ON HIGHER DIMENSIONAL ARITHMETIC PROGRESSIONS IN MEYER SETS

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ABSTRACT. In this paper we study the existence of higher dimensional arithmetic progression in Meyer sets. We show that the case when the ratios are linearly dependent over $\mathbb Z$ is trivial, and focus on arithmetic progressions for which the ratios are linearly independent. Given a Meyer set Λ and a fully Euclidean model set $\Lambda(W)$ with the property that finitely many translates of $\Lambda(W)$ cover Λ , we prove that we can find higher dimensional arithmetic progressions of arbitrary length with k linearly independent ratios in Λ if and only if k is at most the rank of the $\mathbb Z$ -module generated by $\Lambda(W)$. We use this result to characterize the Meyer sets which are subsets of fully Euclidean model sets.

1. Introduction

The Nobel Prize discovery of quasicrystals [21] sparked many questions regarding the nature of solids with long-range aperiodic order. This discovery led to establishment of a new area of mathematics, the area of Aperiodic Order. The goal of this new field is to study objects which show long range order, but they are not necessarily periodic.

The best mathematical models for point sets which show long range order and are typically aperiodic were discovered in the earlier pioneering work of Y. Meyer [15], and have been popularized in the field by R.V. Moody [16, 17] and J. Lagarias [11, 12]. Called model sets, they are constructed via a cut-and-project scheme, a mechanism which starts with a lattice \mathcal{L} in a higher dimensional space which sits at an "irrational slope" with respect to the real space \mathbb{R}^d , cuts a strip around the real space \mathbb{R}^d of bounded width W (called the "window") and projects it on the real space \mathbb{R}^d (see Definition 2.13 for the exact definition). Under various weak conditions, the high order present in the lattice \mathcal{L} shows in the resulting model set, typically via a clear diffraction diagram, for example see [6, 9, 13, 19, 20] just to name a few.

Meyer sets are relatively dense subsets of model sets. As subsets of model sets they inherit part of the high order present in the former, which is evident in their characterisation via discrete geometry, harmonic analysis and algebraic properties [11, 15, 16]. While they typically have positive entropy, and hence are usually not pure point diffractive ([5], compare [4] for a discussion), they still show a highly ordered diffraction diagram [23, 25, 26, 27] with a relatively dense supported pure point spectrum [22, 23, 24, 27].

In [10] we showed a different type of order in Meyer sets in the form of the existence of arbitrarily long arithmetic progressions. More precisely, we proved

that given a Meyer set $\Lambda \subseteq \mathbb{R}^d$, for all $N \in \mathbb{N}$ there exists some R > 0 such that the set $\Lambda \cap B_R(x)$ contains an arithmetic progression of length N for all $x \in \mathbb{R}^d$. Moreover, we showed that van der Waerden type theorems hold in Meyer sets. More recently, related results have been investigated in [1, 18].

Consider now a Meyer set $\Lambda \subseteq \mathbb{R}^d$. While Λ spreads relatively dense in all directions of \mathbb{R}^d , any arithmetic progression is one dimensional, and hence only gives partial information about the structure of Meyer sets. This suggest that one should look for higher dimensional arithmetic progressions, which is the goal of this paper. By a m-dimensional arithmetic progression we understand a set of the form

$$A = \{s + c_1 r_1 + \dots + c_m r_m : 0 \le c_j \le N_j, \forall 1 \le j \le m\}$$

for some fixed $s, r_1, \ldots, r_m \in \mathbb{R}^d$ and $N_1, \ldots, N_m \in \mathbb{N}$. The elements r_1, \ldots, r_m are called the ratios and $\vec{N} = (N_1, \ldots, N_m)$ is the vector length of the progression. The arithmetic progression is proper if all the elements $s + c_1 r_1 + \cdots + c_m r_m$ are distinct.

By a standard application of the Chinese Remainder Theorem we show in Prop. 3.3 that for all $n \in \mathbb{N}$ and $\vec{N} \in \mathbb{N}^m$, every Meyer set contains a proper n-dimensional arithmetic progression of length \vec{N} . While the arithmetic progression is proper, every pair of ratios is linearly dependent over \mathbb{Z} and hence the arithmetic progression is a subset of a 1-dimensional affine \mathbb{Q} -space.

To make the question more interesting and meaningful, we add the extra condition that the ratios are linearly independent over \mathbb{Z} (or equivalently over \mathbb{Q}). Given a fully Euclidean model set $\wedge(W)$ in a CPS $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$, we show in Thm. 4.3 that $\wedge(W)$ has n-dimensional arithmetic progressions of arbitrary length with linearly independent ratios if and only if $n \leq d + m$.

Next, for any Meyer set Λ , it is well known that there exists some fully Euclidean model set $\Lambda(W)$ in some CPS $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$ and a finite set $F \subseteq \mathbb{R}^d$ such that

$$\Lambda \subseteq \Lambda(W) + F.$$

We show in Thm. 5.2 that Λ has *n*-dimensional arithmetic progressions of arbitrary length with linearly independent ratios if and only if $n \leq d + m$. This implies that, while in general there exists multiple fully Euclidean model sets $\wedge(W)$ and finite sets F such that (1) holds, m must be the same for all these model sets.

We complete the paper by answering the following question.

Question 1.1. Which Meyer sets $\Lambda \subseteq \mathbb{R}^d$ are subsets of fully Euclidean model sets?

To our knowledge no characterisation for these sets (which we call fully Euclidean Meyer sets) is known so far. We show in Thm. 6.1 that a Meyer set $\Lambda \subseteq \mathbb{R}^d$ is a subset of a fully Euclidean model set if and only if has n-dimensional arithmetic progressions of arbitrary length with linearly independent ratios, where n is the rank of the \mathbb{Z} -module generated by Λ . The characterisation is of number theory/combinatorics origin, emphasizing once again the nice order present in Meyer sets and model sets.

The paper is structured in the following way. In Section 2.1 we give basic definitions and prove a higher-dimensional version of the van der Waerden's theorem [28]. We prove, in Section 3, that Meyer sets $\Lambda \in \mathbb{R}^d$ contain arithmetic progression of arbitrary length and dimension, albeit with linearly dependent ratios. In Section 4 we establish both the existence of arithmetic progression with linearly independent ratios, and a higher-dimensional van der Waerden-type result for fully Euclidean

model sets. In Section 5, we extend these results to arbitrary Meyer sets in \mathbb{R}^d . We complete the paper by characterising the fully Euclidean Meyer sets.

2. Preliminaries

In this section we review the basic definitions and results needed in the paper.

2.1. Finitely generated free \mathbb{Z} -modules. We start by recalling few basic results about finitely generated free \mathbb{Z} -modules. First recall [8, Thm. VIII.4.12] that if M is a free \mathbb{Z} -module, then all the bases of M have the same cardinality. The common cardinality of these bases is called the **rank** of M and is denoted by $\operatorname{rank}(M)$. Also, a finitely generated \mathbb{Z} -module is free if and only if it is torsion free [7, Thm. 12.5].

Next, let us recall the following two results which we will use a few times in the paper.

Lemma 2.1. [8, Thm. VIII.6.1] Let M be a free \mathbb{Z} -module of rank k. If N is a submodule of M, then N is free and

$$rank(N) \leq rank(M)$$
.

In particular, Lemma 2.1 implies the following result.

Corollary 2.2. Let M be a free \mathbb{Z} -module of rank k. If $v_1, \ldots, v_m \in M$ are linearly independent over \mathbb{Z} then $m \leq k$.

Proof. v_1, \ldots, v_m are a basis for the submodule of M generated by $\{v_1, \ldots, v_m\}$. The claim follows from Lemma 2.1.

Next, let us recall the following result about \mathbb{Z} -submodules of the same rank as the full module.

Lemma 2.3. Let M be a finitely generated free \mathbb{Z} -module and let N be a submodule of M. If

$$rank(M) = rank(N)$$

then, there exists some positive integer n such that

$$\{nv:v\in M\}=:n\cdot M\subseteq N$$
.

Proof. This follows from [8, Thm. VIII.6.1].

We complete the section by proving the following simple result. This result is surely known, but we could not find a good reference for it.

Lemma 2.4. Let M be a finitely generated free \mathbb{Z} -module with $k = \operatorname{rank}(M)$ and let S be a generating set for M. Then, there exists k linearly independent elements $v_1, \ldots, v_k \in S$.

Proof. By a standard application of Zorn's lemma, there exist some $S' = \{v_1, \ldots, v_m\} \subseteq S$ which is a maximal linearly independent subset. Let N be the submodule of M generated by S'. Then, rank(N) = m. To complete the proof, we show that m = k. By [7, Thm. 12.4], there exists a basis y_1, \ldots, y_k of M and $n_1, \ldots, n_m \in \mathbb{Z}$ such that n_1y_1, \ldots, n_my_m is a basis for N.

Now assume by contradiction that m < k. Since S spans M, there exists elements $x_1, \ldots, x_l \in S$ and $k_1, \ldots, k_l \in \mathbb{Z}$ such that

$$y_k = k_1 x_1 + \dots + k_l x_l.$$

Next, note that for each $1 \leq j \leq l$ we either have $x_j \in S'$ or $S' \cup x_j$ is linearly dependent. In both cases, there exists some non-zero $f_j \in \mathbb{Z}$ such that $f_j x_j \in N$. Let $f = f_1 \cdot \ldots \cdot f_l$. Then f is a non-zero integer and

$$fy_k \in N$$
.

Since n_1y_1, \ldots, n_my_m is a basis for N, fy_k can be written as a linear combination of n_1y_1, \ldots, n_my_m . As $f \neq 0$, this gives that y_1, \ldots, y_m, y_k are linearly dependent over \mathbb{Z} , which is a contradiction. Thus, m = k.

Note that one can alternately prove the above lemma by embedding M into a \mathbb{Q} -vector space and looking at the subspace generated by S'.

2.2. **Higher dimensional arithmetic progressions.** In this section we look at higher dimensional arithmetic progressions. Let us start with the following definition.

Definition 2.5. A higher dimensional arithmetic progression is a set of the form

$$A := \{s + c_1 r_1 + c_2 r_2 + \dots + c_n r_n : 0 \le c_1 \le N_1, 0 \le c_2 \le N_2, \dots, 0 \le c_n \le N_n\}$$

for some fixed vectors $s, r_1, \ldots, r_n \in \mathbb{R}^d$ and some arbitrarily fixed natural numbers $N_1, \ldots, N_n \in \mathbb{N}$. If the elements in A are distinct, then the progression is called **proper**. We will call n the **dimension** of the projection and $\bar{N} = (N_1, \ldots, N_n)$ the **vector length** of the progression. The **rank** of the projection is the rank of the \mathbb{Z} -module generated by the ratios $\{r_1, \ldots, r_n\}$. We say that the arithmetic progression is an **li-arithmetic progression** if the ration r_1, \ldots, r_n are linearly independent over \mathbb{Z} .

In the remainder of the paper we will simply refer to a higher dimensional arithmetic progression simply as an "arithmetic progression".

Remark 2.6. The rank of a generalized arithmetic progression is simply the largest cardinality of any \mathbb{Z} -linearly independent subset of $\{r_1, \ldots, r_n\}$. It is obvious that the rank of any arithmetic progression is at most its dimension, with equality if and only if the arithmetic progression is li-arithmetic progression.

Next, let us note here that since our goal is to study the existence of arithmetic progressions of arbitrary length, $\vec{N} = (N_1, \dots, N_n)$, it is sufficient to restrict to the case $N_1 = N_2 = \dots = N_n =: N$. In this case, we will say that the length of the progression is $N \in \mathbb{N}$.

Next, let us review some standard notations. As usual, for $N \in \mathbb{N}$ with $N \ge 1$ we denote by $\lceil N \rceil$ the set

$$[N] := \{0, 1, 2, \dots, N\} = \mathbb{N} \cap [0, N].$$

 $[N]^d$ denotes the Cartesian product of d copies of [N], that is

$$[N]^d = \{(k_1, \ldots, k_d) : k_j \in [N] . \forall 1 \le j \le d\}.$$

We will also need the following definition.

Definition 2.7. A d-dimensional grid of depth n is a set of the form

$$[k_1, \ldots, k_d; l_1, \ldots, l_d; n] := \{(l_1 + m_1 k_1, l_2 + m_2 k_2, \ldots, l_d + m_d k_d) : m_1, \ldots, m_d \in [n]\}$$
 for some fixed positive integers k_1, \ldots, k_d and fixed l_1, \ldots, l_d .

Note that a d-dimensional grid of depth n is simply an arithmetic progression inside \mathbb{Z}^d of dimension d with the ratios

$$r_i = k_i e_i$$

for some $k_j \in \mathbb{N}$, where $e_j = (0, 0, \dots, 1, 0, \dots, 0)$ is the canonical basis.

We now prove the following higher dimensional version of van der Waerden's theorem. For the analogous statement of the one-dimensional version of this theorem, we refer the reader to [10, 28]. Note that are already higher-dimensional generalisations of van der Waerdens's theorem, such as the Gallai-Witt theorem, see [14] for brief discussion and references therein.

Theorem 2.8 (van der Waerden in \mathbb{Z}^d). Given any natural numbers k, r, d, there exists a number W(r, k, d), such that, no matter how we colour \mathbb{Z}^d with r colours, for each $N \geq W(r, k, d)$, we can find a monochromatic d-dimensional grid $[k_1, \ldots, k_d; l_1, \ldots, l_d; k] \subseteq [N]^d$ of depth k.

Proof. We prove the claim via induction on d.

P(1) is the standard van der Waerden theorem [28].

 $P(d) \Rightarrow P(d+1)$: Let r and k be given. Let A be the set of all d-dimensional grids of depth k which are subsets of $[W(r,k,d)]^d$. Now set

$$W(r, k, d+1) = W(|A| \cdot r, k, 1)$$
.

We show that this choice works. Note first that the van der Waerden theorem is equivalent to the fact that given a set X with $|A| \cdot r$ elements, for any function $v : \mathbb{N} \to X$ and any $N \ge W(|A| \cdot r, k, 1)$, there exists an element $x \in X$ such that,

$$v^{-1}(x) \cap [N]$$

contains an arithmetic progression of length k.

Now, consider any colouring of \mathbb{Z}^{d+1} with r colours c_1, \ldots, c_r . Let $N \geq W(r, k, d+1)$. Next, for each $1 \leq j \leq N$ consider the colouring of $\mathbb{Z}^d \times \{j\} \subseteq \mathbb{Z}^{d+1}$. By P(d), the set $[W(r, k, d)]^d \times \{j\}$ contains a monochromatic grid M_j of depth k. Let c(j) be the colour of this grid. We can now define a function

$$v:[N] \to A \times \{c_1, \dots, c_r\}$$
$$v(j) = (M_j, c(j)).$$

Then, there exists some $(M, c_l) \in A \times \{c_1, \dots, c_r\}$ such that,

$$v^{-1}(M,c_l)\cap [N]$$

contains an arithmetic progression of length k. Let l_{d+1}, k_{d+1} be do that $l_{d+1} + mk_{d+1} \in v^{-1}(M, c_l) \cap [N]$ for all $m \in [k]$. Next, since M is a monochromatic grid of depth k, there exists some $k_1, \ldots, k_d; l_1, \ldots, l_d$ such that

$$M = [k_1, \ldots, k_d; l_1, \ldots, l_d; k] \subseteq [W(r, k, d)]^d$$
.

Then, by construction, the grid

$$[k_1, \dots, k_d, k_{d+1}; l_1, \dots, l_d, l_{d+1}; k] \subseteq [W(r, k, d)]^d \times [N] \subseteq [N]^{d+1}$$

is monochromatic of colour c_l . This proves the claim.

Remark 2.9. If we denote by W(r, k, d) the smallest value which satisfies Thm. 2.8, then it is obvious that W(r, k, 1) = W(r, k). Moreover, the proof above yields the terrible upper-bound

$$W(r,k,d+1) \le W(l,k,1)$$

where $l = |A| \cdot r$. Note that,

$$|A| = \left(\sum_{j=0}^{W(n,k,d)} \left\lfloor \frac{W(n,k,d) - j}{d} \right\rfloor \right)^{d}$$

which can be seen by observing that, for each $1 \le i \le d$ and for every particular choice $1 \le j \le W(n, k, d)$ we have

$$1 \le l_j \le \frac{W(n,k,d) - j}{d}.$$

2.3. Meyer sets and model sets. In this subsection we review the notion of model sets and Meyer sets in \mathbb{R}^d . For a more detailed review of these definitions and properties we refer the reader to the monograph [2] and to [16, 17].

We start by reviewing some of the basic definitions for point sets.

Definition 2.10. Let $\Lambda \subseteq \mathbb{R}^d$ be a point set. We say that Λ is

- relatively dense if there exists some R > 0 such that for all $x \in \mathbb{R}^d$ the set $\Lambda \cap B_R(x)$ contains at least one point.
- uniformly discrete if there exists some r > 0 such that for all $x \in \mathbb{R}^d$ the set $\Lambda \cap B_r(x)$ contains at most a point.
- **Delone** if Λ is relatively dense and uniformly discrete.
- locally finite if for all R > 0 and $x \in \mathbb{R}^d$ the set $\Lambda \cap B_R(x)$ is finite.

The above definitions are usually stated for arbitrary locally compact Abelian groups (LCAG) G using open and compact sets, respectively. It is easy to see that in the case of $G = \mathbb{R}^d$ the usual definitions are equivalent to Definition 2.10.

Next, in the spirit of [16] we introduce next the following definition.

Definition 2.11. We say that two Delone sets Λ_1, Λ_2 are equivalent by finite translations if there exists finite sets F_1, F_2 such that

$$\Lambda_1 \subseteq \Lambda_2 + F_2$$
$$\Lambda_2 \subseteq \Lambda_1 + F_1.$$

- **Remark 2.12.** (a) It is easy to see that being equivalent by finite translations is an equivalence relation on the set of Delone subsets of \mathbb{R}^d .
 - (b) By replacing F_1, F_2 by $F = F_1 \cup F_2$ one can assume without loss of generality that $F_1 = F_2$.

Next we review the notion of cut and project schemes and model sets.

Definition 2.13. By a **cut** and **project scheme**, or simply CPS, we understand a triple $(\mathbb{R}^d, H, \mathcal{L})$ consisting of \mathbb{R}^d , a LCAG H, together with a lattice (i.e. a discrete co-compact subgroup) $\mathcal{L} \subset \mathbb{R}^d \times H$, with the following two properties:

• The restriction $\pi^{\mathbb{R}^d}|_{\mathcal{L}}$ of the canonical projection $\pi^{\mathbb{R}^d}: \mathbb{R}^d \times H \to \mathbb{R}^d$ to \mathcal{L} is a one-to-one function.

• The image $\pi^H(\mathcal{L})$ of the \mathcal{L} under the canonical projection $\pi^H: \mathbb{R}^d \times H \to H$ is dense in H.

In the special case where $H = \mathbb{R}^m$ then we refer to $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$ as a **fully Euclidean** CPS.

Next we define in the usual way, $L := \pi^{\mathbb{R}^d}(\mathcal{L})$ and $\star : L \to H$, known as the \star -mapping, by

$$\star = \pi^H \circ (\pi^{\mathbb{R}^d}|_{\mathcal{L}})^{-1}.$$

This allows us rewrite

$$\mathcal{L} = \{(x, x^{\star}) : x \in L\}.$$

Note that the range of the \star -mapping is

$$\star(L) =: L^{\star} = \pi^{H}(\mathcal{L}).$$

We can summarize the CPS in the diagram below.

$$\mathbb{R}^{d} \xleftarrow{\pi^{\mathbb{R}^{d}}} \mathbb{R}^{d} \times H \xrightarrow{\pi^{H}} H$$

$$\cup \qquad \qquad \cup \qquad \qquad \square$$

$$\pi^{\mathbb{R}^{d}}(\mathcal{L}) \xleftarrow{1-1} \mathcal{L} \xrightarrow{\text{dense}} H$$

$$\square \qquad \qquad \cup$$

$$L \xrightarrow{} L^{\star}$$

We can now review the definition of model sets.

Definition 2.14. Given a CPS $(\mathbb{R}^d, H, \mathcal{L})$ and some subset $W \subseteq H$, we denote by $\wedge(W)$ its pre-image under the *-mapping, that is

$$\wedge(W) := \{x \in L : x^* \in W\} = \{x \in \mathbb{R}^d : \exists y \in W \text{ such that } (x, y) \in \mathcal{L}\}.$$

When W is precompact and has non-empty interior, the set $\wedge(W)$ is called a **model set**. If furthermore $H = \mathbb{R}^m$ for some m, then $\wedge(W)$ is called a **fully Euclidean model set**.

We want to emphasize here that the condition W has non-empty interior is essential later in the paper in the proof of Thm. 4.3.

Next, let us recall the following result.

Proposition 2.15. [16, Prop. 2.6][2, Prop. 7.5] Let $(\mathbb{R}^d, H, \mathcal{L})$ be a CPS and $W \subseteq H$.

- (a) If $W \subseteq H$ is pre-compact, then $\wedge(W)$ is uniformly discrete.
- (b) If $W \subseteq H$ has non-empty interior, then $\wedge(W)$ is relatively dense.

In particular, every model set is a Delone set.

Next, we review the concept of Meyer set. We start by recalling the following

Theorem 2.16. [11, 15, 16] Let $\Lambda \subseteq \mathbb{R}^d$ be relatively dense. Then, the following are equivalent.

- (i) Λ is a subset of a model set.
- (ii) There exists a fully Euclidean model set $\wedge(W)$ and a finite set F such that

$$\Lambda \subseteq \Lambda(W) + F$$
.

- (iii) $\Lambda \Lambda$ is uniformly discrete.
- (iv) Λ is locally finite and there exists a finite set F_1 such that

$$\Lambda - \Lambda \subseteq \Lambda + F_1$$
.

Proof. The equivalence (i) \Leftrightarrow (iii) \Leftrightarrow (iv) can be found in [16], while (ii) \Leftrightarrow (iv) is [15, Thm. IV] .

The above theorem gives the concept of a Meyer set. More precisely, we have the following definition.

Definition 2.17. A relatively dense set $\Lambda \subset \mathbb{R}^n$ is called **Meyer set** if it satisfies one (and hence all) equivalent conditions of Thm. 2.16.

A relatively dense set Λ is called **fully Euclidean Meyer set** if there exists some fully Euclidean model set $\Lambda(W)$ such that $\Lambda \subseteq \Lambda(W)$.

One of the goals of this paper is to characterise fully Euclidean Meyer sets.

Next, let us recall the following result from [16].

Lemma 2.18. Let $\Lambda \subseteq \mathbb{R}^d$ be a Meyer set. Then, the group $\langle \Lambda \rangle$ generated by Λ is finitely generated. In particular, $\langle \Lambda \rangle$ is a free \mathbb{Z} -module of finite rank.

Proof. By [16, Thm. 9.1], $\langle \Lambda \rangle$ is finitely generated. It is therefore a finitely generated \mathbb{Z} -module. Moreover, since \mathbb{R} is torsion free as a \mathbb{Z} -module, so is $\langle \Lambda \rangle$. Therefore, $\langle \Lambda \rangle$ is a free \mathbb{Z} -module by [7, Thm. 12.5].

This allows us introduce the following definition.

Definition 2.19. Let $\Lambda \subseteq \mathbb{R}^d$ be any Meyer set. We define the rank of Λ to be $rank(\Lambda) := rank_{\mathbb{Z}}(\langle \Lambda \rangle)$.

We complete the section by showing that each Meyer set is equivalent by finitely many translates with a fully Euclidean model set.

Lemma 2.20. Let Λ be a Meyer set, F finite and $\Lambda(W)$ a fully Euclidean model set such that

$$\Lambda \subseteq \Lambda(W) + F$$
.

Then Λ and $\Lambda(W)$ are equivalent by finite translations.

Proof. Since $\wedge(W)$ is a Meyer set and F is finite, $\wedge(W) + F$ is also a Meyer set [15, 16]. The claim follows now from [25, Lemma 5.5.1].

3. Higher dimensional arithmetic progressions in Meyer sets

In this section we show that Meyer sets contain arithmetic progressions of arbitrary dimensions and length. The proofs will show that the existence of arithmetic progressions of arbitrary length in Meyer sets is interesting only in the case of li-arithmetic progressions. We will study these in the subsequent sections.

We start by recalling the following well-known theorem.

Lemma 3.1 (Chinese Remainder Theorem). [7, Cor. 7.18] If k_1, \ldots, k_n are pairwise coprime and a_1, \ldots, a_n are integers then there exists $x \in \mathbb{Z}^+$ such that

$$\begin{cases} x \equiv a_1 \pmod{k_1} \\ x \equiv a_2 \pmod{k_2} \\ & \vdots \\ x \equiv a_n \pmod{k_n} . \end{cases}$$

Moreover, any two solutions are congruent modulo $k_1k_2...k_n$.

As an immediate consequence we get the following result.

Corollary 3.2. For each $n, N \in \mathbb{N}$ there exists $m_1, \ldots, m_n \in \mathbb{N}$ such that $\sum_{i=1}^n c_i m_i$ are distinct for all $0 \le c_i \le N$.

Proof. Let p_1, \ldots, p_n be distinct primes such that for all $1 \le i \le n$ we have $p_i > N$. By the Chinese Remainder Theorem, there exists m_1, \ldots, m_n such that for each $1 \le i \le n$ we have

$$\begin{cases} m_i &\equiv 1 \pmod{p_i} \\ m_i &\equiv 0 \pmod{p_j} \quad \forall j \neq i. \end{cases}$$
 Now if $\sum_{i=1}^n c_i m_i = \sum_{i=1}^n c_i' m_i$ then, for all $1 \leq k \leq n$, we have

$$\sum_{i=1}^{n} c_i m_i \equiv \sum_{i=1}^{n} c_i' m_i \pmod{p_k}.$$

and hence $c_k \equiv c'_k \pmod{p_k}$. Since

$$0 \le c_k, c_k' \le N < p_k \,,$$

we get $c_k = c'_k$ for all $1 \le k \le n$.

We can now prove the following result.

Proposition 3.3. Let $n, N \in \mathbb{N}$ and let $\Lambda \in \mathbb{R}^d$ be a Meyer set. Then there exists some R > 0 such that $\Lambda \cap B_R(x)$ contains a non-trivial and proper n-dimensional arithmetic progression of length N for all $x \in \mathbb{R}^d$.

Proof. Let $n, N \in \mathbb{N}$ be given. We show that, for some R > 0, there exists

$$s, r_1, r_2, \ldots, r_n \in \Lambda$$

such that

$$s + \sum_{i=1}^{n} c_i r_i \in \Lambda \cap B_R(x)$$
 $\forall 0 \le c_i \le N$,

and that the elements $s + \sum_{i=1}^{n} c_i r_i$ are distinct. Let m_1, \ldots, m_n be as in Cor. 3.2 and set $N' = Nm_1 + \cdots + Nm_n$. By [10, Lemma 4.3] there exists an R > 0 and some non-zero $s, r \in \Lambda$ such that

$$s, s + r, \ldots, s + N'r \in \Lambda \cap B_R(x)$$
.

Now define $r_i := m_i r$. It follows that, for $0 \le \sum_{i=1}^n c_i m_i \le N'$,

$$s + \sum_{i=1}^{n} c_i r_i = s + \sum_{i=1}^{n} c_i m_i r \in \Lambda \cap B_R(x).$$

Moreover, by Cor. 3.2 the progression is proper.

Note here in passing that, by construction the arithmetic progression in Prop. 3.3 has rank 1.

It becomes natural to ask if one can construct arithmetic progressions of higher rank. It is easy to see that one can focus on li-arithmetic progressions. Indeed, exactly as in Prop. 3.3 one can prove the following result.

Lemma 3.4. Let $\Lambda \subseteq \mathbb{R}^d$ be any Meyer set and $k \in \mathbb{N}$. Then, Λ contains liarithmetic progressions of rank k and arbitrary length if and only if for each $d \geq k$, Λ contains arithmetic progressions of rank k, dimension d, and arbitrary length.

Since the proof is similar to the one of Prop. 3.3, we skip it.

4. Higher dimensional arithmetic progressions with linearly independent ratios

In this section we discuss the existence of li-arithmetic progressions in fully Euclidean model sets. We show that the maximal rank of any such progression is the rank of the lattice in the CPS, and that for this rank, we can find li-arithmetic progressions of arbitrary length.

We start by proving the following result (compare [16, Prop. 2.6]).

Lemma 4.1. Let $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$ be a fully Euclidean CPS, and let $W \subseteq \mathbb{R}^m$ be any set with non-empty interior. Then $\wedge(W)$ generates $L = \pi_{\mathbb{R}^d}(\mathcal{L})$. In particular,

$$rank(\wedge(W)) = d + m.$$

In particular, there exists vectors $r_1, \ldots, r_{d+m} \in \Lambda(W)$ which are linearly independent over \mathbb{Z} .

Proof. It suffices to show that $\wedge(W) - \wedge(W) = \wedge(W - W)$ generates L. Note that 0 is an interior point in W - W and hence we can find some r > 0 such that $B_r(0) \subseteq W - W$.

Now, let $x \in L$ be arbitrary. Pick some n such that d(x,0) < nr. Then $\frac{x}{n} \in B_r(0)$. Note here that $\frac{x}{n} \in B_r(0) \cap B_{\frac{r}{n}}(\frac{x}{n})$. Since this set is open, by the density of $\pi_{\mathbb{R}^m}(\mathcal{L})$ in \mathbb{R}^m , there exists some $(z,z^*) \in \mathcal{L}$ such that $z^* \in B_r(0) \cap B_{\frac{r}{n}}(\frac{x}{n})$. Then $z \in \mathcal{N}(W-W)$ and $d(x^*,nz^*) < r$, thus $x^* - nz^* \in B_r(0) \subseteq W-W$. Therefore,

$$z \in \wedge(W) - \wedge(W)$$

 $x - nz \in \wedge(W) - \wedge(W)$.

This gives $x \in \langle \wedge(W) \rangle$. The last claim follows now from Lemma 2.4.

Next, we prove the following generalisation of [10, Prop 4.2].

Proposition 4.2. Let $(\mathbb{R}^d, H, \mathcal{L})$ be any CPS and let $W \subseteq H$ be any set with nonempty interior. Then, for all $M \in \mathbb{N}$, we can find open sets $U_M, V_M \subseteq H$ such that $0 \in V_M$ and

$$\wedge (U_M) + \underbrace{\wedge (V_M) + \wedge (V_M) + \dots + \wedge (V_M)}_{M-\text{times}} \subseteq \wedge (W).$$

Proof. As W has non-empty interior, there exists non-empty open sets $U_M \subseteq H$ and $0 \in V_M \subseteq H$ such that

$$U_M + \underbrace{V_M + V_M + \dots + V_M}_{M\text{-times}} \subseteq W$$
.

Then,

$$\wedge (U_M) + \underbrace{\wedge (V_M) + \wedge (V_M) + \dots + \wedge (V_M)}_{M-\text{times}} \subseteq \wedge (U_M + \underbrace{V_M + V_M + \dots + V_M}_{M-\text{times}}) \subseteq \wedge (W).$$

This completes the proof.

By combining Prop. 4.2 with Cor. 2.2 we get the following result.

Theorem 4.3. Let $\wedge(W)$ be a model set in a fully Euclidean CPS $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$. Then,

- (a) Any arithmetic progression of length N in \land (W) has rank at most d + m.
- (b) For each N there exists some R > 0 such that the set $\wedge(W) \cap B_R(y)$ contains an li-arithmetic progression of length N and rank d + m for all $y \in \mathbb{R}^d$.

Proof. (a) For any arithmetic progression of rank k in $\wedge(W)$ the set $\{r_1, \ldots, r_k\}$, with r_i the ratios of the progression, is \mathbb{Z} -linearly independent in L. Therefore, by Cor. 2.2 we have $k \leq m + d$. This proves (a).

(b) Let N be given. We show that there exists some R > 0, such that, for all $y \in \mathbb{R}^d$ there exists some $s \in \wedge(W)$, and \mathbb{Z} -linearly independent r_1, \ldots, r_{m+d} with $s + \sum_{j=1}^{m+d} c_j r_j \in \wedge(W) \cap B_R(y)$ for all $0 \le c_j \le N$.

Set $M := N \cdot (m+d)$. By Prop. 4.2 there exists open sets $U_M, V_M \subseteq \mathbb{R}^m$ such that $0 \in V_M$ and

$$\wedge (U_M) + \underbrace{\wedge (V_M) + \wedge (V_M) + \dots + \wedge (V_M)}_{M-\text{times}} \subseteq \wedge (W).$$

As U_M has non-empty interior, by Prop. 2.15 there exists R' > 0 such that $\wedge (U_M) + B_{R'}(0) = \mathbb{R}^d$. Next, by Lemma 2.4 there exists \mathbb{Z} -linearly independent vectors $r_1, \ldots, r_{m+d} \in \wedge (V_M)$. Define

$$R\coloneqq N\cdot (\|r_1\|+\cdots+\|r_{m+d}\|)+R'\,.$$

Let $y \in \mathbb{R}^d$ be arbitrary. By the definition of R', there exists some $s \in \land(U_M) \cap B_{R'}(y)$. Then, for all $0 \le c_i \le N$, as $0 \in \land(V_M)$, we have

$$s + \sum_{i=1}^{m+d} c_i r_i = s + \underbrace{r_1 + \dots + r_1}_{c_1 \text{ times}} + \dots + \underbrace{r_{m+d} + \dots + r_{m+d}}_{c_{m+d} \text{ times}} + \underbrace{0 + \dots + 0}_{M - \sum_{i=1}^{m+d} c_i \text{ times}}$$

$$= s + \underbrace{r_1 + \dots + r_1}_{c_1 \text{ times}} + \dots + \underbrace{r_{m+d} + \dots + r_{m+d}}_{C_{m+d} \text{ times}} + \underbrace{0 + \dots + 0}_{M - \sum_{i=1}^{m+d} c_i \text{ times}}$$

$$\in \land (U_M) + \land (V_M) + \land (V_M) + \dots + \land (V_M) + \land (V_M) + \land (V_M) + \dots + \land (V_M)$$

$$c_1 \text{ times}$$

$$+ \dots + \underbrace{\land (V_M) + \land (V_M) + \dots + \land (V_M)}_{C_{m+d} \text{ times}} + \underbrace{\land (V_M) + \land (V_M) + \dots + \land (V_M)}_{M - \sum_{i=1}^{m+d} c_i \text{ times}}$$

$$= \wedge (U_M) + \underbrace{\wedge (V_M) + \wedge (V_M) + \dots + \wedge (V_M)}_{M \text{ times}} \subseteq \wedge (W).$$

Moreover, for all $0 \le c_i \le N$, we have

$$d(s + \sum_{i=1}^{m+d} c_i r_i, y) = \|y - \left(s + \sum_{i=1}^{m+d} c_i r_i\right)\| \le \|y - s\| + \|\sum_{i=1}^{m+d} c_i r_i\|$$

$$\le R' + \sum_{i=1}^{m+d} \|c_i r_i\| = R' + \sum_{i=1}^{m+d} c_i \|r_i\| \le R' + \sum_{i=1}^{m+d} N \|r_i\| = R.$$

This implies that, for each $0 \le c_i \le N$,

$$s + \sum_{i=1}^{m+d} c_i r_i \in \wedge(W) \cap B_R(y).$$

Thm. 4.3 suggests the following definition.

Definition 4.4. Let $\Lambda \subseteq \mathbb{R}^d$ be a Meyer set.

The **ap-rank** of Λ , denoted by ap-rank(Λ), is the largest $k \in \mathbb{N}$ with the property that for all $N \in \mathbb{N}$, there exists an li-arithmetic progression of length N and rank k in Λ .

Note here in passing that by Lemma 3.4, the ap-rank of Λ is the largest positive integer k such that for all $N \in \mathbb{N}$ the set Λ contains an arithmetic progression of length N and rank k.

The next result tells us that the definition of ap-rank makes sense.

Lemma 4.5. Let $\Lambda \subseteq \mathbb{R}^d$ be any Meyer set. Then

$$1 \leq \operatorname{ap-rank}(\Lambda) \leq \operatorname{rank}(\Lambda)$$
.

Proof. The lower bound follows immediately from [10]. Note here that any non-trivial 1-dimensional arithmetic progression in \mathbb{R}^d has rank 1.

Next, consider any arithmetic progression of length $N \geq 2$ and rank k in Λ and let r_1, \ldots, r_k be the linearly independent ratios. Then r_1, \ldots, r_k are linearly independent vectors in $\langle \Lambda \rangle$. The claim follows from Cor. 2.2.

Remark 4.6. (a) Thm. 4.3 says that for a fully Euclidean model set \wedge (W) in the CPS (\mathbb{R}^d , \mathbb{R}^m , \mathcal{L}) we have

$$\operatorname{rank}(\wedge(W)) = \operatorname{ap-rank}(\wedge(W)) = d + m.$$

(b) Since the ap-rank does not change under translates, but the rank changes for translates outside the Z-module generated by the set, it is easy to construct examples of translates of fully Euclidean model sets such that

$$rank(t + \wedge(W)) = ap-rank(t + \wedge(W)) + 1.$$

In Example 5.5, for each $n \in \mathbb{N}$, we will construct a Meyer set Λ such that $\operatorname{rank}(\Lambda) = \operatorname{ap-rank}(\Lambda) + n$.

(c) If $\Lambda \subseteq \mathbb{R}^d$ is a Meyer set, we will show in Cor. 5.4 that

$$\operatorname{ap-rank}(\Lambda) \geq d$$
.

We complete the section by providing a colouring version of Thm. 4.3.

Theorem 4.7. Let $\wedge(W)$ be a model set in a fully Euclidean CPS $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$. Then, for each r, k there exists some R such that, no matter how we colour $\wedge(W)$ with r colours the set $\wedge(W) \cap B_R(y)$ contains a monochromatic li-arithmetic progression of length k and rank d+m for all $y \in \mathbb{R}^d$.

Proof. Let N be such that van der Waerden's Theorem (Thm. 2.8) holds for r, k applied to $[N]^{d+m}$. By Thm. 4.3 there exists some R > 0 such that, for all $y \in \mathbb{R}$, the set

$$\wedge(W) \cap B_R(y)$$
,

contains a non-trivial li-arithmetic progression length N. We show that this R works.

Arbitrarily colour $\wedge(\mathbf{W})$ with r colours and let $y \in \mathbb{R}^d$ be given. By Thm. 4.3 there exists s, r_1, \ldots, r_{m+d} such that r_1, \ldots, r_{m+d} are \mathbb{Z} -linearly independent, and for all $0 \le c_i \le N$,

$$s + \sum_{i=1}^{m+d} c_i r_i \in \wedge(W) \cap B_R(y).$$

Colour the set $[N]^{m+d}$ by colouring (c_1, \ldots, c_{m+d}) with the colour of $s + \sum_{i=1}^{m+d} c_i r_i$. By the choice of N, there exists a monochromatic grid

$$[k_1, \dots, k_{d+m}; l_1, \dots, l_{d+m}; k] \in [N]^{d+m}$$

of length k and dimension d+m. Then, for all $1 \le m_i \le k$, the set

$$s + \sum_{j=1}^{m+d} (l_j + m_j k_j) r_j \in \wedge(W) \cap B_R(y)$$

is monochromatic. Now set $s' := s + \sum_{j=1}^{m+d} l_j r_j$ and $r'_j := k_j r_j$. Then, for all $(m_1, \ldots, m_{m+d}) \in [k]^{m+d}$,

$$s' + \sum_{i=1}^{m+d} m_j r'_j \in \mathcal{N}(W) \cap B_R(y)$$

is a monochromatic li-arithmetic progression of length k and rank m+d.

5. Ap-rank of Meyer sets

In this section we calculate the ap-rank of a Meyer set Λ . We know that Λ is equivalent by finite translates to a fully Euclidean model set $\wedge(W)$, and we use this to show that

$$\operatorname{ap-rank}(\Lambda) = \operatorname{ap-rank}(\Lambda(W)) = \operatorname{rank}(\Lambda(W)).$$

We start by proving the following result.

Lemma 5.1. Let $\Lambda, \Gamma \subseteq \mathbb{R}^d$ be Meyer sets which are equivalent by finite translates. Then

$$\operatorname{ap-rank}(\Lambda) = \operatorname{ap-rank}(\Gamma)$$

Proof. By symmetry it suffices to show that

$$\operatorname{ap-rank}(\Lambda) \leq \operatorname{ap-rank}(\Gamma)$$
.

Let $F = \{f_1, \dots, f_r\}$ be such that

$$\Lambda \subseteq \Gamma + F$$

and let $k = \text{ap-rank}(\Lambda)$. We show that Γ contains arithmetic progressions of length N and rank k.

First, colour F with r colours, such that no two points of F have the same colour. Next, let $N \in \mathbb{N}$ be arbitrary and let N' = W(|F|, N, d) be given by Thm. 2.8. Since $k = \operatorname{ap-rank}(\Lambda)$, by definition, there exists an li-arithmetic progression

$$\{s + c_1 r_2 + \ldots + c_k r_k : 0 \le c_j \le N'\}$$

of length N' and rank k in Λ . Now, for each $(c_1, \ldots, c_k) \in [0, N']^k$ there exists some and $f_j \in F$ so that

$$x \coloneqq s + c_1 r_2 + \ldots + c_k r_k \in \bigcup_{j=1}^r \Gamma + f_j.$$

Pick the smallest j such that $x = y + f_j$ for some $y \in \Gamma$. Colour (c_1, \ldots, c_k) with the colour of this $f_j \in F$. Then, by Thm. 2.8, there exists a monochromatic grid $[c'_1d_1, \ldots, c'_kd_k; k] \subseteq [N']^k$ of depth N. Define

$$s' = s - f_j$$
$$r'_i = d_i r_i$$

Then, for all $0 \le c_j' \le N$

$$x = s' + c_1'r_1' + \ldots + c_k'r_k' \in \Gamma$$

is an li-arithmetic progression of length N and rank k. Since $N \in \mathbb{N}$ is arbitrary, we get $k \leq \operatorname{ap-rank}(\Gamma)$.

Now, by combining all results so far we get the following Theorem, which is the first main result in the paper.

Theorem 5.2. Let $\Lambda \subseteq \mathbb{R}^d$ be a Meyer set, and let $\Lambda(W)$ be any fully Euclidean model set in $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$ and $F \subseteq \mathbb{R}^d$ be finite such that

$$\Lambda \subseteq \wedge(W) + F.$$

Then

$$\operatorname{ap-rank}(\Lambda) = d + m = \operatorname{ap-rank}(\Lambda(W)) = \operatorname{rank}(\Lambda(W)).$$

Moreover, for each N there exists some R > 0 such that the set $\Lambda \cap B_R(y)$ contains an li-arithmetic progression of length N and rank d + m for all $y \in \mathbb{R}^d$.

Proof. By Lemma 2.20, Λ and $\Lambda(W)$ are equivalent by finite translates. Therefore, by Lemma 5.1 we have

$$\operatorname{ap-rank}(\Lambda) = \operatorname{ap-rank}(\Lambda(W))$$
.

Moreover, by Thm. 4.3 we have

$$\operatorname{rank}(\wedge(W)) = \operatorname{ap-rank}(\wedge(W)) = d + m$$
.

Next, since Λ and $\Lambda(W)$ are equivalent by finite translates, there exists some finite set $F = \{t_1, \dots, t_r\}$ such that

$$\wedge(W) \subseteq \Lambda + F$$
.

Colour F with r colours so that each point of F has a different colour. Let R' be the constant given by Thm. 4.7 for $\land(W)$ with r colours and length N. Define

$$R := \max\{R' + ||t_j|| : 1 \le j \le r\}.$$

Colour $\wedge(W)$ the following way: for each $x \in \wedge(W)$, there exists some minimal j such that $x \in t_j + \Lambda$. Colour each x by the colour of t_j for this minimal j. This gives an r-colouring of $\wedge(W)$. Note here that any choice of t_j works, but one needs to make a choice in case some $x \in \wedge(W)$ belongs to $t_j + \Lambda$ for more than one j.

Now let $y \in \mathbb{R}^d$ be arbitrary. By Thm. 4.7 there exists a monochromatic liarithmetic progression $s + \sum_{i=1}^{m+d} c_i r_i \in \Lambda(W) \cap B_{R'}(y)$ of rank m+d, for all $0 \le c_i \le N$.

Since the arithmetic progression is monochromatic, there exists some j such that, for all $0 \le c_i \le N$, we have

$$s + \sum_{i=1}^{m+d} c_i r_i \in t_j + \Lambda.$$

Thus, for all $0 \le c_i \le N$, we have

$$s - t_j + \sum_{i=1}^{m+d} c_i r_i \in \Lambda$$

which is an li-arithmetic progression of length N and rank m+d. Moreover, for each $0 \le c_i, \le N$

$$d(s-t_j+\sum_{i=1}^{m+d}c_ir_i,y)=||s+\sum_{i=1}^{m+d}c_ir_i-t_j-y||\leq ||s+\sum_{i=1}^{m+d}c_ir_i-y||+||t_j||\leq R.$$

This proves the last claim.

We start by listing some consequences of this result.

A first immediate consequence we get that if Λ is a Meyer set, $\Lambda(W)$ a fully Euclidean model set in $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$ and F is a finite set such that

$$\Lambda \subseteq \Lambda(W) + F$$

then m is invariant for Λ .

Corollary 5.3. Let $\Lambda \subseteq \mathbb{R}^d$ be any Meyer set. If $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$ and $(\mathbb{R}^d, \mathbb{R}^n, \mathcal{L})$ are two different CPS, $W \subseteq \mathbb{R}^m, W' \subseteq \mathbb{R}^n$ are precompact windows with non-empty interior and F, F' are finite sets such that

$$\Lambda \subseteq \wedge(W) + F$$

$$\Lambda \subseteq \wedge (W') + F',$$

then m = n.

Proof. By Thm. 5.2 we have

$$\operatorname{ap-rank}(\Lambda) = d + m$$

$$\operatorname{ap-rank}(\Lambda) = d + n$$
.

As an immediate consequence, we get the following improvement on the lower bound from Lemma 4.5.

Corollary 5.4. Let $\Lambda \subseteq \mathbb{R}^d$ be any Meyer set. Then

$$\operatorname{ap-rank}(\Lambda) \geq d$$
.

Next, we show that in general there is no upper bound for ap-rank(Λ) in terms of rank(Λ).

Example 5.5. Let $\wedge(W)$ be any fully Euclidean model set and let r_1, \ldots, r_k be so that

$$\langle \wedge(W) \rangle = \bigoplus_{j=1}^k \mathbb{Z} r_k$$
.

Let s_1, \ldots, s_n be so that $r_1, \ldots, r_k, s_1, \ldots, s_n$ are linearly independent over \mathbb{Z} and let $F = \{s_1, \ldots, s_n\}$. Then

$$\Lambda = \Lambda(W) + F$$

is a Meyer set and

$$\operatorname{ap-rank}(\Lambda) = k$$

 $\operatorname{rank}(\Lambda) = k + n$.

Remark 5.6. If $\wedge(W)$ is a fully Euclidean model set with $k = \operatorname{ap-rank}(\Lambda)$ then, by Thm. 4.3 every li-arithmetic progression in $\wedge(W)$ has rank at most k. The same is not true in Meyer sets.

Indeed let $k \leq d$ and N be any positive integers. Let A be any li-arithmetic progression of rank d and length N and let $\wedge(W)$ be any fully Euclidean model set of rank k such that $0 \in \wedge(W)$. Then

$$\Lambda \coloneqq \wedge(W) + A$$

is a Meyer set, with ap-rank(Λ) = k, which contains the li-arithmetic progression A of rank d and length N.

Note that ap-rank(Λ) = k means that for all d > k, if Λ contains li-arithmetic progressions of rank d, then they are bounded in length. We will see later in Cor. 6.3 that for fully Euclidean Meyer sets, the rank of every **every** li-arithmetic progression is also bounded by the ap-rank.

We complete the section by extending, as usual, Thm. 5.2 to colourings of Λ .

Theorem 5.7. Let $\Lambda \subset \mathbb{R}^d$ be Meyer set, and let $k = \operatorname{ap-rank}(\Lambda)$. Then, for each r, N there exists some R such that, no matter how we colour Λ with r colours, the set $\Lambda \cap B_R(y)$ contains a monochromatic li-arithmetic progression of length N and rank k for all $y \in \mathbb{R}^d$.

Proof. Pick N' such that van der Waerden's Theorem holds for r, k applied to $[N']^{d+m}$. By Thm. 5.2 there exists R > 0 such that for all $y \in \mathbb{R}^d$ the set $\Lambda \cap B_R(y)$ contains an arithmetic progression of length N and dimension k. The rest of the proof is identical to that of Thm. 4.7.

6. A CHARACTERISATION OF FULLY EUCLIDEAN MEYER SETS

We complete this paper by characterizing fully Euclidean Meyer sets. To our knowledge, this is the first result in this direction.

Theorem 6.1. Let $\Lambda \subseteq \mathbb{R}^d$ be a Meyer set. Then Λ is a fully Euclidean Meyer set if and only if

$$\operatorname{ap-rank}(\Lambda) = \operatorname{rank}(\Lambda)$$
.

 $Proof. \Longrightarrow$

Let $\wedge(W)$ be a fully Euclidean model set such that

$$\Lambda \subseteq \Lambda(W)$$
.

Then, we have $\operatorname{rank}(\Lambda) \leq \operatorname{rank}(\Lambda(W))$. Therefore, by Lemma 4.5 and Thm. 5.2,

$$\operatorname{ap-rank}(\Lambda) \leq \operatorname{rank}(\Lambda) \leq \operatorname{rank}(\Lambda(W)) = \operatorname{ap-rank}(\Lambda)$$
.

This gives

$$\operatorname{ap-rank}(\Lambda) = \operatorname{rank}(\Lambda)$$
.

 \leftarrow

Let $k := \operatorname{ap-rank}(\Lambda) = \operatorname{rank}(\Lambda)$. By Thm. 2.16 there exists a CPS $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L})$, a window $W \subseteq \mathbb{R}^m$ and a finite set $F = \{f_1, \dots, f_l\}$ such that

(2)
$$\Lambda \subseteq \Lambda(W) + F.$$

Moreover, without loss of generality we can assume that no proper subset F' of F satisfies (2). We start by showing that there exists some n so that $n\Lambda \subseteq L = \pi_{\mathbb{R}^d}(\mathcal{L})$.

First, note that by Thm. 5.2 we have

$$rank(\wedge(W)) = rank(\wedge(W - W)) = k.$$

By (2), we can partition Λ as

$$\Lambda = \bigcup_{j=1}^{l} \Lambda_{j}$$

$$\Lambda_{j} \subseteq \Lambda(W) + f_{j}.$$

Note that for all $1 \le j \le l$ we have

$$\Lambda_i - \Lambda_i \subseteq \wedge(W) - \wedge(W)$$

and hence

$$\Gamma \coloneqq \bigcup_{j=1}^{l} (\Lambda_j - \Lambda_j) \subseteq \wedge(W) - \wedge(W).$$

We claim that $\Gamma := \bigcup_{i=1}^{l} (\Lambda_i - \Lambda_i)$ is relatively dense. Indeed, set

$$J := \{j : 1 \le j \le l, \Lambda_j \ne \emptyset\}.$$

Then,

$$\Gamma = \bigcup_{j \in J} (\Lambda_j - \Lambda_j) .$$

Now, for each $j \in J$ fix some $x_i \in \Lambda_i$, which exists by the definition of J. Let

$$F' = \{x_j : j \in J\}$$
.

Then, we have

$$\begin{split} & \Lambda = \bigcup_{j \in J} \Lambda_j = \bigcup_{j \in J} (\Lambda_j - x_j + x_j) \subseteq \bigcup_{j \in J} (\Lambda_j - x_j + F') \\ & = (\bigcup_{j \in J} (\Lambda_j - x_j)) + F' \subseteq (\bigcup_{j \in J} (\Lambda_j - \Lambda_j)) + F' = \Gamma + F' \,. \end{split}$$

Since Λ is relatively dense, it follows immediately that Γ is relatively dense. In particular, Γ is a Meyer set. Now, by Thm. 5.2 we have

$$\operatorname{ap-rank}(\Gamma) = \operatorname{rank}(\wedge (W - W)) = k$$
.

Since $\Gamma \subseteq \Lambda(W - W)$ we have

$$k = \operatorname{ap-rank}(\Gamma) \le \operatorname{rank}(\Gamma) \le \operatorname{rank}(\Lambda(W - W)) = k$$

and hence $rank(\Gamma) = k$.

Next, let $L_1 = \langle \Lambda \rangle$ and $L_2 = \langle \Gamma \rangle$ by the \mathbb{Z} -modules generated by Λ and Γ , respectively. Since for each $1 \leq j \leq l$ we have $\Lambda_j \subseteq \Lambda$ and hence $\Lambda_j - \Lambda_j \subseteq \Lambda - \Lambda \subseteq L_1$, we get that L_2 is a \mathbb{Z} -submodule of L_1 .

Now, recall that by Lemma 4.1 we have

$$L = \langle \wedge(W) \rangle = \langle \wedge(W - W) \rangle$$
.

Since $\Gamma \subseteq \Lambda(W - W)$ we have $\Gamma \subseteq L$ and hence L_2 is a submodule of L. Moreover, by the above

$$rank(L_1) = rank(L_2) = k$$

by Lemma 2.3 there exists some positive integer n such that $nL_1 \subseteq L_2$. Therefore

$$n\Lambda \subseteq nL_1 \subseteq L_2 \subseteq L$$
,

as claimed. Next, let v_1, \ldots, v_{d+m} be the vectors so that

$$\mathcal{L} = \mathbb{Z}v_1 \oplus \ldots \oplus \mathbb{Z}v_{m+d}.$$

Now, by enlarging the lattice \mathcal{L} we can make sure that Λ is inside the projection of the lattice on \mathbb{R}^d . Indeed, for each $1 \leq j \leq m+d$ let

$$w_j = \frac{1}{n}v_j$$

and set

$$\mathcal{L}' \coloneqq \mathbb{Z}w_1 \oplus \ldots \oplus \mathbb{Z}w_{m+d}.$$

Then, it is obvious that $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L}')$ is a CPS and that

$$L' = \pi_{\mathbb{R}^d}(\mathcal{L}') = \frac{1}{m}\pi_{\mathbb{R}^d}(\mathcal{L}) = \frac{1}{n}L.$$

Moreover, by construction we have $\mathcal{L} \subseteq \mathcal{L}'$. This gives

$$\Lambda \subseteq \frac{1}{n}L = L'$$
.

We complete the proof by showing that $F \subseteq L'$. The result will follow from this. Let $f \in F$ be arbitrary. By the minimality of F we have

$$\Lambda \not\subseteq \Lambda(W) + (F \setminus \{f\})$$
.

Therefore, there exists some $x \in \Lambda$ such that $x \notin \Lambda(W) + (F \setminus \{f\})$. However, as $x \in \Lambda \subseteq \Lambda(W) + F$, we get that

$$x \in \Lambda(W) + f$$
.

Thus, there exists some $y \in A(W)$ such that x = y + f. It follows that

$$f = x - y \in \Lambda - \Lambda(W) \subseteq L' - L \subseteq L' - L' = L'$$

as claimed. Therefore, for all $1 \leq j \leq l$, there exists some $g_j \in \mathbb{R}^m$ such that $(f_j, g_j) \in \mathcal{L}'$. Define

$$W' = \bigcup_{j=1}^{l} g_j + W \subseteq \mathbb{R}^m.$$

Then, W' is pre-compact and has non-empty interior.

We show that

$$\Lambda \subseteq \{x \in L' : \exists y \in W' \text{ such that } (x,y) \in \mathcal{L}'\} =: \Lambda'(W'),$$

which, as $\wedge'(W')$ is a fully Euclidean model set in the CPS $(\mathbb{R}^d, \mathbb{R}^m, \mathcal{L}')$, completes the proof.

Let $x \in \Lambda$ be arbitrary. Then, as $\Lambda \subseteq \Lambda(W) + F$, there exists some $y \in \Lambda(W)$ and $1 \le j \le l$ such that

$$x = y + f_i$$
.

Since $y \in \Lambda(W)$, there exists some $z \in W$ so that $(y,z) \in L \subseteq L'$. Then, we have

$$(x, z + g_j) = (y, z) + (f_j, g_j) \in \mathcal{L} + \mathcal{L}' \subseteq \mathcal{L}' + \mathcal{L}' = \mathcal{L}'$$
$$z + g_j \in g_j + W \subseteq W',$$

and hence $x \in \Lambda'(W')$.

Remark 6.2. Thm. 6.1 can be equivalently stated as follows.

Let $\Lambda \subseteq \mathbb{R}^d$ be a Meyer set. Then, Λ is a fully Euclidean Meyer set if and only if for $k = \operatorname{rank}(\Lambda)$, for each $N \in \mathbb{N}$ there exist some $s, r_1, \ldots, r_k \in \mathbb{R}^d$ such that r_1, \ldots, r_k are linearly independent over \mathbb{Z} and for all $0 \le C_j \le N$ we have

$$s + C_1 r_1 + \dots + C_k r_k \in \Lambda$$
.

Note here that if A is a li-arithmetic progression of length $N \geq 1$ in a fully Euclidean Meyer set $\Lambda(W)$, with ratios r_1, r_2, \ldots, r_k , then we have $r_1, r_2, \ldots, r_k \in \Lambda$ which gives $k \leq \operatorname{rank}(\Lambda(W))$. Therefore, we get the following.

Corollary 6.3. Let $\Lambda \subseteq \mathbb{R}^d$ be a fully Euclidean Meyer set with rank $(\Lambda) = k$. Then,

- (a) Every li-arithmetic progression in Λ of length $N \ge 1$ has rank at most k.
- (b) For each $N \in \mathbb{N}$ there exists a li-arithmetic progression in Λ of rank k and length N.

We complete the paper by giving an explicit example of a Meyer set Λ which is not fully Euclidean, and we explicitly construct a CPS which produces it as a non-fully Euclidean Meyer set. In fact, Λ will be a regular model in this CPS.

Example 6.4. Let **Fib** denote the well-known Fibonacci model set with its corresponding CPS $(\mathbb{R}, \mathbb{R}, \mathcal{L})$, where $\mathcal{L} = \mathbb{Z}(1,1) \oplus \mathbb{Z}(\tau,\tau')$. We refer the reader to [2] for a full description. Note that **Fib** is a fully Euclidean regular model set within a 2-dimensional CPS.

Now, take $\Lambda = \pi + \mathbf{Fib}$; it follows that Λ is still relatively dense and thus a Meyer set. As the ap-rank is invariant under translates, we have

$$\operatorname{ap-rank}(\Lambda) = \operatorname{ap-rank}(\mathbf{Fib}) = 2$$
.

Now since τ is an algebraic integer and π is transcendental, $1, \tau, \pi$ are linearly independent over \mathbb{Z} . It is easy to see that

$$1, \tau, \pi \subseteq \langle \pi + \mathbf{Fib} \rangle \subseteq \mathbb{Z} + \mathbb{Z}\tau + \mathbb{Z}\pi$$
.

This immediately implies that $\langle \pi + \mathbf{Fib} \rangle = \mathbb{Z} + \mathbb{Z}\tau + \mathbb{Z}\pi$ and hence

$$rank(\Lambda) = 3$$
,

and hence, by Theorem 6.1, Λ is a Meyer set which is not fully Euclidean.

In fact, Λ is a regular model set. Indeed, consider

$$\mathcal{L} := \{ m + n\tau + k\pi : m, n, k \in \mathbb{Z} \} \subseteq \mathbb{R} \times (\mathbb{R} \times (\mathbb{Z}\pi)) .$$

Then, it is easy to see that $(\mathbb{R}, \mathbb{R} \times (\mathbb{Z}\pi), \mathcal{L})$ is a CPS and

$$\Lambda = \Lambda([-1, \tau - 1) \times \{\pi\}).$$

Remark 6.5. Given a CPS (G, H, \mathcal{L}) , a window W and some $a \in G$. As usual let us denote by $L = \pi_G(\mathcal{L})$.

If $a \in L$ then $a + \wedge (W) = \wedge (a^* + W)$ is a model set in the same CPS.

Otherwise, it is shown implicitly in [25, Prop. 5.6.19] that one can make $a+\wedge(W)$ into a model set the following way:

Let H_0 be the cyclic subgroup of G/L generated by a + L. Define

$$\mathcal{L}' \coloneqq \{(x + na, na + L, x^*) : (x, x^*) \in \mathcal{L}, n \in \mathbb{Z}\}.$$

Then, $(G, H \times H_0, \mathcal{L}')$ is a CPS and

$$a + \wedge(W) = \wedge'(W \times \{a + L\}).$$

Let us note here in passing that H_0 is a cyclic group of order at least 2, so it is isomorphic to either some $\mathbb{Z}/n\mathbb{Z}$ or to \mathbb{Z} .

Remark 6.6. Let $\Lambda \subseteq \mathbb{R}^d$ be a Meyer set. Then, by [25, Cor. 5.9.20.] and the structure theorem for compactly generated LCAG, there exists a CPS (\mathbb{R}^d , $\mathbb{R}^m \times \mathbb{Z}^n \times \mathbb{K}$, \mathcal{L}) with \mathbb{K} a compact Abelian group, and some compact $W \subseteq \mathbb{R}^m \times \mathbb{Z}^n \times \mathbb{K}$ such that

$$\Lambda \subseteq \Lambda(W)$$
.

Now, let

$$\pi: \mathbb{R}^m \times \mathbb{Z}^n \times \mathbb{K} \to \mathbb{R}^m \times \mathbb{Z}^n$$

be the canonical projection and let

$$\mathcal{L}' := \{ (x, \pi(x^*)) : (x, x^*) \in \mathcal{L} \}$$

Then, $(\mathbb{R}^d, \mathbb{R}^m \times \mathbb{Z}^n, \mathcal{L}')$ is a CPS and

$$\Lambda \subseteq \wedge(W) \subseteq \wedge'(\pi(W))$$
.

This shows that every Meyer set $\Lambda \subseteq \mathbb{R}^d$ is a subset of a model set in a CPS of the form $(\mathbb{R}^d, \mathbb{R}^m \times \mathbb{Z}^n, \mathcal{L})$, with $n \geq 0$. If the Meyer set is not fully Euclidean, then every such CPS must have n > 0. We suspect that the smallest possible value n can take among all the CPS of this form is exactly

$$n = \operatorname{rank}(\Lambda) - \operatorname{ap-rank}(\Lambda)$$
.

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