The Two Physics Governing the One-Dimensional Cubic Nonlinear Schrödinger Equation

Ana Mucalica

A thesis submitted to the Department of Mathematics and Statistics in partial fulfillment of the requirements for the degree of Bachelor of Science

in

Mathematics Honours

Supervisor: Dr. Ion Bica

Department of Mathematics and Statistics

MacEwan University

Edmonton, Alberta

Winter 2021

Acknowledgements

This thesis and the research behind it would not have been possible without the exceptional support of my faculty mentor, Dr. Ion Bica. His passion, genius and, superb attention to detail have been an inspiration and have kept me motivated throughout my work. The learning I have gained through Dr. Bica's mentorship is truly invaluable in my future academic endeavours. I am also grateful to MacEwan's Department of Mathematics and Statistics for numerous helpful resources, and opportunities to expand on my knowledge and research skills.

Dedications

I dedicate this thesis work to my dearest father, Igor, who has always been a constant source of support and inspiration throughout my life and especially during the challenges of undergraduate school. My ambition, work ethic, innate curiosity, and of course, my wit, are only some of the many virtues my father has inspired in me. For this, I will always be grateful.

The Two Physics Governing the One-Dimensional Cubic Nonlinear Schrödinger Equation

Ana Mucalica

Abstract

In 1926, in his quest to explain the quantum probabilistic nature of particles, Erwin Schrödinger proposed a nonrelativistic wave equation that required only one initial condition, i.e., the initial displacement of an electron. His equation describes the wave-particle duality discovered by Louis de Broglie in 1924. Furthermore, Schrödinger's wave equation is dimensionless, allowing the equation to be a mathematical model describing different physical phenomena. Introducing nonlinearity into the Schrödinger equation, we worked with the so-called self-focusing nonlinear Schrödinger equation. We showed that when the nonlinearity is perfectly balanced with the dispersion, the self-focusing nonlinear Schrödinger model describes the propagation of a soliton. In 1968 Peter Lax introduced the "Lax Pair," a pair of time-dependent matrices/operators describing the nature of a nonlinear evolution partial differential equation, to discuss solitons in continuous media. This procedure is what we call the scattering method for describing mathematically nonlinear processes in physics. We used the scattering method to find the Lax Pair for the nonlinear Schrodinger model, and we showed that the equation is a compatibility condition for the AKNS system. In 1974, Ablowitz, Kaup, Newell, and Segur (AKNS) introduced the inverse scattering transform to solve evolution nonlinear partial differential equations arising from compatibility conditions for the AKNS system. Rather than using the inverse scattering transform, we showed an intuitive approach in revealing the formation and propagation of a soliton for the self-focusing nonlinear Schrodinger equation, using a novel approach via cnoidal waves. The work will also include a novel theorem describing the steepening of the wavefront due to nonlinearity.

Contents

1 Introduction

1.1 Short History of Erwin Schrödingers Scientific Contributions

Based on the references [3], [4], and [6], we provide a short history of Erwin Schrödinger's scientific contributions. Erwin Schrödinger was a Nobel Prize-winning Austrian-Irish physicist who was one of the founders of quantum mechanics. Schrödinger was born in 1887 in Vienna into a wealthy and highly intellectual family. After receiving his doctorate in 1910, he published various papers ranging from magnetism, radioactivity and X-rays to Brownian motion. In 1921 Schrödinger was appointed to the chair for theoretical physics at the University of Zurich, during which period he published on general relativity, probability theory, dielectric phenomena, three- and four-color theories of vision, and atomic theory in particular. Following this fruitful period, in 1926, Schrödinger invented wave mechanics and published what is now known as the Schrödinger wave equation. The invention of wave mechanics is what got Schrödinger a Nobel Prize in Physics in 1933. However this year also brought turmoil in his life when Adolf Hitler assumed power and Schrödinger was forced to settle in Oxford, England. Following this brief and unproductive period, he accepted a position in Graz, Austria, in 1936. Unfortunately he fled Austria again in 1938 when he was abruptly dismissed and Hitler's forces invaded the country. In 1939 Schrödinger was appointed as the first director of the school of theoretical physics at the Dublin Institute for Advanced Studies. What followed was vastly productive period in his career, during which he published many works on the application and statistical interpretation of wave mechanics and problems concerning the relationship between general relativity and wave mechanics. During this time he also published a book "What is Life?" which was of great importance in molecular biology as it presented pioneering work on the relationship between physics and living systems.

In modern-day, Schrödinger is best remembered for his invention of wave mechanics and publishing of his wave equation in 1926. However, in the same year, the development of quantum mechanics was further advanced when another theoretical physicist Werner Heisenberg, established foundations of the matrix mechanics. Heisenberg insisted that quantum mechanics should only use directly observable quantities, which is why he represented the variable describing the position of the electron as Fourier series: the frequencies of the terms in the series were associated with the measurable frequencies of radiation emitted by the atoms, and the amplitudes of these terms were interpreted as the measurable strength of these transitions, [3]. Both frequencies and amplitudes, as well as position variables, were represented by matrices. This new interpretation of quantum mechanics made the new wave mechanics odd and difficult to interpret, but the model still successfully reproduced the hydrogen spectrum. This result was of great importance as the very motivation behind Schrödinger's invention of wave mechanics was to overcome some difficulties associated with Niels Bohr's theory of the hydrogen atom. However in 1926 Schrödinger was thirty-nine, and feeling pressure of the age gap between him and Heisenberg, he was disheartened by the novelty of matrix mechanics. He wrote, [3]:

> I was discouraged (abgeschrekt), if not repelled (abgestossen), by what appeared to me a rather difficult method of transcendental algebra, defying any visualization (Anschaulichkeit).

In response to Heisenberg's and Schrödinger's models, the Danish physicist Niels Bohr suggested that both particle and wave physics were equally valid models, and depending on each context, both models could be used to describe the world. However Schrödinger believed in the fundamental continuity of matter, while Bohr was insistent that electrons made jumps between quantum states. Consequently, there was a dispute over the interpretation of the wave function. Schrödinger attempted to interpret the square of the function in terms of a spatial charge density, however, this interpretation was widely rejected. On the other hand, German physicist Max Born suggested that the wave function should be interpreted as a probability amplitude, where the square of the wave function gives the probability that an electron can be found at a certain point in space. This controversial interpretation implied that the location of a particle can not be ascertained with certainty, but the wave function enables one to work out the probability that the particle will be found in a certain place. Heisenberg, in his Uncertainty Principle, proposed that one can not measure both the position and momentum of an electron at the same time. Since in the Schrödinger's equation only the position of the particle is well defined, the momentum satisfies the Uncertainty Principle, i.e., the particle can start moving in any direction. In this respect, it is commonly accepted that Schrödinger's equation is probabilistic in nature.

One of the more influential works at the time was the work of a French physicist Louis de Broglie. He had had an idea that Einstein's 1905 relation $E = hv$, between the energy of a photon and its frequency, should be generalized to material particles such as electrons to which he assigned a fictitious wave. Inspired by de Broglie's suggestion that a particle is merely a wave crest on a background of waves, Schrödinger used this idea to attempt to eliminate quantum jumps, i.e., an abrupt transition of an electron from one level to another. So, given that it is widely accepted that Schrödinger and Heisenberg fathered quantum theory, de Broglie and Einstein can, consequently, be regarded as its godfathers.

While Schrödinger worked on the problem of wave mechanics, between January and June 1926, he sent a paper setting out his theory of wave mechanics to the Annalen der Physik, [3]. He presented a fundamental equation for the variable Ψ, which described the motion of electron and is now known as wavefunction. Later in 1926, Schrödinger was able to show that there is a formal equivalence between his model of wave mechanics and Heisenberg's model of matrix mechanics.

Schrödinger's legacy extends far greater than the invention of wave mechanics, as he made contributions to nearly every branch of physics. However, Schrödinger was somewhat conservative in his traditional way of thinking about physics, unable to let go of the ideal of continuous and deterministic nature. For these reasons, he was never able to embrace his intellectual child and remained isolated from the mainstream of quantum mechanics until his death in 1961.

1.2 The Physical Nature of the One-dimensional Cubic Nonlinear Schrödinger Equation

Note: Further on, throghout the present work, we will use the acronym NLS to stand for the One-dimensional Cubic Nonlinear Schrödinger Equation.

1.2.1 Dispersive Nature of the One-dimensional Cubic Nonlinear Schrödinger Equation

In his article published in 1926, [15], Schrödinger developed a wave-equation for describing dispersive wave-phenomena, which was suitable for micro-mechanical problems.
The one-dimensional Schrödinger equation can be represented through the following partial differential equation (PDE):

$$iu_t + du_{xx} = 0,\ d \neq 0\ constant \tag{1}$$

where u is a complex-valued function. The complex coefficient i of u_t makes the term $iu_t + du_{xx}$ correspond to dispersion, i.e., different wavelengths travel at different speeds. Indeed, the plane wave

$$u(x,t) = e^{i(kx-\omega t)} \tag{2}$$

is the solution of the equation (1) if and only if it satisfies the dispersion relation

$$\omega = d \cdot k^2 \tag{3}$$

which shows the dispersive behaviour of the wave-phenomena described by the equation (1).

The equation (1) describes the time evolution of a dispersive wave with a potential $|u|^2 = u \cdot \overline{u}$. $\overline{u}$ is the complex conjugate of u, which under only one initial condition that establishes the initial position of the particle, may move in any direction.

The idea of developing such an equation, arose from scientific discussions regarding the non-relativistic wave-phenomena, i.e., wave-phenomena in quantum mechanics.

From relativistic point of view, one can use the classic wave-equation

$$u_{tt} = c^2 u_{xx},\ c > 0 \ constant \tag{4}$$

which requires two initial conditions, specifying the initial position and initial velocity of a particle.

From micro-mechanics point of view, in 1924 De Broglie discovered the wavelike character associated with electrons, [8], which can not assure the initial position and momentum of an electron at the same time. Thus, Schrödinger looked for the simplest wave equation that would require only one initial condition, the initial position of a particle, and to describe the wave-particle duality discovered by de Broglie. In 1927, Heisenberg, through the Uncertainty Principle, formalized de Broglies observation, validating at the same time the Schrödingers wave equation published in 1926.

Before Heisenberg, the scientists were in common accord that if one knows the initial position and initial momentum of a particle, then from a theoretical point of view it is possible to determine the position and momentum at any moment in time for that particle. Through his principle, Heisenberg showed that that might not be the case, as in a micro-mechanics frame one could not know, at the same time, the initial position and initial momentum of a particle. His Uncertainty Principle states as follows, [13]:

> The product of the uncertainty in position and the uncertainty in momentum
> is necessarily greater than a quantity of order $\hbar$

which translates as follows

If ψ_1 is a wave that occupies a region of order Δx in the position space (i.e., x-space), and ψ_2 is a wave that occupies a region of order Δp in the momentum space (i.e., p-space), then

$$\Delta x \cdot \Delta p > O(\hbar) \tag{5}$$

The equation (1) is integrable using the Fourier transform. Let us consider the following initial value problem (IVP)

$$iu_t + du_{xx} = 0,\ d > 0 \tag{6}$$

$$u(x,0) = u_0(x) \tag{7}$$

Remark: The equation (6) is the equation (1) from this section on which we imposed initial Cauchy data, i.e., the initial condition (7).

We choose the function u_0 from the Schwartz space, $S(\mathbb{R}^1)$, therefore it will vanish exponentially at infinity; consequently, we will look for the solution to the IVP (6-7) that is vanishing at infinity.

The integration technique for the IVP (6-7) is the Fourier transform integrating technique, given by the following chain of operations

$$u_0(x) \xrightarrow{\text{fft}} \hat{u}_0(k) \xrightarrow[\text{with respect to time (t)}]{\text{integration}} \hat{u}(k,t) \xrightarrow{\text{ifft}} u(x,t) \tag{8}$$

where $u_0(x)$ is the initial condition, and $u(x,t)$ is the solution of the IVP (6-7).

Remark: fft stands for forward Fourier transform and ifft stands for inverse of Fourier transform.

Applying Fourier transform onto the equation (6) we obtain

$$i\hat{u}_t + d\hat{u}_{xx} = 0 \tag{9}$$

where

$$\hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)e^{-ikx}dx \tag{10}$$

is the Fourier transform of $u(x,t)$ after the variable x.

Using [10], we get

$$\hat{u}_{xx} = ik\hat{u}_x = (ik)^2\hat{u}(k) = -k^2\hat{u}(k) \tag{11}$$

Then the IVP(6-7) becomes

$$i\hat{u}_t - dk^2\hat{u} = 0 \tag{12}$$

$$\hat{u}(k,0) = \hat{u_0}(k) \tag{13}$$

where

$$\hat{u_0}(k) = \int_{-\infty}^{\infty} u_0(x)e^{-ikx}dx \tag{14}$$

The equation (12) is a first order ordinary differential equation whose solution is obtained as follows

$$i\hat{u}_t - dk^2\hat{u} = 0 \implies \hat{u}_t = -idk^2\hat{u} \implies \int \frac{1}{\hat{u}}d\hat{u} = \int -idk^2dt \implies \ln|\hat{u}| = -idk^2t + C,\ C \in \mathbb{R} \tag{15}$$

Solving (15) for $\hat{u}$, and redefining the constant of integration C, we obtain

$$\hat{u} = Ce^{-idk^2t},\ C \in \mathbb{R} \tag{16}$$

Imposing the initial condition (13), we get the solution of the IVP (12-13) to be

$$\hat{u}(k,t) = \hat{u_0}(k)e^{-idk^2t} \tag{17}$$

Applying inverse Fourier Transform to the function in (17), we obtain the solution of the IVP (6-7)

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u_0}(k)e^{i(kx-dk^2t)}dk \tag{18}$$

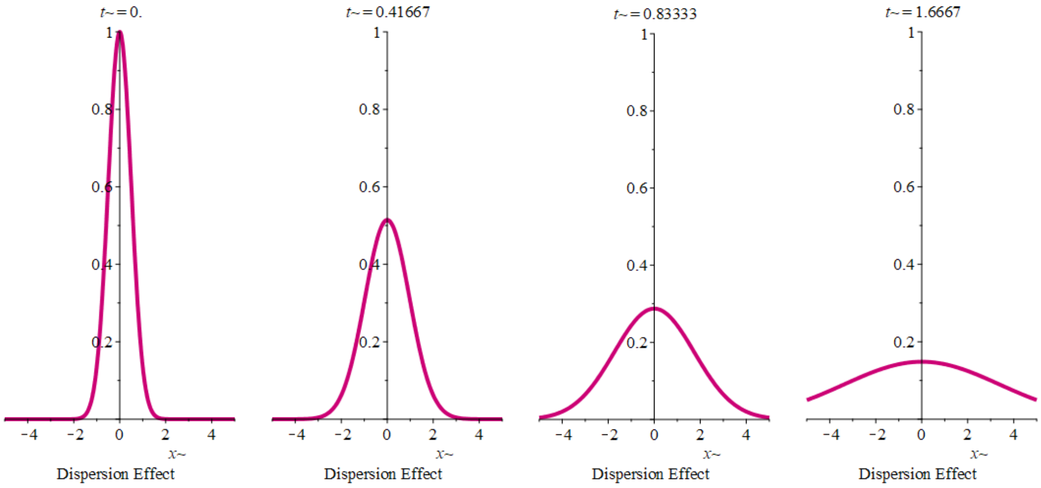
Using the MAPLE software we illustrate the dispersive nature of the PDE (1), see Figure 1, by solving the IVP (6-7) for the following data

$$d = 1 \text{ and } u_0(x) = e^{-x^2} \tag{19}$$

Dispersion - MAPLE simulation

```
restart;
with(plots):
with(PDETools):
with(inttrans):

#1D Linear Schrodinger Equation - LS
LS:=I*diff(u(x,t),t)+d*diff(u(x,t),x$2)=0

#Parameter for LS
print('The value assigned for parameter d');
d:=readstat("Input the value of d");

#Assumptions on Variables
assume(t>0); assume(x,real);

#Forward Fourier of LS
fft:=inttrans[fourier](LS,x,k);

#PDE Derived from Forward Fourier of LS
PDE:=subs(fourier(u(x,t),x,k)=u_fft((k,t),fft);

#Initial Cauchy Data
print('The initial Cauchy data for LS');
u[0]:=readstat("Input the initial Cauchy data for LS");

#Forward Fourier of the Initial Condition
ic:=u_fft(k,0)=inttrans[fourier](u[0],x,k);

#Solving the IVP and Convert Output to ListTools
sol_fft:=convert(pdsolve(PDE,ic),list);
```

```
#Appliying Inverse Fourier
sol_IVP:=simplify(invfourier(sol_fft[2],k,x));

#Potential
V:=simplify(evalc(Re(sol_IVP)^2+Im(sol_IVP)^2);

#Animation
r:=readstat("Input the x-range"):
w:=readstat("Input curve thickness"):
c_a:=readstat("Input curve colour"):
T_max:=readstat("Input maximum range for animation"):
opts:=thickness=w,color=c_:
plots[animate](plot,[V,x=r,opts],t=0..T_max,labels=[x,'V'],
caption="Dispersion Effect");
```



(a) (b) (c) (d)

Figure 1: The dispersion phenomenon described by the Schrödinger equation (1) for $d = 1$ and $u_0(x) = e^{-x^2}$.

1.2.2 Nonlinearity of the One-dimensional Cubic Nonlinear Schrödinger Equation

The second type of physics in NLS is the nonlinearity. The nonlinearity in NLS is expressed through the following PDE:

$$iu_t + \kappa|u|^2 u = 0\ ,\ \ \kappa \in \mathbb{R}^* \tag{20}$$

where the term $|u|^2u$ is so-called cubic nonlinearity. The PDE (20) is nonlinear, therefore the solutions of it will not satisfy the superposition principle, i.e., any change in the input will not be proportional with changes in the output.

The latter statement reveals a fundamental difference between linearity and nonlinearity; while the output of a linear process is predictable, the output of a nonlinear process is unpredictable as "anything can happen", and "anything" is strongly correlated to the nature of the nonlinearity. This is why we cannot create a general theory for nonlinear phenomena, as they are intimately affected by the type of nonlinearity the process is experiencing.

As the next step in the understanding of the nonlinear nature of NLS, let us solve the following IVP

$$iu_t + \kappa|u|^2 u = 0 \tag{21}$$

$$u(x,0) = u_0(x) \tag{22}$$

where u_0 is a complex valued function such that $|u_0|$ belongs to $S(\mathbb{R}^1)$.

Remark: The equation (21) is the equation (20) from this section on which we imposed initial Cauchy data, i.e., the initial condition (22).

To solve the equation (21) we look for solutions of the following form

$$u(x,t) = r(x)e^{i\omega t} \tag{23}$$

Let us begin by first calculating u_t and $|u|^2$.

$$u_t = r(x)e^{i\omega t} i\omega \tag{24}$$

$$|u|^2 = |r(x)e^{i\omega t}|^2 = |r(x)|^2|e^{i\omega t}|^2 = |r(x)|^2|\cos\omega t + i\sin\omega t|^2 = |r(x)|^2(\cos^2\omega t + \sin^2\omega t) = |r(x)|^2 \tag{25}$$

We can solve for $r(x)$ by substituting our initial condition (22) into the given solution form

(23).

$$u(x,0) = r(x)e^{i\omega 0} = r(x) = u_0(x) \tag{26}$$

Next, we will substitute u, u_t and $|u|^2$ into the equation (21) and then solve for ω.

$$\begin{aligned}
i(r(x)e^{i\omega t}i\omega) + \kappa|r(x)|^2 r(x)e^{i\omega t} &= 0\\
-r(x)e^{i\omega t}\omega + \kappa|r(x)|^2 r(x)e^{i\omega t} &= 0\\
-u_0(x)e^{i\omega t}\omega + \kappa|u_0(x)|^2 u_0(x)e^{i\omega t} &= 0\\
-\omega + \kappa|u_0(x)|^2 &= 0\\
\omega = \kappa|u_0(x)|^2&
\end{aligned} \tag{27}$$

Therefore substituting $\omega = \kappa|u_0(x)|^2$ and $r(x) = u_0(x)$ into the given solution form (23) we get the solution of the IVP (20-21):

$$u(x,t) = u_0(x)e^{i\kappa|u_0(x)|^2 t} \tag{28}$$

The nonlinear term $|u|^2$ in the equation (21) is a so-called correction term, [16], to the frequency of a wave-solution for NLS.

For the IVP (20-21) the term $|u|^2u$ will induce a horizontal compression to the profile of the input Cauchy data, adjusted by either a vertical stretch or a vertical compression, i.e., the term $|u|^2u$ will narrow the profile of the input Cauchy data, and the narrowing will be subject to a vertical stretch/compression.

Theorem 1 Let f be a complex valued function such that $|f|$ belongs to $S(\mathbb{R}^1)$. Let x_l be the x-coordinate of the most left hand side local maximum of $|f|$. Consider the following change of coordinate system

$$
\begin{aligned}
X &= x - x_l \\
Y &= y
\end{aligned}
\tag{29}
$$

Then the transformation

$$|f(X)| \to |f(X)|^3 \tag{30}$$

will induce a nonlinear horizontal compression of the profile of $f = f(X)$, i.e., the graph of $|f(X)|$, for $X \in (-\infty, 0)$ followed by a vertical stretch/compression adjustment. Similarly, if x_r is the x-coordinate of the most right hand side local maximum of $|f|$, and considering the change of coordinate system

$$
\begin{aligned}
X &= x - x_r \\
Y &= y
\end{aligned}
\tag{31}
$$

then the transformation (30) will induce a nonlinear horizontal compression of the profile of $f = f(X)$ for $X \in (0, +\infty)$ followed by a vertical stretch/compression adjustment.

Proof We will prove the case for x_l.

Because $|f|$ belongs to $S(\mathbb{R}^1)$, we have that for each positive integer N, there exists a positive constant C_N such that

$$|f(X)| \leq C_N(1 + |X|)^{-N} \ \forall \ \ X < 0 \tag{32}$$

Because both sides of the inequality (32) are nonnegative for all $X < 0$, we have

$$|f(X)|^3 \leq D_N(1 + |X|)^{-3N} \ \forall \ X < 0 \tag{33}$$

where $D_N = C_N^3$.

The inequalities (32) and (33) tell us that for each positive integer N, $|f(X)|$ and $|f(X)|^3$ are at most a positive constant multiple of $(1+|X|)^{-N}$ and $(1+|X|)^{-3N}$ respectively. Then, the way the graph of $|f(X)|$ will be affected by the transformation (30) will be similar to the way the graph of $(1+|X|)^{-N}$ will be affected by the transformation (30).

Let us see how the graph of $(1+|X|)^{-N}$ is affected by the transformation (30).

Let $X_i \in (-\infty, 0)$, $i = 1, 2$ such that

$$(1+|X_2|)^N = (1+|X_1|)^{-3N}$$

$$\Downarrow$$

$$(1-X_2)^{-N} = (1-X_1)^{-3N} \tag{34}$$

We will prove that (34) implies $X_1 \geq X_2$. This means that the graph of $(1+|X|)^{-3N}$ can be viewed as a nonlinear horizontal compression of the graph of $(1+|X|)^{-N}$ for $X < 0$. Therefore, the graph of $|f(X)|^3$ can be viewed as a nonlinear horizontal compression of the graph of $|f(X)|$. The Y-values of the graph of $|f(X)|$ for $X < 0$ will have the "final" impact in determining the graph of $|f(X)|^3$.

If $|f(X)| \in (0,1)$, then the graph of $|f(X)|^3$ will be a nonlinear horizontal compression of the graph of $|f(X)|$ followed by a vertical compression, so the graph of $|f(X)|^3$ will still be to the right with respect to the graph of $|f(X)|$

If $|f(X)| = 1$ then $|f(X)|^3 = 1$, so no vertical stretch/compression

If $|f(X)| > 1$, then the graph of $|f(X)|^3$ will be a nonlinear horizontal compression of the graph of $|f(X)|$ followed by a vertical stretch, which will position the graph of $|f(X)|^3$ to the left with respect to the graph of $|f(X)|$.

Now we are going to prove that $X_1 \geq X_2$.

Let's assume by contradiction that $X_1 < X_2$, where $X_1 < 0$ and $X_2 < 0$.

$$X_1 < X_2 \implies -X_1 > -X_2 \implies 1 - X_1 > 1 - X_2 \implies (1 - X_1)^{-3N} < (1 - X_2)^{-3N} \quad (35)$$

Then using (34) we have $(1 - X_2)^{-N} = (1 - X_1)^{-3N}$. Therefore the inequality 35 becomes

$$(1 - X_2)^{-N} < (1 - X_2)^{-3N} \quad (36)$$

Clearly, (36) is a false inequality since $1 - X_2 > 1 \implies (1 - X_2)^{-N} > (1 - X_2)^{-3N}$. Hence, we have reached our contradiction, which means that our assumption $X_1 < X_2$ was false, i.e., $X_1 \geq X_2$ must be true.

In a very similar manner we can prove that, for x_r, the graph of $|f(X)|^3$ can be viewed as a non-linear horizontal compression of the graph of $|f(X)|$, followed by a vertical stretch/compression adjustment.

Returning to the IVP (20-21) using formula (28), let us evaluate $|u|^2 u$.

We have,

$$\begin{aligned}
|u|^2 u &= |u_0(x) e^{ik|u_0(x)|^2 t}|^2 u_0(x) e^{ik|u_0(x)|^2 t} \\
&= |u_0(x)|^2 |e^{ik|u_0(x)|^2 t}|^2 u_0(x) e^{ik|u_0(x)|^2 t} \\
&= |u_0(x)|^2 |\cos(k|u_0(x)|^2 t) + i \sin(k|u_0(x)|^2 t)|^2 u_0(x) e^{ik|u_0(x)|^2 t} \\
&= |u_0(x)|^2 \sqrt{\cos^2(k|u_0(x)|^2 t) + \sin^2(k|u_0(x|^2 t)} u_0(x) e^{ik|u_0(x)|^2 t} \\
&= |u_0(x)|^2 u_0(x) e^{ik|u_0(x)|^2 t}
\end{aligned} \quad (37)$$

Hence,

$$|u|^2 u = |u_0(x)|^2 u_0(x) e^{ik|u_0(x)|^2 t} \quad (38)$$

The profile of $|u|^2u$ is then

$$
\begin{aligned}
A(x) &= \sqrt{|u|^2u\ \overline{|u|^2u}} \\
&= \sqrt{|u_0(x)|^2u_0(x)e^{ik|u_0(x)|^2t}\ \overline{|u_0(x)|^2u_0(x)e^{ik|u_0(x)|^2t}}} \\
&= ||u_0(x)|^2u_0(x)e^{ik|u_0(x)|^2t}| \\
&= |u_0(x)|^2|u_0(x)||e^{ik|u_0(x)|^2t}| \\
&= |u_0(x)|^3|\cos(k|u_0(x)|^2t) + i\sin(k|u_0(x)|^2t)| \\
&= |u_0(x)|^3\sqrt{\cos^2(k|u_0(x)|^2t) + \sin^2(k|u_0(x)|^2t)} \\
&= |u_0(x)|^3
\end{aligned} \tag{39}
$$

Thus,

$$A(x) = |u_0(x)|^3 \tag{40}$$

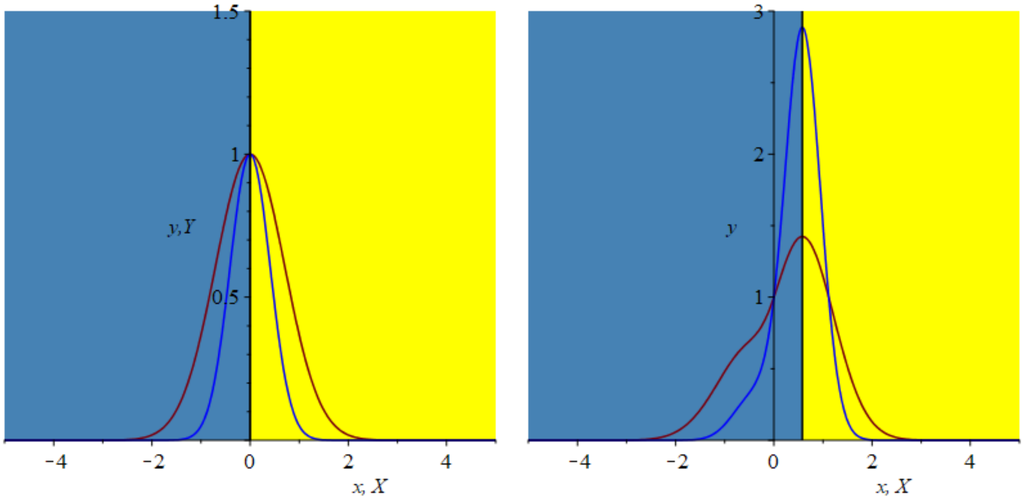
Then, using Theorem 1, the profile of $|u|^2u$, i.e., the graph of $A(x) = |u_0(x)|^3$, is a nonlinear horizontal compression of the profile of the input Cauchy data, i.e., the graph of $|u_0(x)|$, followed by a vertical stretch/compression adjustment.

Using the MAPLE software we exemplify the nonlinear nature of NLS. Figure 2 illustrates the steepening of the wave front due to nonlinearity for the waves described by the equation (20).

Nonlinearity-MAPLE Simulation

```
restart;
with(plots):

#Assumptions on Variables
assume(x,real); assume(t,real);

#The initial condition must vanish at +/- infinity
IC:=u[0][1]+I*u[0][2];

#Example 1
u[0][1]:=cos(x)*exp(-x2);    u[0][2]:=sin(x)*exp(-x2);

#Amplitude of the input data
```

```
A:=simplify(sqrt(evalc(IC*conjugate(IC))));

loc_max:=evalf(maximize(CA,x=-2..2,location));

loc_max_x:=convert(loc_max[2][1][1][1],list);
loc_max_x[2];

#How the nonlinearity affects the input data
A_nonlin:=simplify(CA^3);

p:=plot(CA, x=-5..5):
p_nonlin:=plot(CA_nonlin, x=-5..5, color=blue):
p_axis:=implicitplot(x=loc_max_x[2],
x=-5..5, y=0..1.5, color=black, coloring=["SteelBlue"]=true):
display(p,p_nonlin, p_axis, labels=['x,X','y,Y']);

#Example 2
u[0][1]:=(x2+x+1)*exp(-x2); u[0][2]:=sin(x)*exp(-x2);
IC:=u[0][1]+I*u[0][2] ;

#Amplitude of the input data
A:=simplify(sqrt(evalc(IC*conjugate(IC))));

loc_max:=evalf(maximize(CA,x=-2..2,location));

loc_max_x:=convert(loc_max[2][1][1][1],list);
loc_max_x[2];

#How the nonlinearity affects the input data
A_nonlin:=simplify(CA^3);

p:=plot(CA, x=-5..5):
p_nonlin:=plot(CA_nonlin, x=-5..5, color=blue):
p_axis:=implicitplot(x=loc_max_x[2], x=-5..5,
y=0..3, color=black, coloring=["SteelBlue"]=true):
display(p,p_nonlin, p_axis, labels=['x,X','y']);
```



(a) (b)

Figure 2: (a)Nonlinear horizontal compression of the profile of the input Cauchy data, followed by a vertical stretch adjustment for $u(x,0) = \cos(x)e^{-x^2} + i\sin(x)e^{-x^2}$. (b) Nonlinear horizontal compression of the profile of the input Cauchy data, followed by a vertical stretch adjustment for $u(x,0) = (x^2 + x + 1)e^{-x^2} + i\sin(x)e^{-x^2}$

2 Lax Pair

The Lax pair was introduced by Peter Lax in 1968, [12], to generalize a new method, the scattering method, developed by Gardner, Greene, Kruskal and Miura, [9]. The aim of this method was to find solitary wave (soliton) solutions for "nonlinear equations of evolution".

The concept of Lax's generalization is presented from, [1], as follows:

Consider two differential operators L and M, where L is the operator of the spectral problem

$$L\psi = \lambda\psi \tag{41}$$

and M is the operator governing the associated time evolution of the eigenfunctions $\psi(x,t)$

$$\psi_t = M\psi \tag{42}$$

Differentiating (41) with respect to time, we obtain

$$\frac{\partial}{\partial t}(L\psi) = \frac{\partial}{\partial t}(\lambda\psi) \implies \frac{\partial L}{\partial t}\psi + L\frac{\partial \psi}{\partial t} = \frac{\partial \lambda}{\partial t}\psi + \lambda\frac{\partial \psi}{\partial t} \implies L_t\psi + L\psi_t = \lambda_t\psi + \lambda\psi_t \tag{43}$$

From (42) and (43) we obtain

$$L_t\psi + LM\psi = \lambda_t\psi + \lambda M\psi \implies L_t\psi + LM\psi = \lambda_t\psi + \lambda\psi M \tag{44}$$

Using (41) the equation (44) becomes

$$L_t\psi + LM\psi = \lambda_t\psi + L\psi M \implies L_t\psi + LM\psi - \lambda M\psi = \lambda_t\psi \implies (L_t + LM - ML)\psi = \lambda_t\psi \tag{45}$$

which is equivalent to

$$(L_t + [L, M])\psi = \lambda_t\psi \tag{46}$$

where $[L, M]$ is the operator commutator representing $LM - ML$.

The operators L and M are called Lax pair, and Lax's method was based on the following idea, [12]: Given a linear differential operator L, which satisfies the spectral equation (41), we find an operator M, such that

1. The eigenvalues λ, called the spectral parameter, are time-independent, i.e., $\lambda_t = 0$. Then the equation (46) becomes

$$(L_t + [L, M])\psi = 0 \tag{47}$$

2. The operator $L_t + [L, M]$ is not a differential operator, i.e., it is a multiplicative operator.

Then from equation (47) we obtain

$$L_t + [L, M] = 0 \tag{48}$$

The equation (48) is called Lax equation, and it contains an evolution equation with a first order derivative in time, for suitably chosen differential operators L and M. Lax equation is viewed as a compatibility condition for an integrable evolution equation.

2.1 The AKNS System

In order to find the Lax pair (L, M) for NLS, we need to write the compatibility condition (48) in the matrix form. The matrix form can be viewed as a generalization of Lax equation, and it was developed by Ablowitz, Kaup, Newell, and Segur in their paper published in 1974, [2]. Let us consider two linear equations

$$u_x = Ru \tag{49}$$

$$u_t = Tu \tag{50}$$

where u is a $n-$dimensional vector, and R and T are $n \times n$ matrices.

The compatibility condition between the system (49-50) and Lax equation (48), which is equivalent to the equations (41) and (42), is the existance of a fundamental matrix $\Phi(x, t)$ for the system (49-50), i.e., $\Phi(x, t)$ is a non-singular matrix-valued function whose columns are linearly independent solutions of the system (49-50).

Then we have, [5]

$\Phi(x, t)$ is a fundamental matrix of the system (49-50) if and only if

$$\Phi_x = R\Phi \text{ and } \Phi_t = T\Phi \text{ for all } (x, t) \tag{51}$$

For the compatibility of the equation in (51) we require $\Phi_{xt} = \Phi_{tx}$.

The matrix form of the compatibility condition (48) is obtained by cross differentiation of equations (49) and (50).

$$
\begin{aligned}
u_{xt} &= R_t u + R u_t \\
u_{tx} &= T_x u + T u_x
\end{aligned}
\tag{52}
$$

Equating the cross derivatives, and using (49) and (50), we obtain

$$
R_t u + R u_t - T_x u - T u_x = 0 \implies R_t u + RTu - T_x u - TRu = 0 \tag{53}
$$

which can be written as

$$
R_t - T_x + [R, T] = 0 \tag{54}
$$

where $[R, T] = RT - TR$.

To find the Lax pair (L, M) for a given evolution equation is equivalent to finding the pair (R, T) satisfying the matrix equation (54), which will be expected to generate the same evolution equation. The matrix R is found by considering a linear differential operator L, and assuming that the equations (49) and (41) are equivalent.

Assuring that the equations (50) and (42) are equivalent, we are able to find the matrix T, implicitly to find the operator M, which creates the Lax pair (L, M) of the evolution equation that we study.

As a theoretical exercise, let us consider a linear differential operator L of n-th order, given as follows

$$
L = a_n \frac{\partial^n}{\partial x^n} + a_{n-1} \frac{\partial^{n-1}}{\partial x^{n-1}} + ... + a_1 \frac{\partial}{\partial x} + V \tag{55}
$$

where $a_i \in C\ (\mathbb{R}^2)$, $i = 1..n$, a_n nonvanishing function, and $V \in C^n(\mathbb{R}^2)$. Then the equation (41) is equivalent with the equation (49), where

$$
R = \begin{bmatrix}
0 & 1 & 0 & 0 & \dots & 0 & 0 \\
0 & 0 & 1 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & 1 & \dots & 0 & 0 \\
\multicolumn{7}{c}{\dotfill} \\
0 & 0 & 0 & 0 & \dots & 0 & 1 \\
\frac{1}{a_n}(\lambda - V) & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & -\frac{a_3}{a_n} & \dots & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-1}}{a_n}
\end{bmatrix} \tag{56}
$$

In order to derive the matrix R let us first begin with the $n-$dimensional vector u

$$
u = \begin{bmatrix} u_1 \\ u_2 \\ \cdot\cdot \\ u_n \end{bmatrix} = (u_1, u_2, ..., u_n)^T \tag{57}
$$

Equating equations (49) and (41) we let

$$
u_1 = \psi,\ u_2 = \psi_x,\ u_3 = \psi_{xx}\ , ...,\ u_{n-1} = \psi_{x^{(n-2)}},\ u_n = \psi_{x^{(n-1)}} \tag{58}
$$

Then, taking the partial derivative of u with respect to x, we obtain

$$
u_{1_x} = \psi_x = u_2,\ u_{2_x} = \psi_{xx} = u_3\ , ...,\ u_{(n-1)_x} = \psi_{x^{(n-1)}} = u_n \tag{59}
$$

Now in order to find the term u_{n_x} let us consider equation (41), where L is a differential operator given by (55).

$$a_n\psi_{x^n} + a_{n-1}\psi_{x^{n-1}} + ... + a_1\psi_x + V\psi = \lambda\psi \implies a_n\psi_{x^n} + a_{n-1}u_n + ... + a_1u_2 + Vu_1 = \lambda u_1$$

$$\implies \psi_{x^n} = \frac{\lambda u_1}{a_n} - \frac{Vu_1}{a_n} - \frac{a_1u_2}{a_n} - ... - \frac{a_{n-1}u_n}{a_n} = \frac{(\lambda - V)u_1}{a_n} - \frac{a_1u_2}{a_n} - ... - \frac{a_{n-1}u_n}{a_n} = u_{n_x} \tag{60}$$

Then the $n-$dimensional vector u_x can be written as

$$u_x = (u_{1_x}, u_{2_x}, ..., u_{n_x})^T = (\psi_x, \psi_{xx}, .., \psi_{x^n})^T = (u_2, u_3, ..., \frac{(\lambda - V)u_1}{a_n} - \frac{a_1u_2}{a_n} - ... - \frac{a_{n-1}u_n}{a_n})^T =$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ \frac{1}{a_n}(\lambda - V) & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & -\frac{a_3}{a_n} & \dots & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-1}}{a_n} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ .. \\ u_n \end{bmatrix} = Ru \tag{61}$$

which is the equation (49).

In the next section we will obtain NLS as a compatibility condition for the AKNS system (49-50). The only useful, yet essential information, that we need to learn from the theoretical exercise presented above is that the matrix R is linear in λ. Indeed, from (56), we obtain easily the following

$$R = \lambda R_1 + R_0 \tag{62}$$

where

$$R_1 = \frac{1}{a_n}\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 \end{bmatrix} \text{ and } R_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ -\frac{V}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & -\frac{a_3}{a_n} & \dots & -\frac{a_{n-2}}{a_n} & -\frac{a_{n-1}}{a_n} \end{bmatrix} \tag{63}$$

In the next section we will use R as prescribed by the formula (62), for different R_1 and R_0, and we will consider the matrix T to be a quadratic structure in λ.

2.2 The Cubic Nonlinear Schrödinger Equation as a Compatibility Condition for the AKNS System

Consider the Lax equation in the matrix form

$$R_t - T_x + [R, T] = 0 \tag{64}$$

where 0 has dual nature, as the number zero and as a zero matrix, and in (64) it represents a zero matrix. Let us choose

$$R = \lambda R_1 + R_0,\ R_1 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix},\ R_0 = \begin{bmatrix} 0 & u \\ \pm\bar{u} & 0 \end{bmatrix} \tag{65}$$

where u is a complex-valued function and $\bar{u}$ is its complex-conjugate, i.e., $u\bar{u} = \bar{u}u = |u|^2$, and let us consider

$$T = \lambda^2 T_2 + \lambda T_1 + T_0 \tag{66}$$

where T_2, T_1 and T_0 are matrices to be determined.

Remark: Let us notice that

$$R_1 = -i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -i\sigma_3 \tag{67}$$

where

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{68}$$

is the Pauli matrix, [17]; it is an elementary matrix of type 2 with ${\sigma_3}^{-1} = \sigma_3$.

From (65) we obtain

$$\frac{\partial R}{\partial t} = \frac{\partial}{\partial t}(\lambda R_1 + R_0) \implies R_t = \frac{\partial \lambda}{\partial t} R_1 + \lambda \frac{\partial R_1}{\partial t} + \frac{\partial R_0}{\partial t} \implies R_t = \lambda \frac{\partial R_1}{\partial t} + \frac{\partial R_0}{\partial t} \tag{69}$$

where $\frac{\partial \lambda}{\partial t} = 0$ since λ is time independent.

From (66) we obtain

$$\begin{aligned} \frac{\partial T}{\partial x} &= \frac{\partial}{\partial x}(\lambda^2 T_2 + \lambda T_1 + T_0) \implies T_x = \frac{\partial \lambda^2}{\partial x} T_2 + \lambda^2 \frac{\partial T_2}{\partial x} + \frac{\partial \lambda}{\partial x} T_1 + \lambda \frac{\partial T_1}{\partial x} + \frac{\partial T_0}{\partial x} \\ &\implies T_x = \lambda^2 \frac{\partial T_2}{\partial x} + \lambda \frac{\partial T_1}{\partial x} + \frac{\partial T_0}{\partial x} \end{aligned} \tag{70}$$

Then the commutator bracket in (64) becomes

$$\begin{aligned} &[R, T] = RT - TR = (\lambda R_1 + R_0)(\lambda^2 T_2 + \lambda T_1 + T_0) - (\lambda^2 T_2 + \lambda T_1 + T_0)(\lambda R_1 + R_0) = \\ &\lambda^3 R_1 T_2 + \lambda^2 R_1 T_1 + \lambda R_1 T_0 + \lambda^2 R_0 T_2 + \lambda R_0 T_1 + R_0 T_0 - \lambda^3 T_2 R_1 - \lambda^2 T_1 R_1 - \lambda T_0 R_1 - \lambda^2 T_2 R_0 + \\ &\lambda T_1 R_0 - T_0 R_0 = \lambda^3 [R_1, T_2] + \lambda^2([R_1, T_1] + [R_0, T_2]) + \lambda([R_1, T_0] + [R_0, T_1]) + [R_0, T_0] \end{aligned} \tag{71}$$

Using (69),(70) and, (71) the equation (64) becomes

$$\lambda^3[R_1, T_2] + \lambda^2\left([R_1, T_1] + [R_0, T_2] - \frac{\partial T_2}{\partial x}\right) + \lambda\left([R_1, T_0] + [R_0, T_1] - \frac{\partial T_1}{\partial x} + \frac{\partial R_1}{\partial t}\right) \\ + \frac{\partial R_0}{\partial t} - \frac{\partial T_0}{\partial x} + [R_0, T_0] = 0 \tag{72}$$

The left hand side of the equation (72) is a cubic polynomial in λ, and the equation (72) is satisfied for all λ if and only if the coefficients of the polynomial are equal to zero matrix. Then we have

For the coefficient of λ^3

$$[R_1, T_2] = 0 \tag{73}$$

For the coefficient of λ^2

$$-\frac{\partial T_2}{\partial x} + [R_1, T_2] + [R_0, T_2] = 0 \tag{74}$$

For the coefficient of λ

$$\frac{\partial R_1}{\partial t} - \frac{\partial T_1}{\partial x} + [R_0, T_1] + [R_1, T_0] = 0 \tag{75}$$

For the free term

$$\frac{\partial R_0}{\partial t} - \frac{\partial T_0}{\partial x} + [R_0, T_0] = 0 \tag{76}$$

From (73) we have

$$[R_1, T_2] = 0 \Longleftrightarrow R_1T_2 - T_2R_1 = 0 \Longleftrightarrow R_1T_2 = T_2R_1 \tag{77}$$

Because $R_1 = -i\sigma_3$ and σ_3 is idempotent (i.e., ${\sigma_3}^2 = I$), we will choose

$$T_2 = i \cdot D = i \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \tag{78}$$

where $d_1, d_2 \in \mathbb{R}$, $d_1 \neq d_2$ (i.e., nonzero diagonal matrix), and we show that (73) is satisfied.

Let us show that $R_1T_2 = T_2R_1$

$$\begin{aligned} R_1T_2 = -i\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \cdot i\begin{bmatrix}d_1 & 0\\ 0 & d_2\end{bmatrix} = -i^2\begin{bmatrix}d_1 & 0\\ 0 & -d_2\end{bmatrix} = \begin{bmatrix}d_1 & 0\\ 0 & -d_2\end{bmatrix}\\ T_2R_1 = i\begin{bmatrix}d_1 & 0\\ 0 & d_2\end{bmatrix} \cdot (-i)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = -i^2\begin{bmatrix}d_1 & 0\\ 0 & -d_2\end{bmatrix} = \begin{bmatrix}d_1 & 0\\ 0 & -d_2\end{bmatrix} \end{aligned} \tag{79}$$

Thus $[R_1, T_2] = 0$, which means that (73) is satisfied under the choices of R_1 and T_2.

For (74) to be satisfied, let us choose

$$T_1 = \alpha R_0,\ \alpha = \frac{d_2 - d_1}{2} \tag{80}$$

With the choice of T_1 and T_2 from above, let us show that (74) is satisfied.

$$\begin{aligned} &-i\frac{\partial}{\partial x}\begin{bmatrix}d_1 & 0\\ 0 & d_2\end{bmatrix} + [R_1, T_1] + [R_0, T_2] = R_1T_1 - T_1R_1 + R_0T_2 - T_2R_0 = \\ &\begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix} \cdot \begin{bmatrix}0 & \frac{(d_2-d_1)}{2}u\\ \pm\frac{(d_2-d_1)}{2}\bar{u} & 0\end{bmatrix} - \begin{bmatrix}0 & \frac{(d_2-d_1)}{2}u\\ \pm\frac{(d_2-d_1)}{2}\bar{u} & 0\end{bmatrix} \cdot \begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix} + \begin{bmatrix}0 & u\\ \pm\bar{u} & 0\end{bmatrix} \cdot i\begin{bmatrix}d_1 & 0\\ 0 & d_2\end{bmatrix} \\ &-i\begin{bmatrix}d_1 & 0\\ 0 & d_2\end{bmatrix} \cdot \begin{bmatrix}0 & u\\ \pm\bar{u} & 0\end{bmatrix} = \begin{bmatrix}0 & -i\frac{(d_2-d_1)}{2}u\\ \pm i\frac{(d_2-d_1)}{2}\bar{u} & 0\end{bmatrix} - \begin{bmatrix}0 & i\frac{(d_2-d_1)}{2}u\\ \mp i\frac{(d_2-d_1)}{2}\bar{u} & 0\end{bmatrix} + \begin{bmatrix}0 & id_2u\\ \pm id_1\bar{u} & 0\end{bmatrix} \\ &-\begin{bmatrix}0 & id_1u\\ \pm id_2\bar{u} & 0\end{bmatrix} = \begin{bmatrix}0 & -i(d_2-d_1)u\\ \pm i(d_2-d_1)\bar{u} & 0\end{bmatrix} + \begin{bmatrix}0 & i(d_2-d_1)u\\ \mp i(d_2-d_1)\bar{u} & 0\end{bmatrix} = 0 \end{aligned} \tag{81}$$

We now need to find T_0 such that (75) and (76) to be satisfied. Let us consider

$$T_0 = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \tag{82}$$

Let us use (80) and (82) to simplify equation (75) as follows

$$\begin{aligned}
&\frac{\partial}{\partial t}\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} - \frac{\partial}{\partial x}\begin{bmatrix} 0 & \alpha u \\ \pm\alpha\bar{u} & 0 \end{bmatrix} + [R_0, T_1] + [R_1, T_0] = 0 \implies \\
&-\begin{bmatrix} 0 & \alpha u_x \\ \pm\alpha\bar{u}_x & 0 \end{bmatrix} + R_0T_1 - T_1R_0 + R_1T_0 - T_0R_1 = 0 \implies \\
&-\begin{bmatrix} 0 & \alpha u_x \\ \pm\alpha\bar{u}_x & 0 \end{bmatrix} + \begin{bmatrix} \pm\alpha\bar{u}u & 0 \\ 0 & \pm\alpha u\bar{u} \end{bmatrix} - \begin{bmatrix} \pm\alpha\bar{u}u & 0 \\ 0 & \pm\alpha u\bar{u} \end{bmatrix} - i\begin{bmatrix} t_{11} & t_{12} \\ -t_{21} & -t_{22} \end{bmatrix} + i\begin{bmatrix} t_{11} & -t_{12} \\ t_{21} & -t_{22} \end{bmatrix} = 0 \implies \\
&-\begin{bmatrix} 0 & \alpha u_x \\ \pm\alpha\bar{u}_x & 0 \end{bmatrix} + i\begin{bmatrix} 0 & -2t_{12} \\ 2t_{21} & 0 \end{bmatrix} = 0 \implies \begin{bmatrix} 0 & \alpha u_x \\ \pm\alpha\bar{u}_x & 0 \end{bmatrix} + i\begin{bmatrix} 0 & 2t_{12} \\ -2t_{21} & 0 \end{bmatrix} = 0
\end{aligned} \tag{83}$$

The equation (75) is satisfied unconditionally if and only if

$$\begin{aligned}
\alpha u_x + 2it_{12} = 0 &\implies t_{12} = \frac{\alpha}{2} i u_x \\
\pm\alpha\bar{u}_x - 2it_{21} = 0 &\implies t_{21} = \mp\frac{\alpha}{2} i\bar{u}_x
\end{aligned} \tag{84}$$

Hence, we found the off-diagonal entries of T_0. To find the main diagonal entries of T_0, we will work with the equation (76). Replacing $R_0 = \begin{bmatrix} 0 & u \\ \pm\bar{u} & 0 \end{bmatrix}$, $T_0 = \begin{bmatrix} t_{11} & \frac{\alpha}{2} i u_x \\ \mp\frac{\alpha}{2} i\bar{u}_x & t_{22} \end{bmatrix}$ into (76) we obtain

$$\frac{\partial}{\partial t}\begin{bmatrix} 0 & u \\ \pm\bar{u} & 0 \end{bmatrix} - \frac{\partial}{\partial x}\begin{bmatrix} t_{11} & \frac{\alpha}{2}iu_x \\ \mp\frac{\alpha}{2}i\bar{u}_x & t_{22} \end{bmatrix} + R_0T_0 - T_0R_0 = 0 \implies$$

$$\begin{bmatrix} 0 & u_t \\ \pm\bar{u}_t & 0 \end{bmatrix} - \begin{bmatrix} t_{11_x} & \frac{\alpha}{2}iu_{xx} \\ \mp\frac{\alpha}{2}i\bar{u}_{xx} & t_{22_x} \end{bmatrix} + \begin{bmatrix} 0 & u \\ \pm\bar{u} & 0 \end{bmatrix} \cdot \begin{bmatrix} t_{11} & \frac{\alpha}{2}iu_x \\ \mp\frac{\alpha}{2}i\bar{u}_x & t_{22} \end{bmatrix} - \begin{bmatrix} t_{11} & \frac{\alpha}{2}iu_x \\ \mp\frac{\alpha}{2}i\bar{u}_x & t_{22} \end{bmatrix} \cdot \begin{bmatrix} 0 & u \\ \pm\bar{u} & 0 \end{bmatrix} = 0$$

$$\implies \begin{bmatrix} -t_{11_x} & u_t - \frac{\alpha}{2}iu_{xx} \\ \pm\bar{u}_t \pm \frac{\alpha}{2}i\bar{u}_{xx} & -t_{22_x} \end{bmatrix} + \begin{bmatrix} \mp\frac{\alpha}{2}i\bar{u}_x u & t_{22}u \\ \pm t_{11}\bar{u} & \pm\frac{\alpha}{2}iu_x\bar{u} \end{bmatrix} - \begin{bmatrix} \pm\frac{\alpha}{2}iu_x\bar{u} & t_{11}u \\ \pm t_{22}\bar{u} & \mp\frac{\alpha}{2}i\bar{u}_x u \end{bmatrix} = 0 \implies$$

$$\begin{bmatrix} -t_{11_x} \mp \frac{\alpha}{2}i(u\bar{u}_x + u_x\bar{u}) & u_t - \frac{\alpha}{2}iu_{xx} + u(t_{22} - t_{11}) \\ \pm\bar{u}_t \pm \frac{\alpha}{2}i\bar{u}_{xx} \pm \bar{u}(t_{11} - t_{22}) & -t_{22_x} \pm \frac{\alpha}{2}i(u_x\bar{u} + u\bar{u}_x) \end{bmatrix} = 0 \tag{85}$$

Choose t_{11} and t_{22} such that

$$-t_{11_x} \mp \frac{\alpha}{2}i(u\bar{u}_x + u_x\bar{u}) = 0 \tag{86}$$

and

$$-t_{22_x} \pm \frac{\alpha}{2}i(u\bar{u}_x + u_x\bar{u}) = 0 \tag{87}$$

From (86) we get

$$t_{11_x} = \mp\frac{\alpha}{2}i\frac{\partial}{\partial x}(u\bar{u}) \tag{88}$$

Choose

$$t_{11} = \mp\frac{\alpha}{2}iu\bar{u} = \mp\frac{\alpha}{2}i|u|^2 \tag{89}$$

From (87) we get

$$t_{22_x} = \pm\frac{\alpha}{2}i\frac{\partial}{\partial x}(u\bar{u}) \tag{90}$$

Choose

$$t_{22} = \pm\frac{\alpha}{2}iu\bar{u} = \pm\frac{\alpha}{2}i|u|^2 \tag{91}$$

Then the equation (86) is satisfied.

Hence, the equation (85) becomes

$$\begin{bmatrix} 0 & u_t - \frac{\alpha}{2} i u_{xx} \pm u(\alpha i |u|^2) \\ \pm \bar{u}_t \pm \frac{\alpha}{2} i \bar{u}_{xx} \pm \bar{u}(\mp \alpha i |u|^2) & 0 \end{bmatrix} = 0 \tag{92}$$

The equation (92) represents the equation (76), then the equation (76) will be satisfied unconditionally if and only if

$$u_t - \frac{\alpha}{2} i u_{xx} \pm \alpha i u |u|^2 = 0 \tag{93}$$

and

$$\pm \bar{u}_t \pm \frac{\alpha}{2} i \bar{u}_{xx} \pm \bar{u}(\mp \alpha i |u|^2) = 0 \tag{94}$$

The equations in (94) are equivalent with the following equations

$$\bar{u}_t + \frac{\alpha}{2} i \bar{u}_{xx} \mp \bar{u}(\alpha i |u|^2) = 0 \tag{95}$$

Notice that the equations in (95) are the complex conjugates of the equations in (93). Hence, (76) will be satisfied unconditionally if and only if the equations in (93) are satisfied, i.e.,

$$u_t - \frac{\alpha}{2} i u_{xx} + \alpha i u |u|^2 = 0$$

$$\Updownarrow$$

$$i u_t + \frac{\alpha}{2} u_{xx} - \alpha u |u|^2 = 0 \tag{96}$$

and

$$u_t - \frac{\alpha}{2} i u_{xx} - \alpha i u |u|^2 = 0$$

$$\Updownarrow$$

$$iu_t + \frac{\alpha}{2}u_{xx} + \alpha u|u|^2 = 0 \tag{97}$$

The equations (96) and (97) are the compatibility condition for the AKNS system (Lax equation in the matrix form) (64), and they are called the defocussing (96), and self-focussing, (97), cubic nonlinear Schrödinger equations, respectively.

3 Soliton Solution for the Self-Focusing Cubic Nonlinear Schrödinger Equation

Let us consider the self-focusing NLS model derived in section 2.2.

$$iu_t + \frac{\alpha}{2}u_{xx} + \alpha u|u|^2 = 0 \tag{98}$$

where $\alpha \in \mathbb{R}^*$. The linearization (i.e., the linear part) of the equation (98) is

$$iu_t + \frac{\alpha}{2}u_{xx} = 0 \tag{99}$$

which is the one-dimensional Schrödinger equation that describes the dispersive nature of the NLS model (98), and it was thoroughly discussed in section 1.2.1. Introducing the cubic nonlinearity $u|u|^2$ onto the dispersive waves described by (99), the waves will be subject to a "force" (i.e., the nonlinear effect) that will act against the dispersion. The waves will be forced to "tighten up," and the final result will be dispersive waves with steepend wave front. In the case that a "perfect" balance has been created between the dispersion and nonlinearity we will have the formation of a soliton.

In this chapter, we will intuitively show the formation of a soliton for the self-focusing NLS model (98).

Let us consider the self-focusing NLS model (98) with $\alpha < 0$, and let us rewrite α as follows

$$\alpha \longrightarrow -\alpha, \ \alpha > 0$$

Then the equation (98) becomes

$$iu_t - \frac{\alpha}{2}u_{xx} - \alpha u|u|^2 = 0, \ \alpha > 0 \tag{100}$$

Following the same idea as in section 1.2.2, first we will seek solutions for the equation (100) of the following form

$$u(x,t) = r(x)e^{-it} \tag{101}$$

Let us begin by first calculating u_t, u_{xx}, and $|u|^2$.

$$u_t = -ir(x)e^{-it} \tag{102}$$

$$u_{xx} = r''(x)e^{-it} \tag{103}$$

$$\begin{aligned} |u|^2 &= |r(x)e^{-it}|^2 = |r(x)|^2|e^{-it}|^2 = |r(x)|^2|\cos(-t) + i\sin(-t)|^2 \\ &= |r(x)|^2(\cos^2(-t) + \sin^2(-t)) = |r(x)|^2 = r(x)^2 \end{aligned} \tag{104}$$

Hence, we can write the equation (100) as the following second order ordinary differential equation

$$re^{-it} - \frac{\alpha}{2}r''e^{-it} - \alpha re^{-it}r^2 = 0 \implies r - \frac{\alpha}{2}r'' - \alpha r^3 = 0 \implies r'' + 2r^3 - \frac{2}{\alpha}r = 0 \tag{105}$$

We want to find a solution of the ODE (105) with the following boundary conditions

$$r'(0) = 0 \tag{106}$$

$$r(x) \longrightarrow 0 \; as \; |x| \longrightarrow \infty \tag{107}$$

The boundry condition (107) reads as r is decreasing sufficiently fast at infinity. Intuitively, the boundry condition (106) and (107) respectively, lead us to search for a solution of the equation (105) of the following form

$$r(x) = \beta \operatorname{sech}(\lambda x), \;\; \beta \neq 0, \;\; \lambda \neq 0 \tag{108}$$

We notice right away that the functions of the form (108) satisfy the boundary conditions (106) and (107) respectively, for any $\beta \neq 0$ and $\lambda \neq 0$.

Note: The intuitive thinking in searching for solutions of the type (108) is strongly based on the theory Korteweg and de Vries developed in 1895, [11], in explaining Russell's solitary wave, [14].

Substituting $r(x)$ given in (108) into the ODE (105), we obtain the following

$$\begin{aligned}
&\lambda^2\beta \operatorname{sech}(\lambda x)\tanh^2(\lambda x) - \lambda^2\beta \operatorname{sech}^3(\lambda x) - \frac{2}{\alpha}\beta \operatorname{sech}(\lambda x) + 2\beta^3 \operatorname{sech}^3(\lambda x) = 0 \implies \\
&\lambda^2\beta \operatorname{sech}(\lambda x)(\tanh^2(\lambda x) - \operatorname{sech}^2(\lambda x)) - \frac{2}{\alpha}\beta \operatorname{sech}(\lambda x) + 2\beta^3 \operatorname{sech}^3(\lambda x) = 0 \implies \\
&\lambda^2\beta \operatorname{sech}(\lambda x)(1 - 2\operatorname{sech}^2(\lambda x)) - \frac{2}{\alpha}\beta \operatorname{sech}(\lambda x) + 2\beta^3 \operatorname{sech}^3(\lambda x) = 0 \implies \\
&\lambda^2\beta \operatorname{sech}(\lambda x) - 2\lambda^2\beta \operatorname{sech}^3(\lambda x)) - \frac{2}{\alpha}\beta \operatorname{sech}(\lambda x) + 2\beta^3 \operatorname{sech}^3(\lambda x) = 0
\end{aligned} \tag{109}$$

In order for the left hand side of the equation (109) to equal to 0, we need

$$\lambda^2\beta = \frac{2}{\alpha}\beta \tag{110}$$

and

$$2\lambda^2\beta = 2\beta^3 \tag{111}$$

Looking at equation (111), we conclude that $\lambda = \beta$ must hold, and from the equation (110) we find that the values of our parameters are $\lambda = \beta = \sqrt{\frac{2}{\alpha}}$.

Hence a solution of the ODE (105) satisfying the boundary conditions (106) and (107) is

$$r(x) = \lambda\, \mathsf{sech}(\lambda x),\ \lambda = \sqrt{\frac{2}{\alpha}} \tag{112}$$

From (101) and (112), we obtain the following solution of the NLS (101)

$$u(x,t) = \lambda\, \mathsf{sech}(\lambda x) e^{-it},\ \lambda = \sqrt{\frac{2}{\alpha}} \tag{113}$$

While the one-dimensional Schrödinger equation (99) is used to describe the wave function for a free particle, the nonlinear model (98) is far from describing the quantum state of a particle. While the typical use of NLS is in nonlinear optics, the model is used as well in describing phenomena like surface gravity waves in deep water, or acoustic waves propagating in deep water, etc., [7]. Thus, in the present case, we view the nonlinear model (100) in an inertial reference frame where Galilean invariance is applicable, i.e., the laws of motion of an object describe the same motion of the object in all inertial reference frames.

Let us rewrite the solution (113) as follows

$$u(x,t) = r(x) e^{-it} \tag{114}$$

where

$$r(x) = \lambda\, \mathsf{sech}(\lambda x),\ \lambda = \sqrt{\frac{2}{\alpha}} \tag{115}$$

Applying scaling symmetry onto the solution (114), we obtain the following scaled solution for the NLS (100)

$$u(x,t) \longrightarrow u(x,t|\delta) = \delta u(\delta x, \delta^2 t),\ \delta \neq 0 \tag{116}$$

The NLS (100) is Galilean invariant as follows: if $u(x,t)$ is a solution of the NLS (100), then we can obtain a new solution by changing the inertial reference frame, and adding a phase factor as follows

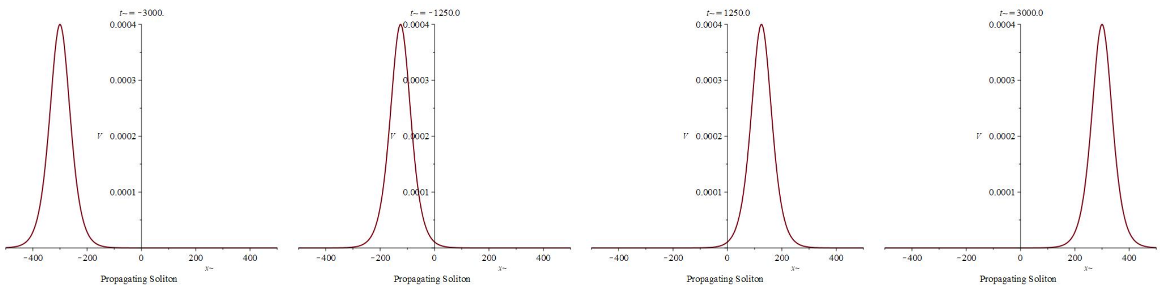
$$u(x,t) \longrightarrow u(x,t|v) = u(x-vt,t)e^{-\frac{i}{2}\lambda^2 v(x-\frac{v}{2})t}, \ v \in \mathbb{R} \tag{117}$$

where the parameter λ is given in (115).

Applying the Galilean invariance (117) onto the scaled solution (116), we obtain the soliton solution of the NLS (100) as follows

$$\begin{aligned} u(x,t) :&= u(x,t|\delta|v) = u(x-vt,t|\delta)e^{-\frac{i}{2}\lambda^2 v(x-\frac{v}{2})t} = \delta u(\delta(x-vt),\delta^2 t)e^{-\frac{i}{2}\lambda^2 v(x-\frac{v}{2})t} \\ &= \delta r(\delta(x-vt))e^{-i\delta^2 t}e^{-\frac{i}{2}\lambda^2 v(x-\frac{v}{2})t} = \delta r(\delta(x-vt))e^{-\frac{i}{4}(2\lambda^2 vx+(4\delta^2-\lambda^2 v^2))t} \end{aligned} \tag{118}$$

where r is given in (115). Thus the formula (118) gives us an exact solution of the self-focusing NLS model (100), describing the time evolution of a soliton profile in deep water, see Figure 3.



(a) (b) (c) (d)

Figure 3: The time evolution of a soliton for the NLS model (100) following the formula (118) for $\alpha = 2$ (therefore $\lambda = 1$), $v = 0.1$, and $\delta = 0.02$.

4 Conclusion

The NLS models (96) and (97) are of great interest in studying nonlinear waves emerging from areas of physics such as nonlinear optics or deep water wave propagation phenomena. As mentioned, Zakharov and Shabat described the soliton profiles for the NLS models in 1972, [16], which was the culminating point of the present work. Though they were first described in 1834, much of the theory of solitons is still unknown. With their widespread applications in areas such as fibre optics, nuclear physics, in magnets and biology, more extensive research and advance studies of solitons, is of surging importance. With this, we conclude the current work with the following quintessential remarks that summarize the successful outcomes of this thesis.

- Schrödinger was a prominent scientist who contributed to the wave theory of matter and to other fundamentals of quantum mechanics.

- This work leads us to discover the effect of the dispersion and nonlinearity in much more depth, i.e., the steepening of the wavefront of the dispersive waves.

- We succeeded in showing how using the same type of intuitive grounds that Korteweg and de Vries used in 1895 helped us to obtain the soliton profile for the self-focusing NLS model.

References

1. Ablowitz, A. M., Clarkson A. P., (1991). Solitons, Nonlinear Evolution Equations and Inverse Scattering. USA: Cambridge University Press.

2. Ablowitz, A. M., Kaup J. D., Newell C. A, Segur H., (1974). The Inverse Scattering TransformFourier Analysis for Nonlinear Problems, Studies in Applied Mathematics, 53, doi:10.1002/sapm1974534249.

3. Aschman, D., (1989). The Legacy of Erwin Schrödinger: Quantum Mechanics. Trans Roy. Soc. S. Afr., 47(1), 81-101, ISSN 0035-919X.

4. Baigrie, B. S., (2019). Erwin Schrödinger: Salem Press Biographical Encyclopedia. 3p.

5. Chen C., (1998). Linear System Theory and Design. USA: Oxford University Press.

6. Dronamraju K. R., (Nov99). Erwin Schrodinger and the Origins of Molecular Biology: Genetics. 153(3), 1071-1076, ISSN 00166731

7. Faddeev, L. D., Takhtajan, L. A., (2007). Hamiltonian Methods in the Theory of Solitons. USA: Springer.

8. Feynman R. P., (1985). QED: The Strange Theory of Light and Matter. New Jersey: Princeton University Press.

9. Gardner, Clifford S.; Greene, John M.; Kruskal, Martin D.; Miura, Robert M., (1967). Method for Solving the Korteweg-deVries Equation. Physical Review Letters, 19(19), 1095-1097.

10. J. F. James. (2011). A Student's Guide to Fourier Transforms - With Applications in Physics. Cambridge University Press.

11. Korteweg, D. J., de Vries, G., On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, Phil. Mag. S. 5. Vol. 39, No. 240, 422-443, 1895.

12. Lax D. P., (1968). Integrals of Nonlinear Equations of Evolution and Solitary Waves. AEC Reaserch and Development Report, New York University, Courant Institute Library, NYO-1480-87.

13. Messiah A., (1961). Quantum Mechanics. 129-149, New York: Dover Publications.

14. Russell, J. R., Report on Waves, Report of the Fourteenth Meeting of the British Association of the Advancement of Science, York, September 1844, 311-391, London, 1845.

15. Schödinger E., (1926). An Undulatory Theory of the Mechanics of Atoms and Molecules. The Physical Review, 28(6), 1049-1070.

16. Zakharov V. E., Shabat A. B., (1972). Exact Theory of Two-Dimensional Self-Focusing and One-Dimensional Self-Modulation of Waves in Nonlinear Media. Soviet Physics JETP. 34(1), 118-134.

17. Weisstein, Eric W. "Pauli Matrices." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/PauliMatrices.html