relatively compact (left or right) fundamental domain.

Proposition 6.1 (Density of a uniform lattice) *Let Γ be a uniform lattice in a unimodular amenable locally compact group G. Then Γ has positive finite Leptin density*

$$L_\Gamma := L_{\delta_\Gamma} = \frac{1}{m(F)} = \frac{1}{\mathrm{covol}(\Gamma)} \, ,$$

where F is any measurable relatively compact (left or right) fundamental domain for Γ.

Proof Fix a measurable relatively compact left-fundamental domain F of Γ. Define the compact set $K = \overline{F^{-1}F}$. Fix an arbitrary compact set $A \subseteq G$ and consider

$$\Gamma_A = \{\gamma \in \Gamma : A \cap F^{-1}\gamma \neq \varnothing\} \, , \qquad A_F = \bigcup_{\gamma \in \Gamma_A} F^{-1}\gamma \, .$$

Note that Γ_A is a finite set and A_F is a disjoint union. We have

$$A \subseteq A_F \subseteq KA \, . \tag{6.2}$$

For the second inclusion, assume that $x \in A_F$. Then $x \in F^{-1}\gamma$ for some $\gamma \in \Gamma_A$ and there exists some $a \in A$ such that $a \in F^{-1}\gamma$. Hence $x \in F^{-1}\gamma \subseteq F^{-1}Fa \subseteq F^{-1}FA \subseteq KA$. With equality (6.1) we thus have

$$m(A) \leq m(A_F) = m(F) \cdot \mathrm{card}(\Gamma_A) = m(F) \cdot \sum_{\gamma \in \Gamma_A} \mathrm{card}(\Gamma \cap F^{-1}\gamma)$$

$$= m(F) \cdot \mathrm{card}(\Gamma \cap A_F) \leq m(F) \cdot \mathrm{card}(\Gamma \cap KA) = m(F) \cdot \delta_\Gamma(KA).$$

By Definition 3.3 this implies $L^- \geq 1/m(F)$. We can similarly estimate

$$m(KA) \geq m(A_F) = m(F) \cdot \mathrm{card}(\Gamma_A) = m(F) \cdot \mathrm{card}(\Gamma \cap A_F) \geq m(F) \cdot \delta_\Gamma(A) \, .$$

This leads to $L^+ \leq 1/m(F)$ by Definition 3.3. Finally, since G is amenable, we can use Lemma 3.6 in order to conclude $L^- = L^+$. This and the fact that the expression $1/m(F)$ does not depend on F show that Γ has Leptin density $1/\mathrm{covol}(\Gamma) = 1/m(F)$. $\square$

The next result, whose proof uses a standard argument [59, Lemma 2], will be needed in the following section.

Lemma 6.2 *Let G be a locally compact group and let Γ be a uniform lattice in G. Consider any measurable relatively compact set $U \subseteq G$ such that $\Gamma U = G$. Then U contains a measurable relatively compact left-fundamental domain for Γ in G.*

Proof Let F be a measurable relatively compact left-fundamental domain for Γ in G. By relative compactness of both F and U, the set $\Gamma_U = \{\gamma \in \Gamma : U \cap \gamma F \neq \varnothing\}$ is finite, and we can thus write $\Gamma_U = \{\gamma_1, \ldots, \gamma_n\}$. Define $F_k = F \cap \gamma_k^{-1} U \subseteq F$ and note $F = \bigcup_{k=1}^n F_k$ as $\Gamma U = G$. Define $F_1' = \gamma_1 F_1$ and $F_k' = (\gamma_k F_k) \setminus \bigcup_{j=1}^{k-1} (\Gamma F_j) \subseteq U$ for $k \geq 2$ and note that $F_U := \dot{\bigcup}_{k=1}^n F_k'$ is a disjoint union. Furthermore, F_U is a measurable relatively compact set which satisfies $F_U \subseteq U$. We show that F_U is a left-fundamental domain for Γ. Consider any $z \in G$ and choose $k \in \{1, \ldots, n\}$ such that $z \in \Gamma F_k$ but $z \notin \Gamma F_j$ for all $j < k$. Then $z \in \Gamma F_k' \subseteq \Gamma F_U$ and thus $\Gamma F_U = G$. Assume $F_U^{-1} \gamma_1' \cap F_U^{-1} \gamma_2' \neq \varnothing$ for some $\gamma_1', \gamma_2' \in \Gamma$. Then $F_r'^{-1} \gamma_1' \cap F_s'^{-1} \gamma_2' \neq \varnothing$ for some r and some s. We may assume $r \leq s$ without loss of generality. By construction of F_s' we then have $r = s$. Since $F_r' \subseteq \gamma_r F$, we have $F^{-1} \gamma_r^{-1} \gamma_1' \cap F^{-1} \gamma_r^{-1} \gamma_2' \neq \varnothing$. Since F is a left-fundamental domain for Γ, this implies $\gamma_1' = \gamma_2'$. □

7 Uniform Model Sets in Amenable Groups

We analyse the upper and lower Leptin densities of weak model sets in amenable groups. As an application we give a simple geometric proof of the density formula for regular model sets.

7.1 Definition and Elementary Properties

Let $(G, H, \mathcal{L})$ be a *cut-and-project scheme*, i.e., both G and H are unimodular locally compact groups, and $\mathcal{L}$ is a uniform lattice in $G \times H$, which projects injectively to G and densely to H. Let us denote the canonical projections by π^G and π^H. If $W \subseteq H$ is relatively compact, then the set $\Lambda_W = \pi^G(\mathcal{L} \cap (G \times W))$ is called a *weak model set*. Any weak model set is left-uniformly discrete and right-uniformly discrete. A *model set* is a weak model set which satisfies $\mathring{W} \neq \varnothing$. Here and in the sequel of this paper, we denote by $\mathring{S}$ the interior of a set S in a topological space. Any model set is both right-relatively dense and left-relatively dense. We call a model set *regular* if W is Riemann measurable, i.e., if $m_H(\partial W) = 0$, a nomenclature that is nowadays common in the abelian case [2, Definition 7.2]. Recall that the topological boundary of set S in a topological spaces is defined as $\partial S := \overline{S} \cap \overline{S^c}$. Note that in other contexts, the definition of regular model set can be slightly different. For instance, in [11, 12], regular model sets are defined for non-abelian groups and homogeneous spaces arising from them via cut-and-project schemes with a slightly more restrictive notion of window but also for possibly non-uniform lattices. We refer to [2, 45, 46] for further background on model sets in mostly abelian situations.

As $\pi^H(\mathcal{L})$ is dense in H, the lattice $\mathcal{L}$ admits fundamental domains in $G \times H$ that are "closely aligned to G". This is formalised in the following lemma, which combines [29, Lemma 2.3] and Lemma 6.2.

Lemma 7.1 *Let $(G, H, \mathcal{L})$ be a cut-and-project scheme and consider any non-empty, open and relatively compact set $U \subseteq H$. Then there exists an open relatively compact*

set $V \subseteq G$ satisfying

$$\mathcal{L}(V \times U) = G \times H.$$

Moreover $V \times U$ contains a measurable relatively compact left-fundamental domain for $\mathcal{L}$. □

7.2 Leptin Density of Weak Model Sets

The above property allows to derive bounds on the strong upper and lower densities of any weak model sets in the same way as in the lattice case.

Theorem 7.2 (Leptin density estimates for weak model sets) *Let $(G, H, \mathcal{L})$ be a cut-and-project scheme, where G is amenable. Fix a relatively compact set $W \subseteq H$. We then have*

$$\frac{m_H(\mathring{W})}{\text{covol}(\mathcal{L})} \leq L^-_{\Lambda_{\mathring{W}}} \leq L^-_{\Lambda_W} \leq L^+_{\Lambda_W} \leq L^+_{\Lambda_{\overline{W}}} \leq \frac{m_H(\overline{W})}{\text{covol}(\mathcal{L})}.$$

In particular, if W is Riemann measurable, then

$$L_{\Lambda_W} = \frac{m_H(W)}{\text{covol}(\mathcal{L})}.$$

Remark 7.3 The so-called density formula $L_{\Lambda_W} = m_H(W)/\text{covol}(\mathcal{L})$ for Riemann measurable windows has been proven in the literature in various contexts, compare the discussion in the introduction. In [11, Section 4.5] the notion of covolume of a regular model set over locally compact second countable groups is defined. We emphasize that regular model sets as of [11, 12] are defined via cut-and-project schemes with additional assumptions on W, but $\mathcal{L}$ does not need to be uniform, and the construction works beyond the realm of amenable groups. In the language of the present paper, the covolume of a regular model set is the inverse of its Leptin density. Sticking to the framework of [11], a density approximation formula for regular model sets in amenable locally compactond countable groups can be derived by combining Proposition 4.13 with the ergodic theorem in Cor. 5.4 of [11].

Proof (i) Consider any compact unit neighborhood $U \subseteq H$. By Lemma 7.1, there exists a compact set F such that $\mathcal{L}(F \times U) = G \times H$, and we can pick a relatively compact measurable left-fundamental domain $\mathcal{F}_U \subseteq F \times U$. Define $\mathcal{K}_U = \mathcal{F}_U^{-1}\mathcal{F}_U$, write $K = \overline{F^{-1}F}$ and note that $\mathcal{K}_U \subseteq K \times U^{-1}U$. For an arbitrary compact set $\mathcal{A} \subseteq G \times H$, consider

$$\mathcal{L}_{\mathcal{A}} = \{\ell \in \mathcal{L} : \mathcal{A} \cap \mathcal{F}_U^{-1}\ell \neq \varnothing\}, \qquad \mathcal{A}_U = \bigcup_{\ell \in \mathcal{L}_{\mathcal{A}}} \mathcal{F}_U^{-1}\ell.$$

Then $\mathcal{L}_{\mathcal{A}}$ is a finite set and $\mathcal{A}_U$ is a disjoint union. We have

$$\mathcal{A} \subseteq \mathcal{A}_U \subseteq \mathcal{K}_U \mathcal{A} \, .$$

For the second implication, assume that $x \in \mathcal{A}_U$. Then $x \in \mathcal{F}_U^{-1}\ell$ for some $\ell \in \mathcal{L}_{\mathcal{A}}$ and there exists some $a \in \mathcal{A}$ such that $a \in \mathcal{F}_U^{-1}\ell$. Hence $x \in \mathcal{F}_U^{-1}\ell \subseteq \mathcal{F}_U^{-1}\mathcal{F}_U a \subseteq \mathcal{F}_U^{-1}\mathcal{F}_U \mathcal{A} \subseteq \mathcal{K}_U \mathcal{A}$.

(ii) We first prove the last two upper inequalities. Fix arbitrary $\varepsilon > 0$. Choose U in (i) sufficiently small such that $m_H(U^{-1}U\overline{W}) \leq (1+\varepsilon) \cdot m_H(\overline{W})$, which is possible due to continuity of the Haar measure on H. For arbitrary compact $A \subseteq G$ and $\mathcal{A} = A \times \overline{W}$ we estimate with equality (6.1)

$$
\begin{aligned}
m_{G \times H}(\mathcal{K}_U \mathcal{A}) &\geq m_{G \times H}(\mathcal{A}_U) = m_{G \times H}(\mathcal{F}_U) \cdot \mathrm{card}(\mathcal{L}_{\mathcal{A}}) \\
&= m_{G \times H}(\mathcal{F}_U) \cdot \sum_{\ell \in \mathcal{L}_{\mathcal{A}}} \mathrm{card}(\mathcal{L} \cap \mathcal{F}_U^{-1}\ell) \\
&= m_{G \times H}(\mathcal{F}_U) \cdot \mathrm{card}(\mathcal{L} \cap \mathcal{A}_U) \geq m_{G \times H}(\mathcal{F}_U) \cdot \mathrm{card}(\mathcal{L} \cap \mathcal{A}) \, .
\end{aligned}
$$

Combining the latter estimate with $m_{G \times H}(\mathcal{K}_U \mathcal{A}) \leq m_G(KA) \cdot m_H(U^{-1}U\overline{W})$, we arrive at

$$
(1+\varepsilon) \cdot \frac{m_H(\overline{W})}{m_{G \times H}(\mathcal{F}_U)} \cdot m_G(KA) \geq \frac{m_H(U^{-1}U\overline{W})}{m_{G \times H}(\mathcal{F}_U)} \cdot m_G(KA) \geq \mathrm{card}(\mathcal{L} \cap \mathcal{A})
$$
$$
\geq \mathrm{card}(\Lambda_{\overline{W}} \cap A) \geq \mathrm{card}(\Lambda_W \cap A) \, .
$$
$$(7.1)$$

Now the last two upper inequalities claimed in the theorem follow by definition of the upper Leptin density.

(iii) To show the first two lower inequalities, fix arbitrary $\varepsilon > 0$. Take a compact set $V \subseteq \mathring{W}$ sufficiently large such that $m_H(V) \geq (1-\varepsilon) \cdot m_H(\mathring{W})$, which is possible by regularity from below of the Haar measure.

Choose a compact zero neighborhood $U \subseteq H$ such that $U^{-1}UV \subseteq \mathring{W}$, compare [15, Lemma 4.1.3]. Consider arbitrary compact $A \subseteq G$ and write $\mathcal{A} = A \times V$. Define $\mathcal{F}_U$ and $\mathcal{K}_U$ as in step (i) above. Using the equality (6.1), we compute

$$
\begin{aligned}
m_{G \times H}(\mathcal{A}) &\leq m_{G \times H}(\mathcal{A}_U) = m_{G \times H}(\mathcal{F}_U) \cdot \mathrm{card}(\mathcal{L}_{\mathcal{A}}) \\
&= m_{G \times H}(\mathcal{F}_U) \cdot \sum_{\ell \in \mathcal{L}_{\mathcal{A}}} \mathrm{card}(\mathcal{L} \cap \mathcal{F}_U^{-1}\ell) \\
&= m_{G \times H}(\mathcal{F}_U) \cdot \mathrm{card}(\mathcal{L} \cap \mathcal{A}_U) \leq m_{G \times H}(\mathcal{F}_U) \cdot \mathrm{card}(\mathcal{L} \cap \mathcal{K}_U \mathcal{A}) \, .
\end{aligned}
$$

Noting that $\mathcal{K}_U \mathcal{A} \subseteq KA \times U^{-1}UV \subseteq KA \times \mathring{W}$, we infer

$$\mathrm{card}(\mathcal{L} \cap \mathcal{K}_U \mathcal{A}) \leq \mathrm{card}(\Lambda_{\mathring{W}} \cap KA) \leq \mathrm{card}(\Lambda_W \cap KA) \, .$$

Note that in the first inequality above we used the injectivity of the projection π^G. Putting everything together yields

$$
(1 - \varepsilon) \cdot m_H(\mathring{W}) \cdot m_G(A) \leq m_H(V) \cdot m_G(A) \leq m_{G \times H}(\mathcal{F}_U) \cdot \text{card}(\Lambda_{\mathring{W}} \cap KA)
$$
$$
\leq m_{G \times H}(\mathcal{F}_U) \cdot \text{card}(\Lambda_W \cap KA). \tag{7.2}
$$

Now the first two lower inequalities claimed in the theorem follow by definition of the lower Leptin density.

(iv) Recall that $L_{\Lambda_W}^{-} \leq L_{\Lambda_W}^{+}$ by amenability of G, see Lemma 3.6. Now the "In particular"-part of the theorem follows from $m_H(\mathring{W}) = m_H(W) = m_H(\overline{W})$ by Riemann measurability of W. $\qquad \square$

The inequalities (7.1) and (7.2) in the preceding proof lead to the following uniform estimates.

Corollary 7.4 *Let $(G, H, \mathcal{L})$ be a cut-and-project scheme as in the previous theorem. Let $(A_i)_{i \in \mathbb{I}}$ be a strong Følner net in G. Then for all $\varepsilon > 0$ there is some $i_0 \in \mathbb{I}$ such that for all $i \succ i_0$ and all $(x, h) \in G \times H$, one gets the uniform estimates*

$$
\frac{m_H(\mathring{W})}{\text{covol}(\mathcal{L})} - \varepsilon \leq \frac{\text{card}(\Lambda_{Wh} \cap A_i x)}{m_G(A_i)} \leq \frac{m_H(\overline{W})}{\text{covol}(\mathcal{L})} + \varepsilon.
$$

In particular, if W is Riemann measurable then the convergence to the limit is uniform in (x, h).

Proof Let $\varepsilon > 0$. It follows from the inequality (7.1) with W replaced by Wh, from the unimodularity of the group G and from the Følner property of $(A_i)_{i \in \mathbb{I}}$ that there must be some $i_0 \in \mathbb{I}$ such that

$$
\sup_{(x,h) \in G \times H} \frac{\text{card}(\Lambda_{Wh} \cap A_i x)}{m_G(A_i)} \leq (1 + \varepsilon)^2 \cdot \frac{m_H(\overline{W})}{\text{covol}(\mathcal{L})}
$$

for all $i \succ i_0$. Adjusting ε gives the claimed upper bound.

As for the lower bound, we consider the compact set K in inequality (7.2) in the proof of Theorem 7.2 and claim that for given $\varepsilon > 0$ there must be some $i_1 \succ i_0$ such that

$$
\inf_{(x,h) \in G \times H} \frac{\text{card}(\Lambda_{Wh} \cap A_i x)}{m_G(A_i)} \geq \inf_{(x,h) \in G \times H} \frac{\text{card}(\Lambda_{Wh} \cap K A_i x)}{m_G(A_i)} - \varepsilon
$$

for all $i \succ i_1$. To this end, fix a finite constant C and a symmetric unit neighborhood $B \subseteq G \times H$ such that $\delta_{\mathcal{L}}(B^2(x, h)) \leq C$. Using Lemma 2.4 (which is justified as $G \times H$ is unimodular) and the comparison Lemma 5.2 (which is justified as K is

symmetric and contains the identity), we estimate

$$\text{card}(\Lambda_{Wh} \cap (Ax) \Delta (KAx)) = \text{card}(\Lambda_{Wh} \cap \delta^K(Ax)) \leq \text{card}(\Lambda_{Wh} \cap \partial_K(Ax))$$
$$\leq \frac{C}{m_{B \times H}(B)} \cdot m_{G \times H}(\pi^G(B)\partial_K(Ax) \times \pi^H(B)Wh)$$
$$= \frac{C}{m_{G \times H}(B)} \cdot m_G(\pi^G(B)\partial_K A) \cdot m_H(\pi^H(B)W) .$$

Note that the above estimate is uniform in $(x, h) \in G \times H$. Thus dividing by $m_G(A)$ and using the strong Følner property of Lemma 5.11 yields the claim. But inequality (7.2) yields

$$\inf_{(x,h) \in G \times H} \frac{\text{card}(\Lambda_{Wh} \cap KA_i x)}{m_G(A_i)} \geq (1 - \varepsilon) \cdot \frac{m_H(\mathring{W})}{\text{covol}(\mathcal{L})}$$

for all i. Now adjusting ε finishes the proof. $\square$

8 Almost Periodicity of Regular Model Sets

8.1 Almost Periodicity for Point Sets and Measures

With respect to mathematical diffraction theory, almost periodicity of the underlying point set has become a central notion in recent years. For upper translation bounded measures in σ-compact locally compact abelian groups, this is analysed in [37]. In particular, pure point diffraction of a measure is shown to be equivalent to so-called mean almost periodicity of the measure [37, Theorem 2.13], see also below. The discussion is based on density notions using van Hove sequences. Here we will work with the stronger uniform version of mean almost periodicity, thereby avoiding averaging sequences or nets.

Definition 8.1 A model set Λ in an amenable locally compact group G is called *right-uniformly mean almost periodic* if for every $\varepsilon > 0$ there exists a both left- and right-relatively dense set $T \subseteq G$ such that $L^+_{\Lambda \Delta (\Lambda t)} \leq \varepsilon$ for every $t \in T$.

Remark 8.2 Left-uniformly mean almost periodic model sets can be defined analogously. Recall that groups containing a model set are necessarily unimodular. Due to invariance of the Leptin densities with respect to translations, see Lemma 3.5, the set T in the above definition can be chosen to be symmetric, i.e., $T = T^{-1}$. In this situation, T is right-relatively dense if and only it if left-relatively dense.

Note that, for σ-compact locally compact abelian groups, the latter is a weaker notion than Weyl almost periodicity [37, Definition 4.1], which is defined via approximation by trigonometric polynomials. Also note that Meyer's and Guihéneuf's definition of almost periodic pattern [26, Definition 7], [25, Definition 4] and almost periodic measure [26, Definition 10] coincide with Weyl almost periodicity. Note

finally that the notion of Weyl almost periodicity is based on the Weyl seminorm [37, Section 1.3], which can be rephrased using the Leptin upper density, without resorting to a van Hove sequence or net.

8.2 Uniform Mean Almost Periodicity and Model Sets

We analyse (right-)uniform mean almost periodicity for regular model sets.

Theorem 8.3 *Any regular model set in an amenable locally compact group is right-uniformly mean almost periodic.*

This is an immediate consequence of the following result, which may be seen as weak variant of Lemma 3 from [26], however without resorting to an averaging net.

Lemma 8.4 *Let Λ_W be a regular model set. Then for every $\varepsilon > 0$ there exists a compact symmetric unit neighborhood $U \subseteq H$ such that $L^+_{\Lambda_{(WU)\cap(W^c U)}} \leq \varepsilon$, and $t \in \Lambda_U$ implies $\Lambda_W \triangle (\Lambda_W t) \subseteq \Lambda_{(WU)\cap(W^c U)}$.*

Proof Take any compact symmetric unit neighborhood $U \subseteq H$. Consider $t \in \Lambda_U$, write $t = \pi^G(\ell)$ for $\ell \in \mathcal{L} \cap (G \times U)$ and note

$$(\Lambda_W t) \triangle \Lambda_W = \Lambda_{(W\pi^H(\ell))} \triangle \Lambda_W \subseteq \Lambda_{(WU)\cap(W^c U)} .$$

Now note that $m_H(\overline{WU \cap W^c U}) \to m_H(\partial W) = 0$ as $U \to \{e\}$ due to continuity of m_H. We thus have by Theorem 7.2 that $L^+_{\Lambda_{WU\cap W^c U}} \leq \mathrm{covol}(\mathcal{L})^{-1} m(\overline{WU \cap W^c U}) \to 0$ as $U \to \{e\}$. □

To put this result into perspective, take any strong Følner net $(A_i)_{i \in \mathbb{I}}$ in G. We then have the estimate

$$d(t) := \limsup_{i \in \mathbb{I}} \frac{\mathrm{card}((\Lambda_W t) \triangle \Lambda_W \cap A_i)}{m(A_i)} \leq L^+_{\Lambda_{(WU)\cap(W^c U)}} ,$$

see Proposition 5.14. In particular, the set $P_\varepsilon(\Lambda_W) = \{t \in G : d(t) \leq \varepsilon\}$ is both right- and left-relatively dense for all $\varepsilon > 0$.

In the context of σ-compact locally compact abelian groups, the property of $P_\varepsilon(\Lambda_W)$ being relatively dense for all $\varepsilon > 0$ is called mean almost periodicity of Λ_W, see [37, Theorem 2.18]. Note that for Λ being an arbitrary Meyer set, mean almost periodicity of Λ is equivalent to pure point diffraction of Λ. Moreover, any regular model set Λ_W is a Meyer set and thus, Λ_W has pure point diffraction. Whereas this has been known since [4, Theorem 5] and [22, Theorem 1.1], it has recently been put into broader perspective in [37, Theorems 2.18, 2.13]. We also point out that uniform mean almost periodicity of Λ is a stronger notion than mean almost periodicity, that typically results in continuous dynamical eigenfunctions, compare [35, Theorem 5]. In light of these results, it might also be interesting to study variants and aspects of uniformly mean almost periodic point sets beyond the abelian situation.

Appendix A. Nets and Convergence

We briefly explain our definition of net and collect some basic properties. We adopt the setting of Moore–Smith convergence as described in [31, Chap. 2].

A.1. Directed Sets

Let $\mathbb{I}$ be a set, and let $\prec$ be a binary relation on $\mathbb{I}$. Recall the following properties that $(\mathbb{I}, \prec)$ might have.

- Reflexiveness: For all $i \in \mathbb{I}$ we have $i \prec i$.
- Anti-symmetry: For all $i, j \in \mathbb{I}$, we have that $i \prec j$ and $j \prec i$ implies $i = j$.
- Transitivity: For all $i, j, k \in \mathbb{I}$ we have that $i \prec j$ and $j \prec k$ implies $i \prec k$.
- Directedness: For all $i, j \in \mathbb{I}$ there exists $k \in \mathbb{I}$ such that $i \prec k$ and $j \prec k$.

We say that $(\mathbb{I}, \prec)$ is a *partial order* if $\prec$ is reflexive, transitive and antisymmetric. In that case we call $\mathbb{I}$ a partially ordered set or poset. Examples of directed partial orders are $(\mathbb{N}, \leq)$, $(\mathbb{R}, \leq)$, $(\mathbb{R}, \geq)$, $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(X), \supseteq)$, where $\mathcal{P}(X)$ is the collection of subsets of the set X.

A.2. Nets, Cluster Points and Limits

Let $(\mathbb{I}, \prec)$ a directed poset and let X be a topological space. A *net* in X is a map $\mathbb{I} \to X$. We denote nets by $(x_i)_{i \in \mathbb{I}}$. The *range of a net* is the set $\{x_i : i \in \mathbb{I}\} \subseteq X$. We call $x \in X$ a *cluster point* of a net $(x_i)_{i \in \mathbb{I}}$ if for every neighborhood U of x and for every $i_0 \in \mathbb{I}$ there exists $i \succ i_0$ such that $x_i \in U$. A net $(x_i)_{i \in \mathbb{I}}$ *converges* to $x \in X$ if for every neighborhood U of x there exists $i_0 \in \mathbb{I}$ such that $x_i \in U$ for all $i \succ i_0$.

In this case we say that x is a *limit point* of $(x_i)_{i \in \mathbb{I}}$. Any limit point is a cluster point. Limit points are unique if and only if X is a Hausdorff space [31, Chap. 2, Theorem 3].

A.3. Subnets and Cluster Points

Let X be a topological space and let $(\mathbb{I}, \prec)$ and $(\mathbb{J}, \prec)$ be directed preorders. Then a net $(y_j)_{j \in \mathbb{J}}$ in X is a *subnet* of $(x_i)_{i \in \mathbb{I}}$ in X if $y_j = x_{\phi(j)}$ for some function $\phi : \mathbb{J} \to \mathbb{I}$ which is *strictly cofinal*, i.e., for every $i_0 \in \mathbb{I}$ there exists $j_0 \in \mathbb{J}$ such that $j \succ j_0$ implies $\phi(j) \succ i_0$. Note that if $(x_i)_{i \in \mathbb{I}}$ converges to x, then every subnet $(y_j)_{j \in \mathbb{J}}$ converges to x. This is a direct consequence of the subnet definition. Also note that the composition of two cofinal maps is a cofinal map. Hence a subnet of a subnet is a subnet of the original net. The following characterisation of a cluster point is standard [31, Chap. 2, Theorem 6].

Proposition A.1 *A point $x \in X$ is a cluster point of a net $(x_i)_{i \in \mathbb{I}}$ if and only if there exists a subnet $(x_{\phi(j)})_{j \in \mathbb{J}}$ that converges to x.* □

A.4. Nets in $\overline{\mathbb{R}}$

Let us consider nets in the affinely extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$. A net $(x_i)_{i \in \mathbb{I}}$ in $\overline{\mathbb{R}}$ is *increasing* if $x_i \leq x_j$ for all $i \prec j$. It is *decreasing* if $x_i \geq x_j$ for all $i \prec j$. As in the sequence case, we have the following result.

Lemma A.2 *Let $(x_i)_{i \in \mathbb{I}}$ be an increasing net in $\overline{\mathbb{R}}$. Then $(x_i)_{i \in \mathbb{I}}$ converges to its supremum $x = \sup\{x_i : i \in \mathbb{I}\} \leq \infty$.* □

Let $(x_i)_{i \in \mathbb{I}}$ be a net in $\overline{\mathbb{R}}$. Then the net $(s_i)_{i \in \mathbb{I}}$ where $s_i = \inf\{x_j : j \succ i\}$ is increasing, and the net $(t_i)_{i \in \mathbb{I}}$ where $t_i = \sup\{x_j : j \succ i\}$ is decreasing. Note $s_i \leq x_i \leq t_i$ for all $i \in \mathbb{I}$ as $\prec$ is reflexive. As the nets $(s_i)_{i \in \mathbb{I}}$ and $(t_i)_{i \in \mathbb{I}}$ both have a unique limit point, we can define

$$\liminf_{i \in \mathbb{I}} x_i = \lim_{i \in \mathbb{I}} \inf_{j \succ i} x_j = \sup_{i \in \mathbb{I}} \inf_{j \succ i} x_j \geq -\infty, \qquad \limsup_{i \in \mathbb{I}} x_i = \lim_{i \in \mathbb{I}} \sup_{j \succ i} x_j = \inf_{i \in \mathbb{I}} \sup_{j \succ i} x_j \leq \infty.$$

(8.1)

As in the sequence case it is seen that $\liminf_{i \in \mathbb{I}} x_i$ is the smallest cluster point of $(x_i)_{i \in \mathbb{I}}$. Moreover one may easily construct a subnet $(x_{\phi(n)})_{n \in \mathbb{N}}$ which is a sequence and converges to $\liminf_{i \in \mathbb{I}} x_i$. Analogous results hold for $\limsup_{i \in \mathbb{I}} x_i$.

Appendix B. Box Decompositions in σ-Compact Groups

Some σ-compact locally compact groups admit Følner sequences of hierarchically nested monotiles. Such sequences can be used for box decomposition arguments. We indicate in this section how to prove equality of uniform Leptin and Beurling densities via this method, as was done in [24, Section 7] for $G = \mathbb{R}^d$ and $\nu = \delta_\Lambda$ being a Dirac comb over a uniformly discrete set Λ.

In a σ-compact group we call a compact set A a *right monotile for G* if there is a countable set T such that $A \cdot T = G$ and for each $s, t \in T$ with $s \neq t$, we have $\mathring{A}s \cap \mathring{A}t = \varnothing$, where as above, $\mathring{A}$ denotes the interior of the set A. Analogously, one defines left monotiles for G. In the following we will exclusively deal with right monotiles and refer to those as *monotiles for G*.

Definition B.1 We call a Følner sequence $(A_n)_{n \in \mathbb{N}}$ a *monotile Følner sequence* if for each $n \in \mathbb{N}$, the set A_n a is a monotile for G. We say that a monotile Følner sequence is *topologically nested* if for all $n \in \mathbb{N}$ there is some $s_n \in G$ such that $A_n \subseteq \mathring{A}_{n+1}s_n$.

Monotile Følner sequences can be found in many amenable groups. In [61], Weiss has shown that all countable linear amenable groups and all residually finite amenable groups admit such sequences. In the abelian situation, Emerson used in [17, Theorem 5] the structure theorem for compactly generated LCA groups for finding monotile Følner sequences. By passing to a subsequence if necessary one can make sure that these sequences are topologically nested. The construction of Emerson has been used in the context of mathematical quasicrystals, see for example the proof of Lemma 1.1 in [56]. Moreover, topologically nested monotile Følner sequences can be constructed explicitly for many homogeneous Lie groups such as the Heisenberg group.

Example B.2 Let $G = H_3(\mathbb{R})$ be the 3-dimensional Heisenberg group with its group multiplication given by

$$(a, b, c) \cdot (x, y, z) = (a + x, b + y, c + z + ay).$$

For each $n \in \mathbb{N}$, the set $F_n := [0, n)^2 \times [0, n^2)$ is a fundamental domain for the (uniform) lattice

$$\Gamma_n := \{(k, l, m) : k, l \in n\mathbb{Z}, m \in n^2\mathbb{Z}\}.$$

Now it is easy to see that by setting $A_n := \overline{F}_n$ for $n \in \mathbb{N}$, one obtains a monotile strong Følner sequence for G. Moreover, we have $A_n \cdot t_n \subseteq \mathring{A}_{n+1}$ when setting $t_n = (\varepsilon, \varepsilon, \varepsilon)$, with $0 < \varepsilon < 1$. Therefore, $(A_n)_{n \in \mathbb{N}}$ is also topologically nested.

For illustration, let us give a box decomposition proof of $B_{\mathcal{A}}^- \leq L^-$ when $\mathcal{A} = (A_n)_{n \in \mathbb{N}}$ is a topologically nested monotile strong Følner sequence in a σ-compact amenable unimodular group.

Proof of $B_{\mathcal{A}}^- \leq L^-$ Let $\mathcal{A} = (A_n)_{n \in \mathbb{N}}$ be a Følner sequence of monotiles as of Definition B.1 and suppose that $\mathcal{A}$ is also a strong Følner sequence. We show

$$L^- = \sup_{K \in \mathcal{K}} \inf_{A \in \mathcal{K}_p} \frac{\nu(KA)}{m(A)} \geq B_{\mathcal{A}}^- = \lim_{n \to \infty} \inf_{s \in G} \frac{\nu(A_n s)}{m(A_n)}.$$

Setting $B_n := \mathring{A}_n = A_n \setminus \partial A_n$ we note that we have

$$B_{\mathcal{A}}^- = \lim_{n \to \infty} \inf_{s \in G} \frac{\nu(B_n s)}{m(A_n)}.$$

Indeed, this follows from standard estimates, using $A_n s \setminus B_n s = \partial A_n s \subseteq \partial_U A_n s$ for any relatively compact unit neighborhood U, and using the Følner property together with

$$\lim_{n\to\infty} \sup_{s\in G} \frac{\nu(\partial_U A_n s)}{m(A_n)} = 0 \,,$$

cf. Lemma 5.11. Fix arbitrary $\varepsilon > 0$. Take $n \in \mathbb{N}$ large enough such that for all $s \in G$ we have

$$\frac{\nu(B_n s)}{m(A_n)} \geq B_{\mathcal{A}}^- - \varepsilon \,.$$

Consider an arbitrary compact set $A \in \mathcal{K}_p$. Since $B_n t_n = \mathring{A}_n t_n \supseteq A_{n-1}$ for some $t_n \in G$, by assumption on a topologically nested monotile Følner sequence, we find a finite set I and $s_i \in G$ for $i \in I$ such that

$$A \subseteq \bigcup_{i\in I}^{\cdot} B_n s_i \subseteq \bigcup_{i\in I} A_n s_i \subseteq KA,$$

where $K = A_n A_n^{-1}$, compare the argument for the inclusions (6.2) in the proof of Proposition 6.1. Now we can estimate for n large enough

$$\nu(KA) \geq \nu\left(\bigcup_{i\in I} B_n s_i\right) = \sum_{i\in I} \nu(B_n s_i) \geq |I| \cdot m(A_n) \cdot (B_{\mathcal{A}}^- - \varepsilon)$$

$$\geq m\left(\bigcup_{i\in I} A_n s_i\right) \cdot (B_{\mathcal{A}}^- - \varepsilon) \geq m(A) \cdot (B_{\mathcal{A}}^- - \varepsilon) \,.$$

As A was arbitrary, we infer

$$\sup_{K\in\mathcal{K}} \inf_{A\in\mathcal{K}_p} \frac{\nu(KA)}{m(A)} \geq B_{\mathcal{A}}^- - \varepsilon \,.$$

As $\varepsilon > 0$ was arbitrary, we get $L^- \geq B_{\mathcal{A}}^-$.   □

Appendix C. The Densities of Gröchenig, Kutyniok, Seip

We review the density notion from [24], which has inspired our definition of Leptin density. Consider any locally finite measure ν on G. For example, ν might be the counting measure on some right uniformly discrete point set in G. Assume that G admits a uniform lattice. Then densities of ν are defined via point counting with respect to Γ.

Definition C.1 Let G be a unimodular locally compact group. Assume that Γ is a uniform lattice in G. For locally finite measures μ, ν on G we write $\mu \leq \nu$ if for every $\varepsilon > 0$ there exists $K \in \mathcal{K}$ such that for all $A \in \mathcal{K}$ we have $(1 - \varepsilon) \cdot \mu(A) \leq \nu(KA)$. We define

$$\mathcal{D}_\nu^- = \sup\{\alpha \in [0, \infty) : \alpha \delta_\Gamma \leq \nu\}, \qquad \mathcal{D}_\nu^+ = \inf\{\alpha \in [0, \infty) : \nu \leq \alpha \delta_\Gamma\}.$$

Here δ_Γ is the point counting measure associated to Γ via $\delta_\Gamma(A) = \mathrm{card}(A \cap \Gamma)$.

Proposition C.2 *Let G be a unimodular locally compact group which admits a uniform lattice Γ. Let ν be any locally finite measure on G. We then have*

$$L_\Gamma \cdot \mathcal{D}_\nu^- = L_\nu^-, \qquad L_\Gamma \cdot \mathcal{D}_\nu^+ = L_\nu^+,$$

where L_Γ denotes the Leptin density of the lattice Γ.

Proof We argue $L_\Gamma \cdot \mathcal{D}^- = L^-$, the proof for the upper density is analogous.
(i) We collect some prerequisites from the proof of Proposition 6.1. Let F be a measurable relatively compact left-fundamental domain for Γ and note $L_\Gamma \cdot m(F) = 1$. Defining $K = \overline{F^{-1}F}$, we have for any $A \in \mathcal{K}$ the estimates

$$m(F) \cdot \delta_\Gamma(A) \leq m(KA), \qquad m(F) \cdot \delta_\Gamma(KA) \geq m(A).$$

(ii) To show $\mathcal{D}^- \leq m(F) \cdot L^-$, we assume without loss of generality $\mathcal{D}^- > 0$. Consider any $0 < d < \mathcal{D}^-$. Then by Definition C.1 there exists $K' \in \mathcal{K}$ such that for all $A \in \mathcal{K}$ we have $d\, \delta_\Gamma(A) \leq \nu(K'A)$. This implies $\nu(K'KA) \geq d\, \delta_\Gamma(KA) \geq d \cdot m(A)/m(F)$. We thus have $d/m(F) \leq L^-$. As $0 < d < \mathcal{D}^-$ was arbitrary, we conclude $\mathcal{D}^- \leq m(F) \cdot L^-$.
(iii) To show $m(F) \cdot L^- \leq \mathcal{D}^-$, we assume without loss of generality $L^- > 0$. Take any $0 < d < L^-$. Then by definition of the lower Leptin density, cf. Definition 3.3, there exists $K' \in \mathcal{K}$ such that for all $A \in \mathcal{K}$ we have $d \cdot m(KA) \leq \nu(K'KA)$. This implies $m(F) \cdot d\, \delta_\Gamma(A) \leq d \cdot m(KA) \leq \nu(K'KA)$. We thus obtain $m(F) \cdot d < \mathcal{D}^-$. As $0 < d < L^-$ was arbitrary, we conclude $m(F) \cdot L^- \leq \mathcal{D}^-$. $\qquad\square$

References

1. Argabright, L.N., de Lamadrid, J.G.: Fourier Analysis of Unbounded Measures on Locally Compact Abelian Groups, vol. 145. Memoirs of the American Mathematical Society, Providence (1974)
2. Baake, M., Grimm, U.: Aperiodic Order. Vol. 1. A Mathematical Invitation, Encyclopedia of Mathematics and its Applications, vol. 149. Cambridge University Press, Cambridge (2013)
3. Baake, M., Lenz, D.: Dynamical systems on translation bounded measures: pure point dynamical and diffraction spectra. Ergod. Theory Dyn. Syst. **24**, 1867–1893 (2004)
4. Baake, M., Moody, R.V.: Weighted Dirac combs with pure point diffraction. J. Reine Angew. Math. **573**, 61–94 (2004)
5. Bader, U., Caprace, P.-E., Gelander, T., Mozes, S.: Lattices in amenable groups. Fund. Math. **246**, 217–255 (2019)
6. Beckus, S., Hartnick, T., Pogorzelski, F.: Linear repetitivity beyond abelian groups. arxiv: 2001.10725 (2020)

7. Beckus, S., Pogorzelski, F.: Delone dynamical systems and spectral convergence. Ergod. Theory Dyn. Syst. **40**, 1510–1544 (2020)
8. Beurling, A.: The Collected Works of Arne Beurling. Vol. 2 Harmonic Analysis. Birkhäuser, Boston (1989)
9. Björklund, M., Hartnick, T.: Approximate lattices. Duke Math. J. **167**, 2903–2964 (2018)
10. Björklund, M., Hartnick, T.: Spectral theory of approximate lattices in nilpotent Lie groups. Math. Annalen (2021) https://doi.org/10.1007/s00208-021-02329-w
11. Björklund, M., Hartnick, T., Pogorzelski, F.: Aperiodic order and spherical diffraction, I: auto-correlation of regular model sets. Proc. Lond. Math. Soc. **116**, 957–996 (2018)
12. Björklund, M., Hartnick, T., Pogorzelski, F.: Aperiodic order and spherical diffraction, II: translation bounded measures on homogeneous spaces. Math. Z. **300**, 1157–1201 (2022)
13. Björklund, M., Hartnick, T., Pogorzelski, F.: Aperiodic order and spherical diffraction, III: the shadow transform and the diffraction formula. J. Funct. Anal. **281**, 109265 (2021)
14. Ceccherini-Silberstein, T., Coornaert, M., Krieger, F.: An analogue of Fekete's lemma for subadditive functions on cancellative amenable semigroups. J. Anal. Math. **124**, 59–81 (2014)
15. Deitmar, A., Echterhoff, S.: Principles of Harmonic Analysis. Springer, New York (2009)
16. Downarowicz, T., Huczek, D., Zhang, G.: Tilings of amenable groups. J. Reine Angew. Math. **747**, 277–298 (2019)
17. Emerson, W.: Ratio properties in locally compact amenable groups. Transactions of the American Mathematical Society **133**, 179–204 (1968)
18. Feldman, L., Greenleaf, F.P.: Existence of Borel transversals in groups. Pac. J. Math. **25**, 455–461 (1968)
19. Fischer, V., Ruzhansky, M.: Quantization on Nilpotent Lie Groups. Birkhäuser/Springer, Cham (2016)
20. Führ, H., Gröchenig, K., Haimi, A., Klotz, A., Romero, J.L.: Density of sampling and interpolation in reproducing kernel Hilbert spaces. J. Lond. Math. Soc. **96**, 663–686 (2017)
21. Gabardo, J.P.: Convolution Inequalities for Positive Borel Measures on $\mathbb{R}^d$ and Beurling Density. Excursions in Harmonic Analysis, vol. 2, pp. 23–47. Appl. Numer. Harmon. Anal., Birkhäuser/Springer, New York (2013)
22. Gouéré, J.-B.: Quasicrystals and almost periodicity. Commun. Math. Phys. **255**, 655–681 (2005)
23. Greenleaf, F.: Ergodic theorems and the construction of summing sequences in amenable locally compact groups. Commun. Pure Appl. Math. **26**, 29–46 (1973)
24. Gröchenig, K., Kutyniok, G., Seip, K.: Landau's necessary density conditions for LCA groups. J. Funct. Anal. **255**, 1831–1850 (2008)
25. Guihéneuf, P.-A.: Model sets, almost periodic patterns, uniform density and linear maps. Rev. Mat. Iberoam. **34**, 455–474 (2018)
26. Guihéneuf, P.-A., Meyer, Y.: Almost periodic patterns. Preprint (2014)
27. Hauser, T.: Relative topological entropy for actions of non-discrete groups on compact spaces in the context of cut and project schemes. J. Dyn. Differ. Equ. **33**, 891–912 (2021)
28. Hof, A.: On diffraction by aperiodic structures. Commun. Math. Phys. **169**, 25–43 (1995)
29. Huck, C., Richard, C.: On pattern entropy of weak model sets. Discret. Comput. Geom. **54**, 741–757 (2015)
30. Keller, G., Richard, C.: Dynamics on the graph of the torus parametrisation. Ergod. Theory Dyn. Syst. **28**, 1048–1085 (2018)
31. Kelley, J.L.: General Topology. D. Van Nostrand Company, Toronto-New York-London (1955)
32. Krieger, F.: Le lemme lemme d'Ornstein-Weiss d'après Gromov. In: Hasselblatt, B. (ed.) Dynamics, Ergodic Theory, and Geometry, Math. Sci. Res. Inst. Publ. vol. 54, pp. 99–111. Cambridge Univ. Press, Cambridge (2007)
33. Lagarias, J.C., Wang, Y.: Substitution Delone sets. Discret. Comput. Geom. **29**, 175–209 (2003)
34. Landau, H.J.: Necessary density conditions for sampling and interpolation of certain entire functions. Acta Math. **117**, 37–52 (1967)
35. Lenz, D.: Continuity of eigenfunctions of uniquely ergodic dynamical systems and intensity of Bragg peaks. Commun. Math. Phys. **287**, 225–258 (2009)
36. Lenz, D., Richard, C.: Pure point diffraction and cut and project schemes for measures: the smooth case. Math. Z. **256**, 347–378 (2007)
37. Lenz, D., Spindler, T., Strungaru, N.: Pure point diffraction and mean, Besicovitch and Weyl almost periodicity. http://arxiv.org/abs/2006.10821v1 (2020)
38. Leptin, H.: On a certain invariant of a locally compact group. Bull. Am. Math. Soc. **72**, 870–874 (1966)

39. Lindenstrauss, E., Weiss, B.: Mean topological dimension. Isr. J. Math. **115**, 1–24 (2000)
40. Machado, S.: Approximate lattices and Meyer sets in nilpotent Lie groups. Discr. Anal., No. 1, 18 pp. (2020)
41. Meyer, Y.: A propos de la formule de Poisson. Fac. Sci., Univ. Paris, Orsay (1969)
42. Meyer, Y.: Nombres de Pisot, Nombres de Salem et Analyse Harmonique, Lecture notes in mathematics **117**, Springer Berlin-New York, 1970. Cours Peccot donné au Collège de France en avril-mai (1969)
43. Meyer, Y.: Algebraic Numbers and Harmonic Analysis. North-Holland Publishing Co, American Elsevier Publishing Co, Inc, Amsterdam-London, New York (1972)
44. Mitkovski, M., Ramirez, A.: Density results for continuous frames. J. Fourier Anal. Appl. **26**, 56 (2020)
45. Moody, R.V.: Meyer sets and their duals. In: Moody, R.V. (ed.) The Mathematics of Long-Range Aperiodic Order, pp. 403–441. NATO ASI Series vol. C 489, Kluwer, Dordrecht (1997)
46. Moody, R.V.: Model sets: a survey. In: Axel, F., Dénoyer, F., Gazeau, J. (eds.) From Quasicrystals to More Complex Systems, EDP Sciences, pp. 145–166. Les Ulis and Springer, Berlin (2000)
47. Moody, R.: Uniform distribution in model sets. Can. Math. Bull. **45**, 123–130 (2002)
48. Müller, P., Richard, C.: Ergodic properties of randomly coloured point sets. Can. J. Math. **65**, 349–402 (2013)
49. Ornstein, D.S., Weiss, B.: Entropy and isomorphism theorems for actions of amenable groups. J. Anal. Math. **48**, 1–141 (1987)
50. Paterson, A.L.T.: Amenability, Mathematical Surveys and Monographs, vol. 29. American Mathematical Society, Providence (1988)
51. Pogorzelski, F., Schwarzenberger, F.: A Banach space-valued ergodic theorem for amenable groups and applications. J. Anal. Math. **130**, 19–69 (2016)
52. Richard, C., Schumacher, C.: On sampling and interpolation by model sets. J. Fourier Anal. Appl. **26**, 39 (2020)
53. Richard, C., Strungaru, N.: A short guide to pure point diffraction in cut-and-project sets. J. Phys. A: Math. Theor. **50**, 154003 (2017)
54. Richard, C., Strungaru, N.: Pure point diffraction and Poisson summation. Ann. Henri Poincaré **18**, 3903–3931 (2017)
55. Schlottmann, M.: Cut-and-project sets in locally compact Abelian groups. In: Quasicrystals and discrete geometry (Toronto, ON, 1995), Fields Inst. Monogr. 10, American Mathematical Society, Providence, RI, pp. 247–264 (1998)
56. Schlottmann, M.: Generalized model sets and dynamical systems. In: Baake, M. (ed.) Directions in Mathematical Quasicrystals, pp. 143–159. CRM Monograph Series, American Mathematical Society, Providence (2000)
57. Schreiber, J.-P.: Sur la notion de modèles dans les groupes abéliens localement compacts. C.R. Acad. Sci. Paris Sér. A-B **272**, A30–A32 (1971)
58. Schreiber, J.-P.: Approximations diophantiennes et problèmes additifs dans les groupes abéliens localement compacts. Bull. de la S.M.F. 101, 297–332 (1973)
59. Siegel, C.L.: Discontinuous groups. Ann. Math. **44**, 674–689 (1943)
60. Tempelman, A.: Ergodic Theorems for Group Actions, Mathematics and Its Applications, vol. 78. Kluwer Academic Publishers Group, Dordrecht (1992)
61. Weiss, B.: Monotileable amenable groups. Topology, ergodic theory, real algebraic geometry, 257–262, Amer. Math. Soc. Transl. Ser. 2, 202, Adv. Math. Sci., **50**, Americam Mathematical Society, Providence (2001)
62. Zorin-Kranich, P.: Return times theorem for amenable groups. Isr. J. Math. **204**, 85–96 (2014)