Van der Pol Oscillator - Analysis of a Non-conservative System

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Abstract

The Van der Pol oscillator was introduced by Balthasar van der Pol, who was "a famous scholar, a famous scientist, a famous administrator at the international level, he was equally well known for the clarity of his lectures (in several languages), his knowledge of the classics, his warm personality and his talents for friendship, and his love for music." [2] The oscillator describes the nonlinear oscillations for systems like a triode circuit, which produce self-sustained oscillations known as relaxation oscillations. Extensive studies have been done on the oscillator, for understanding it and for using it as an applied model for the heartbeat, for example. In this thesis, we will explain the nature of the oscillator from an original point of view, in the low-friction regime. First, we will give an intuitive physical explanation of the first order averaging method, a perturbation theory method, applied onto the oscillator. We will follow with an analytical approach of the first order averaging method, and we will show the mathematical complexity of it. We will conclude with the application of the first order averaging method to the Van der Pol oscillator, confirming the findings from the intuitive approach.

Contents

1 Introduction

1.1 Introduction to self-sustaining oscillating systems

In the nineteenth century, scientists became increasingly interested in systems with self-sustaining oscillations, and how to model the behaviour of such systems. A simple example of this is the idealized mass on a spring. This model describes hanging some mass, m, on a spring with spring constant k, and perturbing this system by some distance, A, from equilibrium. This is modeled through an ordinary differential equation (ODE) by describing the force acting on this system in two distinct ways.

Let x be the distance of the mass from equilibrium at time t. First, we describe the force exerted on the mass through Newton's second law of motion:

$$F = m \cdot \ddot{x} \tag{1.1.1}$$

The second way is to view the force as proportional to the mass' current distance from equilibrium. This is known as Hooke's law:

$$F = -k \cdot x \tag{1.1.2}$$

combining (1.1.1) and (1.1.2) we get:

$$m\ddot{x} = -kx$$

$$\Downarrow$$

$$m\ddot{x} + kx = 0 \tag{1.1.3}$$

thus arriving at the ODE that describes the motion of the idealised mass on a spring.

The general solution for the ODE (1.1.3) is:

$$x(t) = c_1 \cdot \cos\left(\sqrt{\frac{k}{m}} \cdot t\right) + c_2 \cdot \sin\left(\sqrt{\frac{k}{m}} \cdot t\right)$$

Thus, we can see that the above solution is a sinusoidal function.

Our focus for this thesis will be on self-sustaining oscillations arising from electrical circuits. The electrical equivalent of the mass on a spring model is an LC circuit. That is, a circuit containing an inductor and a capacitor. The LC circuit takes the form of a simple harmonic oscillator meaning it is modelled by a similar ODE with a similar sinusoidal solution.

1.2 History of Relaxation Oscillations

Self-sustaining oscillations do not always come with simple sinusoidal solutions. More complex systems will have behaviours that are more difficult to model. Examples of such systems are what Balthasar Van der Pol called relaxation oscillations, named after the "relaxation" of the charge in the capacitor in the circuit that he was studying. These are self-sustaining systems that result from nonlinear restoring forces. The use of nonlinear refers to whether the restoring force of a system takes the form of a linear polynomial. If we look at the restoring force of the mass on a spring model, $F = -k \cdot x$, we can see that this function is linear with respect to x. The restoring force refers to the force that tries to reset the system back to equilibrium.

These systems were studied since the late nineteenth century [4]. With one early example coming from London in the nineteenth century [4]. At the time, street lights and lighthouses used electrical arcs to produce light. Since electrical arcs create loud buzzing sounds, the government needed to find a solution to appease the citizens of London. This is when William Du Bois Duddel got involved. Duddel was an English electrical engineer and physicist. He noticed that if he placed an oscillating circuit, that being an LC circuit, in the circuit for the electric arc, he could change the sound being produced by the arc by varying the frequency of the oscillations in the circuit. He used this to turn up the frequency to the point that the buzzing sound was no longer audible. Through further experimentation, he noted that this change in frequency would correspond to different pitch in the sound created. With this in mind, he realized he could produce particular musical notes. Duddel dubbed this the ”musical arc” and it was one of the earliest examples of an electrical instrument [3].

Later, in 1903, Danish engineers Peder Oluf Pederson and Valdemar Poulsen improved this same technology[8][6]. They discovered that they could make the device take in direct current and output radio waves that allowed the transmission of wireless audio waves. This took the form of one of the first devices to transmit sound through radio waves.

Both of these inventions represent the importance of understanding relaxation oscillations. At that time, inventions regrading such oscillations were still in their infancy, and before proper theory of how to model and understand these systems was created. Yet, inventions with applications in telecommunications were arising, allowing progress in the technology of long range communications.

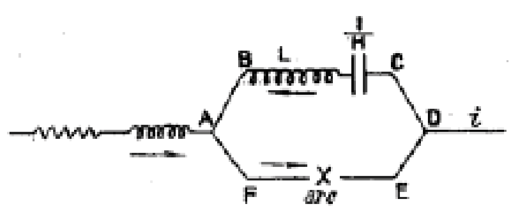
1.3 The Van der Pol equation

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0 \tag{1.3.1}$$

The equation (1.3.1) is called the Van der Pol equation. The Van der Pol equation is an ODE derived by Balthasar Van der Pol in 1926 [4]. Balthasar Van der Pol (27 January 1889 – 6 October 1959) was a dutch electrical engineer, physicist and mathematician who made substantial contributions to the theory of radio communications. Van der Pol was studying a triode circuit (a circuit in a vacuum tube consisting of three electrodes, the cathode, the anode, and a grid. The grid affects current between the cathode and the anode). This triode circuit exhibits relaxation oscillation behaviour, and Van der Pol sought to model it. More generally, Van der Pol's equation is dimensionless. This means, it is able to model a much wider variety of oscillatory behaviours. It should be noted that if $\mu = 0$ in the Van der Pol equation, we arrive to the equation of the simple harmonic oscillator, which is described in the equation (1.1.3) for the idealised mass on a spring.

Leading up to the derivation of Van der Pol's equation, there were other differential equations all with similar aims, to model relaxation oscillations in particular electrical systems.

First was Poincaré's equation for Duddel's musical arc [History of Van der Pol equation]. To derive this, Poincaré needed to utilise his newly developed theory of limit cycles. A limit cycle is a closed trajectory in the phase plane of a dynamical system. A particle on this trajectory will return to its initial displacement indefinitely, and the trajectories of other particles in the vicinity of the limit cycles will asymptotically approach the trajectory of the cycle. In order to construct the differential equation for the musical arc, Poincaré studied the block diagram of the circuit for the musical arc [4]



Using this diagram and denoting x as the current charge of the capacitor at time t, he used Kirchoff's law of voltage of a closed loop, which says that the sum of voltages in a closed loop is equal to zero. With this, and noting that $\frac{dx}{dt} = I$ is the current in the loop, Poincaré derived the differential equation:

$$L\ddot{x} + \rho\dot{x} + \theta(\dot{x}) + Hx = 0 \tag{1.3.2}$$

where L is the inductance of the inductor, $H = \frac{1}{C}$ where C is the capacitance of the capacitor, $\rho\dot{x}$ is a general term to describe resistance in the inductor and other instances of damping.

Next came French scientist Paul Janet. Janet was the first to show that the musical arc and the triode could both be described by the same differential equation [4]. This was what led Van der Pol to developing his dimensionless equation, which was described at the opening of the section.

1.4 Applications of the Van der Pol equation

Thus far we have seen that the Van der Pol equation has its origins in circuitry, but this is not the only application of this model. More modern research sees the Van der Pol model being applied to the pacemaker centers of the heart [9]. Paper [9] notes that the sino-atrial and the atrioventricular nodes can be viewed as relaxation oscillations. These nodes are responsible for generating and transmitting the electrical signal that causes the heart to beat[Alberta health]. With this in mind, the author creates a modified version of the Van der Pol equation:

$$\ddot{x} + \alpha(x^2 - \mu) + \frac{x(x+d)(x+2d)}{d^2} = 0 \tag{1.4.1}$$

Using this modified equation, the author investigates the dynamical features of the model, in order to try to create a model to describe the activity of a healthy heart.

2 Liénard Equation

2.1 RLC Series Circuit

An RLC series circuit is a circuit containing a resistor, an inductor and a capacitor. Similarly to an LC circuit, an RLC circuit has oscillatory behaviour with respect to the charge of the capacitor and the electric current through the circuit.

In Figure 1 we have an RLC series circuit supplied by a generator providing $v(t)$ volts

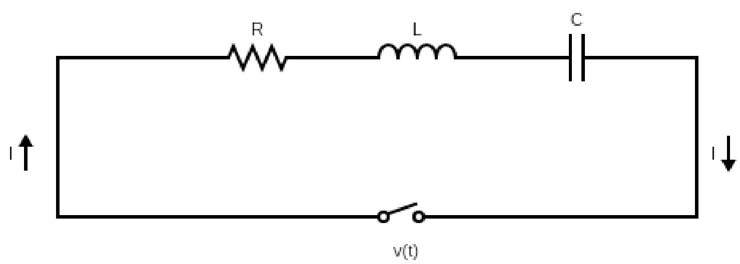


Figure 1: Example of an RLC series circuit

(V) of electricity. This generator is in series with a resistor with resistance R ohms (Ω), an inductor with inductance L henrys (H), and a capacitor with capacitance C farads (F). When the switch is closed, an electrical current of $I = I(t)$ amperes (A) will flow through the circuit. The intensity of the electric current in the capacitor is a function of time and of the charge in the capacitor, $Q = Q(t)$, and it is defined by

$$I = \frac{dQ}{dt} \tag{2.1.1}$$

To describe the oscillations in this circuit, we derive two ODEs. The first will describe the electric charge in the capacitor and the second will describe the intensity of the electric current on the circuit.

We apply Kirchhoff's voltage law to the circuit in Figure 1, which tells us that the sum of all voltage drops in the resistor, inductor and capacitor is equal to the voltage of the source:

$$V_R + V_L + V_C = V(t) \tag{2.1.2}$$

From physics, we know the following formulas for voltage drops:

- The voltage drop on the resistor is:

$$V_R = V_R(t) = R \cdot I \text{ (Ohm's law)} \tag{2.1.3}$$

- The voltage drop on the inductor is:

$$V_L = V_L(t) = L\frac{dI}{dt} \tag{2.1.4}$$

  Remark: The inductance, L, is defined as the ratio of the magnetic flux produced, Φ_B, to the intensity of the electrical current in the inductor producing it.

$$L := \frac{\Phi_B(I)}{I}$$

- The voltage drop on the capacitor is:

$$V_C = V_C(t) = \frac{1}{C}Q \tag{2.1.5}$$

Combining (2.1.2 - 2.1.5), we arrive at:

$$RI + L\frac{dI}{dt} + \frac{1}{C}Q = v(t) \tag{2.1.6}$$

From (2.1.1) and (2.1.6) we get:

$$R\frac{dQ}{dt} + L\frac{d(\frac{dQ}{dt})}{dt} + \frac{1}{C}Q = v(t) \tag{2.1.7}$$

or equivalently

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = v(t) \tag{2.1.8}$$

thus giving us our ODE for the electrical charge $Q = Q(t)$. If we want to solve the ODE (2.1.8), we need initial conditions. Thus, we create the initial value problem

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = v(t), Q(0) = Q_0, Q'(0) = 0 \tag{2.1.9}$$

Note: The condition on $Q'(0)$ is always valid since $Q' = I$ and the intensity of the electric current will be 0 the moment the switch is closed (t=0).

To obtain the ODE for the intensity of the electric current in the circuit, we start with (2.1.6) and we take the derivative of both sides of the equation with respect to time, to get:

$$R\frac{dI}{dt} + L\frac{d^2I}{dt^2} + \frac{1}{C}\frac{dQ}{dt} = \frac{dv(t)}{dt}$$

Once more we use (2.1.1) in the latter equation to obtain:

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dv(t)}{dt} \tag{2.1.10}$$

which is the ODE describing the intensity of the electric current in the circuit.

2.2 Derivation of the Liénard Equation

In order to derive the Liénard equation, we need to modify the previously discussed RLC circuit. We replace the resistor with a semiconductor or vacuum tube (triode) and replace the capacitor with a linear time-varying one. By replacing the resistor with a semiconductor or vacuum tube, we introduce a varying amount of resistance to the circuit. The resistance in the semiconductor is inversely proportional to the temperature in it, i.e. as the temperature increases, the resistance decreases. Similarly, the vacuum tube (or triode) will have a varying resistance, but the resistance will vary based off of the ”grid bias” instead of temperature. Grid bias describes the relative charge between the control grid and the cathode of the triode. A more negative charge will repel the electrons back to the cathode, thus reducing the intensity of the electric current. A more positive grid bias will repel fewer electrons, thus increasing the intensity of electric current up to some threshold. [1]

This distinction between semiconductors and vacuum tubes being able to alter the intensity of the electric current versus the resistors having an unchanging effect on the current is why semiconductors and vacuum tubes are considered ”active” components whereas resistors are considered ”passive” components. Active describes that the component can induce or alter electric current in the circuit, contrastingly passive components are unable to induce more electric current, nor actively alter it, i.e. there is no change that can be made to the passive component that will effect the electric current. [5]

Semiconductors can switch between conductive and insulative behaviour due to the gaps between the energy levels of the electrons in the material. Conducting of electricity is carried out by valence electrons jumping between different energy levels in the conducting material. For a conductor, such as copper wire, it is easy for valence electrons to transfer between different energy levels; however, for a semiconductor, such as silicone, it is more difficult for the electrons to move to higher energy levels, thus they need some extra energy (in our case heat) in order to nudge them to higher energy states. Without the extra energy, the semiconductor will act as an insulator.

For our derivation of the Liénard equation, we will specifically replace the resistor with a semiconductor and not a vacuum tube. The voltage drop on the semiconductor will thusly differ from the resistor.

When viewing the voltage drop in the semiconductor, we have to consider that the intensity of the electric current is influenced by the temperature, thus variations in temperature will create variations in the intensity of the electric current passing through the semiconductor. Since the current effects the voltage drop, the voltage drop will be varying non-linearly with respect to the intensity of the electric current.

Since we have changed circuit components, the ODE (2.1.10) will be changed accordingly;

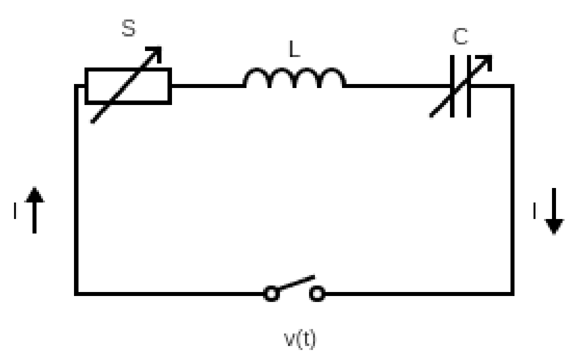


Figure 2: RLC circuit with the resistor exchanged with a semiconductor and the capacitor exchanged with a linear time-varying capacitor

the first consequence is a change in intensity of the electrical current in the capacitor. Since the capacitance is now a function of time, we get:

$$I = \frac{dQ}{dt}$$

Solving (2.1.5) for $Q = Q(t)$, we get:

$$I = \frac{d(V_C \cdot C)}{dt} = (V_C \cdot C)' \tag{2.2.1}$$

Keeping in mind that $C = C(t)$ we get:

$$I = \frac{dV_C}{dt}C + \frac{dC}{dt}V_C \tag{2.2.2}$$

which is our equation for intensity of the electrical current in a capacitor with nonconstant capacitance.

Note: We can describe the voltage drop in a time constant capacitance capacitor by using (2.2.2). Since capacitance is constant, $\frac{dC}{dt} = 0$, thus (2.2.2) becomes:

$$I = \frac{dV_C}{dt}C \tag{2.2.3}$$

Dividing (2.2.3) by C, we get

$$\frac{dV_C}{dt} = \frac{1}{C}I \tag{2.2.4}$$

Integrating both sides of (2.2.4) on $[0, t]$, we obtain:

$$\int_0^t V_C'(\theta)d\theta = \frac{1}{C}\int_0^t I(\theta)d\theta \tag{2.2.5}$$

which gives us:

$$V_C(t) - V_C(0) = \frac{1}{C}\int_0^t I(\theta)d\theta$$

$$\Downarrow$$

$$V_C(t) = V_C(0) + \frac{1}{C}\int_0^t I(\theta)d\theta \tag{2.2.6}$$

Thus we get an equation for the voltage in a time-constant capacitance capacitor.

Now we need to derive an ODE for the intensity of the electric current $I = I(t)$ from Figure 8, thus we start by applying Kirchhoff's voltage law to the closed loop in Figure 8.

$$V_S + V_L + V_C = v(t) \tag{2.2.7}$$

As discussed prior, we can view the voltage drop, V_S, as a non-linear function of the intensity of the electrical current. Thus we get

$$V_S = F(I) \tag{2.2.8}$$

where F is a nonlinear function of I, which we assume to be continuous and differentiable.

Combining (2.1.4), (2.2.7) and (2.2.8), we get

$$F(I) + L\frac{dI}{dt} + V_C = v(t). \tag{2.2.9}$$

Multiplying both sides of the equation by C in order to group V_C and C, we get

$$CF(I) + CLI' + C \cdot V_C = Cv(t).$$

Taking the derivative of both sides with respect to t gives us:

$$C'F(I) + CF'(I)I' + C'LI' + CLI'' + (C \cdot V_C)' = (Cv(t))'$$

Applying (2.2.1), we get:

$$C'F(I) + CF'(I)I' + C'LI' + CLI'' + I = (Cv(t))'$$
$$CLI'' + (CF'(I) + C'L)I' + (C'F(I) + I - (Cv(t'')) = 0$$

Putting this in standard form, we get:

$$I'' + \frac{1}{CL}(CF'(I) + C'L)I' + \frac{1}{CL}(C'F(I) + I - (Cv(t))') = 0 \tag{2.2.10}$$

Which is the ODE for the intensity of the electric current $I = I(t)$ known as the Liénard equation.

2.3 Derivation of the Van der Pol equation

To derive the Van der Pol equation, we consider the circuit in Figure 8 in which we modify the electrical components such that the voltage source is constant and the capacitor is of fixed capacitance.

Now we want to find the Liénard equation under these conditions. Combining the equations (2.1.5) and (2.2.9), and taking the derivative with respect to time, we get:

$$F'(I)I' + LI'' + \frac{1}{C}Q' = 0 \tag{2.3.1}$$

$$\Downarrow$$

$$F'(I)I' + LI'' + \frac{1}{C}I = 0 \tag{2.3.2}$$

Now let us consider

$$F(I) = \frac{1}{3}I^3 - aI,\ a > 0 \text{ positive constant.}$$

Thus, (2.3.2) becomes:

$$(I^2 - a)I' + LI'' + \frac{1}{C}I = 0 \tag{2.3.3}$$

Next, consider the one-to-one transformation

$$(I, t) \to (\alpha x, \delta s) \tag{2.3.4}$$

$$\alpha > 0 \text{ and } \delta > 0 \text{ such that } LC = \delta^2 \text{ and } a = \alpha^2$$

Applying (2.3.4) to (2.3.3), we get the following:

$$\begin{aligned}
L(\frac{d^2\alpha x}{dt^2}) + ((\alpha x)^2 - \alpha^2)\frac{d(\alpha x)}{dt} + \frac{1}{C}(\alpha x) = 0 \\
L\Big(\frac{d^2\alpha x}{d(\delta s)^2}\Big) + ((\alpha x)^2 - \alpha^2)\frac{d(\alpha x)}{d\delta s} + \frac{1}{C}(\alpha x) = 0 \\
\frac{LC\alpha}{\delta^2}\Big(\frac{d^2 x}{ds^2}\Big) + \frac{C\alpha^3}{\delta}((x)^2 - 1)\frac{d(x)}{ds} + (\alpha x) = 0 \\
\frac{\delta^2\alpha}{\delta^2}\Big(\frac{d^2 x}{ds^2}\Big) + \frac{C\alpha^3}{\delta}((x)^2 - 1)\frac{d(x)}{ds} + (\alpha x) = 0 \\
\frac{d^2 x}{ds^2} + \frac{C\alpha^2}{\delta}((x)^2 - 1)\frac{dx}{ds} + x = 0
\end{aligned} \tag{2.3.5}$$

Letting $\mu = \frac{C\alpha^2}{\delta}$ and $\frac{dx}{ds} = \dot{x}$ in (2.3.5) we get

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0 \tag{2.3.6}$$

which is the Van der Pol equation.

The formula for μ in terms of the constants a, C and L is given as follows:

$$\mu^2 = \frac{C^2\alpha^4}{\delta^2} = \frac{C^2 a^2}{LC} = a^2\frac{C}{L}$$

$$\Downarrow$$

$$\mu = a\sqrt{\frac{C}{L}} \tag{2.3.7}$$

3 Van der Pol Equation

3.1 First Order Averaging Method, Intuitive Approach

3.1.1 Energy in the Van der Pol Oscillator

To begin our discussion of the first order averaging method, we will once again view the simple harmonic oscillator:

$$\ddot{x} + x = 0 \tag{3.1.1}$$

$$x(0) = \frac{x_0}{2} \tag{3.1.2}$$

$$\dot{x}(0) = \frac{x_0}{2} \tag{3.1.3}$$

This system describes a particle following simple harmonic oscillatory motion between $-x_0$ and x_0. The initial conditions of the system tell us that the starting position of the particle matches the starting velocity. We can track the position of the particle by solving the initial value problem.

The unique solution of (3.1.1)-(3.1.3) comes to:

$$x(t) = \frac{1}{2}x_0 \cos(t) + \frac{1}{2}x_0 \sin(t) \tag{3.1.4}$$

The simple harmonic oscillator can also be represented through a system of first order ODEs. To do this we start by letting $y(t) = \dot{x}(t)$, then we create the system:

$$\dot{y}(t) = \ddot{x}(t) = -x(t)$$

$$\dot{x}(t) = y(t)$$

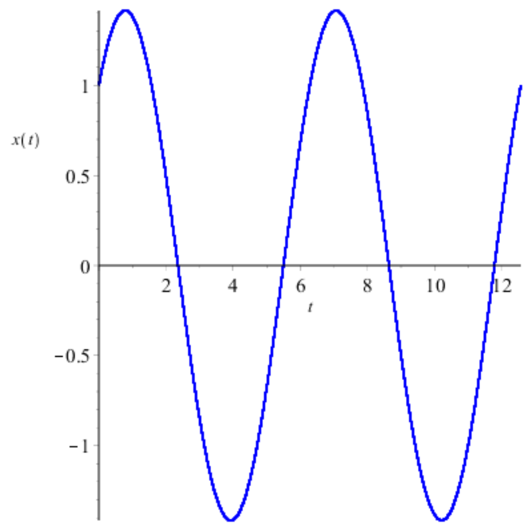


Figure 3: Plot of the solution to the simple harmonic oscillator with $x_0 = 2$.

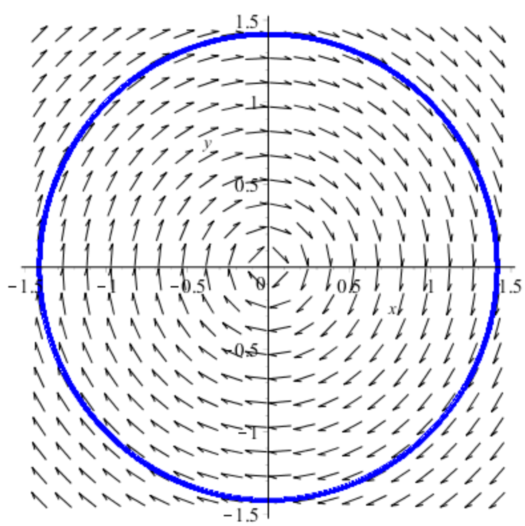


Figure 4: Phase Plot of the solution to the simple harmonic oscillator with $x_0 = 2$.

giving us the system:

$$\begin{cases} \dot{x}(t) = y(t) \\ \dot{y}(t) = -x(t) \end{cases} \tag{3.1.5}$$

This gives us the directional field for the simple harmonic oscillator:

$$\vec{V} =< \dot{x}(t), \dot{y}(t) >=< y(t), -x(t) >= y(t)\vec{e}_x - x(t)\vec{e}_y \tag{3.1.6}$$

In order to discover whether the system is conservative or non-conservative, we need to first discuss the energy of the system. First, we use the fact that the energy in a mechanical system is:

$$E = E_p + E_k \tag{3.1.7}$$

where E_p and E_k are the potential energy and kinetic energy respectively.

From Physics, we know the kinetic and potential energy of our particle to be

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}\dot{x}^2 \tag{3.1.8}$$

and

$$E_p = \frac{1}{2}x^2 \tag{3.1.9}$$

Now, to check whether the system is conservative or not, we need to see if the derivative of the energy with respect to time is 0. Thus, we take the derivative of (3.1.7) with respect to time:

$$\begin{aligned} \frac{dE}{dt} &= \frac{dE_p}{dt} + \frac{dE_k}{dt} \\ &= \dot{x}\ddot{x} + \dot{x}x \end{aligned}$$

$$= -\dot{x}x + \dot{x}x$$

$$= 0$$

Hence the system is conservative.

Next, we introduce a friction coefficient dependent upon x to our ODE:

$$\ddot{x} - \phi\dot{x} + x = 0 \tag{3.1.10}$$

where $\phi = \mu\left(1 - \frac{x^2}{{x_0}^2}\right)$, $\mu, x_0 \in \mathbb{R}$. Typically μ is referred to as the control parameter, and it measures the magnitude of nonlinearity in the oscillatory system described by (3.1.10).

We need to check whether the addition of the friction coefficient has changed the simple harmonic oscillator to no longer be conservative. To do this, we start with our new system (3.1.10) and multiply both sides by $\dot{x}$.

$$\ddot{x}\dot{x} - \phi\dot{x}^2 + x\dot{x} = 0$$

$$\ddot{x}\dot{x} + \dot{x}x = \phi\dot{x}^2$$

$$\frac{d(\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2)}{dt} = \phi\dot{x}^2 \tag{3.1.11}$$

Noting that the left hand side of (3.1.11) is the derivative of the energy of the system with respect to time, we arrive at:

$$\frac{dE}{dt} = \phi\dot{x}^2 \tag{3.1.12}$$

The equation (3.1.12) shows that the total energy in the system is no longer conserved, thus the system is non-conservative.

Now, let us make the following substitution:

$$x_{old} = x_0 \cdot x_{new} \tag{3.1.13}$$

Under this substitution, the equation (3.1.10) becomes:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0 \tag{3.1.14}$$

The equation (3.1.14) is the Van der Pol equation derived through physical means in Chapter 2. Thus we get that the rate of change of the energy in the Van der Pol oscillator is:

$$\frac{dE}{dt} = \mu(1 - x^2)\dot{x}^2 \tag{3.1.15}$$

We can note that there are two distinct cases for the energy in (3.1.15):

- Case 1: $-1 < x < 1$

  If $-1 < x < 1$, then we get that $x^2 < 1$ or $0 < 1 - x^2$, thus we can see that $\frac{dE}{dt}$ is positive. This tells us that the energy in the system is increasing, therefore the Van der Pol oscillator generates energy

- Case 2: $x < -1$ or $x > 1$

  In this case, $x^2 > 1$. Thus, we get $1 - x^2 < 0$ which tells us that the energy in the system is decreasing, therefore the Van der Pol oscillator dissipates energy

This shows us that, in Case 1, the amplitudes of the displacement will be increasing in time, i.e. amplitudes will be amplified. In Case 2, we can see that the amplitudes in the oscillator will be decreasing in time, or that the amplitudes will be damped. When $x \to \pm 1$, intuitively we understand that these should be a limit cycle that would separate the two distinct behaviours of the Van der Pol oscillator.

In the limit cycle we expect that, averaged over one period $T > 0$, the energy that was lost through the dissipation is gained through the injection of energy, i.e the energy is balanced in the limit cycle over one period of it, and we have

$$\overline{\frac{dE}{dt}} = 0 \tag{3.1.16}$$

where $\overline{\frac{dE}{dt}} = 0$ is the average energy over a period $[t_0, t_0 + T]$.

Next, we need to prove that the average energy over a period is constant. To start our derivation, we begin with the Van der Pol equation.

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$$

Next, we multiply both sides by $\dot{x}$ to get:

$$\ddot{x}\dot{x} - \mu(1 - x^2)\dot{x}^2 + x\dot{x} = 0$$

Rearranging, we get:

$$\mu(1 - x^2)\dot{x}^2 = \ddot{x}\dot{x} + x\dot{x}$$

Next, to represent averaging over one period, we take the integral over $[t_0, t_0 + T]$

$$\begin{aligned}\int_{t_0}^{t_0+T} \mu(1 - x^2)\dot{x}^2 dt &= \int_{t_0}^{t_0+T} \ddot{x}\dot{x} + x\dot{x}dt \\ \int_{t_0}^{t_0+T} \frac{dE}{dt}dt &= \int_{t_0}^{t_0+T} \frac{d(\dot{x}^2 + x^2)}{dt}dt \end{aligned} \tag{3.1.17}$$

Since we intuitively understand that we are on the limit cycle, we can say that, after one period a particle returns to the same location with the same velocity. Thus, in (3.1.17) we can say that the right hand side of the equation is 0.

$$\begin{aligned}\int_{t_0}^{t_0+T} \frac{dE}{dt} dt &= 0 \\ \frac{1}{T}\int_{t_0}^{t_0+T} \frac{dE}{dt} dt &= 0 \\ \frac{1}{T}(E(t_0+T) - E(t_0)) &= 0 \end{aligned} \tag{3.1.18}$$

The left hand side of (3.1.18) is the average value of $\frac{dE}{dt}$. Thus, we get:

$$\overline{\frac{dE}{dt}} = 0 \tag{3.1.19}$$

Our next step in intuitively understanding the averaging method is to view the Van der Pol equation as perturbations from the simple harmonic oscillator.

For $0 < \mu << 1$, we can view the Van der Pol oscillator as perturbations by μ from the simple harmonic oscillator, i.e.

$$x = x_h + \mu u(t, x_h) \tag{3.1.20}$$

where $u \in C^2[t_0, t_0+T] \times D$, $D \subset \mathbb{R}$ compact set.

Next, we want to see what happens to the average energy under this new view of x.

Starting with (3.1.18), and combining it with (3.1.20), we get:

$$\begin{aligned}\overline{\frac{dE}{dt}} &= 0 \\ \frac{1}{T}\int_{t_0}^{t_0+T} \frac{dE}{dt} dt &= 0 \\ \int_{t_0}^{t_0+T} \mu(1-x^2)\dot{x}^2 dt &= 0\end{aligned}$$

$$\int_{t_0}^{t_0+T} \mu(1-(x_h+\mu u(t,x_h))^2)\frac{(d(x_h+\mu u(t,x_h)))^2}{dt}dt = 0$$

$$\mu\int_{t_0}^{t_0+T}[1-(x_h^2+2x_h\mu u(t,x_h)+(\mu u(t,x_h))^2)(\dot{x_h}+\mu\dot{u}(t,x_h)\dot{x_h})^2]dt = 0$$

$$\mu\int_{t_0}^{t_0+T}\dot{x}^2[(1-x_h^2)-2x_h\mu u(t,x_h)-(\mu u(t,x_h))^2)\cdot$$

$$(1+2\mu u(t,x_h)+(\mu\dot{u}(t,x_h))^2)]dt = 0$$

$$\mu\int_{t_0}^{t_0+T}\dot{x}^2(1-x_h^2)dt-\mu^2\int_{t_0}^{t_0+T}\dot{x}^2[2x_h u(t,x_h)-\mu(u(t,x_h))^2)\cdot$$

$$(1+2\mu u(t,x_h)+(\mu\dot{u}(t,x_h))^2)]dt = 0$$

$$\mu\int_{t_0}^{t_0+T}\dot{x}^2(1-x_h^2)dt+O(\mu^2) = 0 \tag{3.1.21}$$

Thus, we arrived at the following equation that we will use later on, to determine the radius of the unit circle:

$$\mu\int_{t_0}^{t_0+T}(1-x^2)\dot{x_h}^2dt+O(\mu^2) = 0 \tag{3.1.22}$$

3.1.2 The Van der Pol Equation as an Autonomous System

The Van der Pol equation can be written as an autonomous system of ordinary differential equations as follows:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x+\mu(1-x^2)y \end{cases} \tag{3.1.23}$$

The only equilibrium point of (3.1.23) is the origin. To show this, let's solve the system

$$\begin{cases} y = 0 \\ -x + \mu(1 - x^2)y = 0. \end{cases} \tag{3.1.24}$$

Combining both equations in the system (3.1.24), we get

$$-x = 0. \tag{3.1.25}$$

Thus, the only solution to system (3.1.24) is $x = 0$ and $y = 0$, therefore the only equilibrium point of (3.1.23) is the origin.

Next, we need to linearize the system about the origin. This requires finding the Jacobian matrix of the system, and finding the eigenvalues of the Jacobian at the origin:

$$\text{Let } f_1(x, y) = y \text{ and } f_2(x, y) = -x + \mu(1 - x^2)y$$

Thus the Jacobian matrix of our system is:

$$J_{f_1,f_2}(x, y) = \begin{bmatrix} 0 & 1 \\ -1 - 2\mu xy & \mu(1 - x^2) \end{bmatrix}$$

$$\text{Let } A = J_{f_1,f_2}(0, 0) = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix} \tag{3.1.26}$$

Now we need to find the eigenvalues of A.

$$\lambda I - A = \begin{bmatrix} \lambda & -1 \\ 1 & \lambda - \mu \end{bmatrix}$$

$$
\begin{aligned}
|\lambda I - A| &= \lambda(\lambda - \mu) + 1 \\
&= \lambda^2 - \lambda\mu + 1
\end{aligned}
$$

Thus, the eigenvalues of A are:

$$
\lambda_{1,2} = \frac{\mu}{2} \pm \sqrt{\left(\frac{\mu}{2}\right)^2 - 1} \tag{3.1.27}
$$

From here, we have to consider 3 cases:

- Case 1: $0 < \mu < 2$

  If $\mu < 2$, then $\left(\frac{\mu}{2}\right)^2 < 1$. This shows that A has imaginary eigenvalues both with positive real parts. The origin will be an unstable focus.

- Case 2: $\mu = 2$

  If $\mu = 2$, then $\left(\frac{\mu}{2}\right)^2 - 1 = 0$, and A will have repeated positive eigenvalues, which gives us an unstable node. In order to classify which type of node we have, we have to check the dimension of the eigenspace of A with respect to $\lambda = \frac{\mu}{2} = 1$:

  We need to solve the equation $(\lambda I - A)\vec{v} = 0$ for $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$
\lambda I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \tag{3.1.28}
$$

  We can see that v_2 is free. Let $v_2 = t$. Combining $v_2 = t$ and (3.1.28) we get

$$
\begin{aligned}
v_1 - v_2 &= 0 \\
v_1 &= t, \ t \in \mathbb{R} \\
v_1 &= t
\end{aligned} \tag{3.1.29}
$$

Thus,

$$\vec{v} = \begin{bmatrix} t \\ t \end{bmatrix} \tag{3.1.30}$$

is an eigenvector for A corresponding to eigenvalue $\lambda = 1$ for any $t \in \mathbb{R}$.

Thus, the eigenspace of A with respect to $\lambda = 1$ is $span((1,1)^T)$, therefore the eigenspace is of dimension 1. Since the eigenspace is of dimension 1 and the eigenvalue is positive, we can say that the origin is an improper unstable node.

- Case 3: $\mu > 2$

  If $\mu > 2$, then $\left(\frac{\mu}{2}\right)^2 > 1$, and thus the eigenvalues of A are real. Now we have to check the sign of both eigenvalues. Thus, we have to check whether $\frac{\mu}{2} > \sqrt{\left(\frac{\mu}{2}\right)^2 - 1}$:

$$\frac{\mu}{2} > \sqrt{\left(\frac{\mu}{2}\right)^2 - 1}$$
$$\left(\frac{\mu}{2}\right)^2 > \left(\frac{\mu}{2}\right)^2 - 1$$
$$0 > -1$$

  Thus, the latter inequality is always true. Now we can conclude that, when $\mu > 2$ we have two distinct real positive eigenvalues. This means that the origin is an unstable node.

We can conclude, from these three cases, that no matter the value of $\mu > 0$, we have an unstable equilibrium point at the origin.

Another consequence of the above study is that the Van der Pol oscillator generates energy away from its stationary solution towards the limit cycle, while away from the limit cycle high energies dissipate as they approach the limit cycle, therefore the energy gained compensates the energy loss. Hence, in the limit cycle a balanced energy (E_b) is created for each period of the limit cycle.

In the limit cycle we view the oscillatory process as being conserved, therefore intuitively we can claim the following:

$$x_h^2 + \dot{x}_h^2 = E_b \tag{3.1.31}$$

For simplicity we can rewrite (3.1.31) as follows:

$$x_h^2 + \dot{x}_h^2 = \phi_0^2 \text{ where } \phi_0 = \sqrt{E_b} \neq 0 \tag{3.1.32}$$

Returning to the equation (3.1.22), and noting that we are in the regime of $0 < \mu << 1$, so we can say that $O(\mu^2)$ can be approximated to be zero. We obtain:

$$\int_{t_0}^{t_0+T} (1 - x_h^2)\dot{x}_h^2 dt = 0 \tag{3.1.33}$$

We are interested in finding the radius of the circular orbit described in (3.1.32), ϕ_0, such that the integral in (3.1.33) is zero, independently of the period $T = 2\pi$ of the limit cycle.

Using (3.1.32), we know that in the $x_h\dot{x}_h-$plane we have a circular orbit. We can parameterize this orbit as follows:

$$\begin{cases} x_h = \phi_0 \cos(\theta) \\ \dot{x}_h = \phi_0 \sin(\theta) \end{cases} \tag{3.1.34}$$

From (3.1.34), we can find $\frac{d\theta}{dt}$:

$$x_h = \phi_0 \cos(\theta)$$
$$\dot{x}_h = -\phi_0 \sin(\theta)\frac{d\theta}{dt}$$

and

$$\dot{x}_h = \phi_0 \sin(\theta)$$

Therefore, we get

$$\frac{d\theta}{dt} = -1. \tag{3.1.35}$$

By (3.1.35), we can find $\theta(t)$:

$$\begin{aligned} \frac{d\theta}{dt} &= -1 \\ \theta(t) &= -t + c,\, c \in \mathbb{R} \end{aligned} \tag{3.1.36}$$

Now we have enough information to solve (3.1.33) for ϕ_0:

$$\begin{aligned} \int_{t_0}^{t_0+T} (1 - x_h^2)\dot{x}_h^2 dt &= \int_{t_0}^{t_0+T} (1 - \phi_0^2 \cos^2(\theta))\phi_0^2 \sin^2(\theta) dt \\ &= \phi_0^2 \int_{t_0}^{t_0+T} (\sin^2(\theta) - \phi_0^2 \sin^2(\theta)\cos^2(\theta)) dt \frac{d\theta}{d\theta} \\ &= \phi_0^2 \int_{t=t_0}^{t=t_0+T} (\sin^2(\theta) - \phi_0^2 \sin^2(\theta)\cos^2(\theta)) \frac{dt}{d\theta} d\theta \\ &= -\phi_0^2 \left(\int_{-t_0+c}^{-t_0-T+c} \sin^2(\theta) d\theta - \phi_0^2 \int_{-t_0+c}^{-t_0-T+c} \sin^2(\theta)\cos^2(\theta) d\theta \right) \\ &= -\phi_0^2 \left(-\frac{\sin(2\theta) - 2\theta}{4} - \phi_0^2 \left(-\frac{\sin(4\theta) - 4\theta}{32} \right) \right) \Bigg|_{-t_0+c}^{-t_0-T+c} \\ &= -\phi_0^2 \left(\frac{-8\sin(2\theta) + 16\theta}{32} + \phi_0^2 \left(\frac{\sin(4\theta) - 4\theta}{32} \right) \right) \Bigg|_{-t_0+c}^{-t_0-T+c} \\ &= -\phi_0^2 \left(\frac{-8\sin(2\theta) + \phi_0^2 \sin(4\theta) + 4\theta(4 - \phi_0^2)}{32} \right) \Bigg|_{-t_0+c}^{-t_0-T+c} \end{aligned} \tag{3.1.37}$$

Since

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\frac{x_h}{\phi_0}\frac{\dot{x}_h}{\phi_0}, \tag{3.1.38}$$

$$\sin(4\theta) = 2\sin(2\theta)\cos(2\theta) = 4\sin(\theta)\cos(\theta)(\cos^2(\theta) - \sin^2(\theta)) \tag{3.1.39}$$

$$= 4\frac{x_h}{\phi_0}\frac{\dot{x}_h}{\phi_0}\left(\frac{x_h^2}{\phi_0^2} - \frac{\dot{x}_h^2}{\phi_0^2}\right), \tag{3.1.40}$$

and x_h and $\dot{x}_h$ are T-periodic, then $\sin(2\theta)$ and $\sin(4\theta)$ are T-periodic.

Hence, the trigonometric terms in the equation (3.1.37) are equal at the end points of evaluation. This tells us that

$$-\phi_0^2\left(\frac{-8\sin(2\theta) + \phi_0^2\sin(4\theta)}{32}\right)\Bigg|_{-t_0+c}^{-t_0-T+c} = 0. \tag{3.1.41}$$

Now, let us investigate what is left in (3.1.37):

$$\begin{aligned}
&-\phi_0^2\left(\frac{-8\sin(2\theta) + \phi_0^2\sin(4\theta) + 4\theta(4-\phi_0^2)}{32}\right)\Bigg|_{-t_0+c}^{-t_0-T+c} \\
&= -\phi_0^2\left(\frac{4\theta(4-\phi_0^2)}{32}\right)\Bigg|_{-t_0+c}^{-t_0-T+c} \\
&= -\phi_0^2\left(\frac{\theta(4-\phi_0^2)}{8}\right)\Bigg|_{-t_0+c}^{-t_0-T+c} \\
&= -\phi_0^2\left(\frac{(4-\phi_0^2)}{8}\right)(-t_0 - T + c - (-t_0 + c)) \\
&= \phi_0^2\left(\frac{(4-\phi_0^2)}{8}\right)T
\end{aligned} \tag{3.1.42}$$

In order for (3.1.42) to equal zero, we need $\phi_0 = 2$. Hence, for $0 < \mu << 1$, the limit cycle is a circle of radius 2.

In this section we focused on the analysis of the Van der Pol oscillator, intuitively showing the existence of the limit cycle that separates the two states of the system, the dissipation of energy state and the generation of energy state. The study was done in the regime of $0 < \mu << 1$.

If μ is large, the Van der Pol system will still have its two states mentioned above, but they will "meet" at a limit cycle that will be distorted from the shape of a circle. The Liénard Theorem [7] implies that for any $\mu > 0$, the Van der Pol system has a unique limit cycle separating two states.

In this section we intended to show original work, based on the intuitive understanding of the Van der Pol system, rather than to reproduce a classic result in Dynamical Systems, i.e. Liénard Theorem,

In the set of figures below we show, for different values of μ, the phase portrait of some solution curves of the Van der Pol equation. Because Van der Pol equation is a second order ODE, which can written as an autonomous system of two first order ODE's, we plotted the solution curves together with the direction field of the equation.

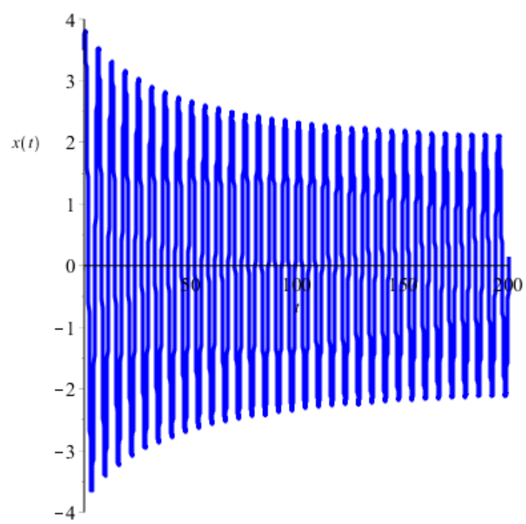


Figure 5: Energy dissipation for the Van der Pol oscillator with $\mu = 0.01$ and initial conditions $x(0) = 3.5$ and $\dot{x}(0) = 1.5$.

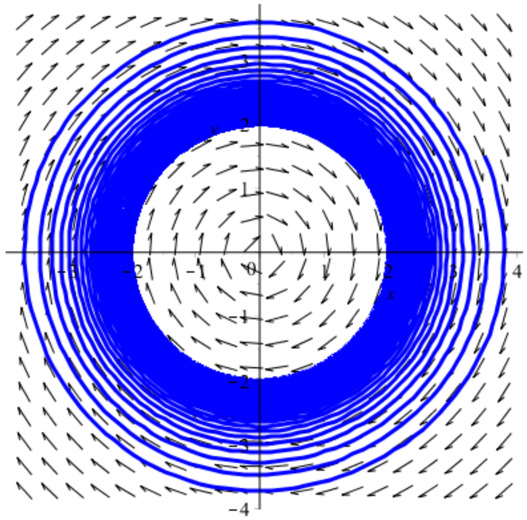


Figure 6: Phase portrait for energy dissipation for the Van der Pol oscillator with $\mu = 0.01$ and initial conditions $x(0) = 3.5$ and $\dot{x}(0) = 1.5$.

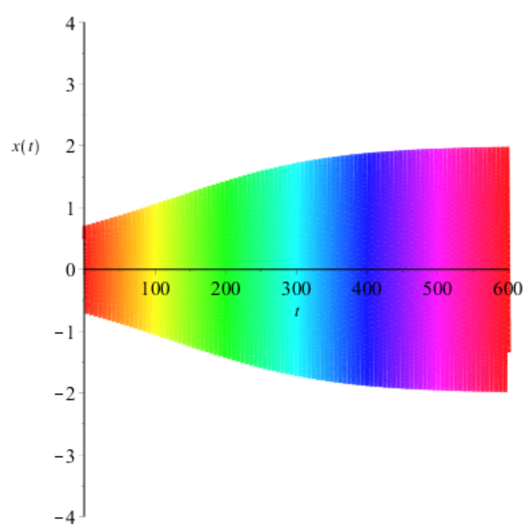


Figure 7: Energy generation for the Van der Pol oscillator with $\mu = 0.01$ and initial conditions $x(0) = 0.5$ and $\dot{x}(0) = 0.5$.

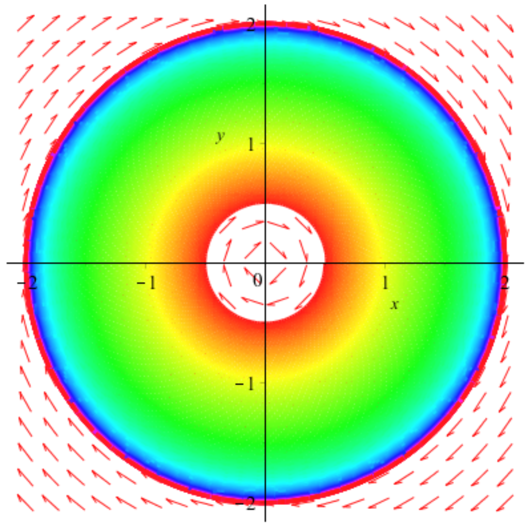


Figure 8: Phase portrait for energy generation for the Van der Pol oscillator with $\mu = 0.01$ and initial conditions $x(0) = 0.5$ and $\dot{x}(0) = 0$. The colours of the plot show the density of the trajectory of the particle in a section of the graph, i.e. where more colours appear, the smaller the gap between paths of the particle over distinct periods. This shows the most density at the circle of radius two, which is the limit cycle.

3.2 First Order Averaging Method, Analytical Approach

Consider the system

$$\dot{y} = \mu g(t, y) \tag{3.2.1}$$

$g : S^1 \times D \to \mathbb{R}^n, D \subset \mathbb{R}^n$ compact set, which satisfies the following conditions:

i. $0 < \mu << 1$

ii. $g \in C^2(S^1 \times D)$

iii. g is periodic, of period T, in its first argument.

Consider the following transformation

$$L = y + \mu u(t, y) \tag{3.2.2}$$

where u is called generating function.

If L has one-to-one correspondence, then (3.2.2) can be solved uniquely for y

$$y = L + \mu v(t, L). \tag{3.2.3}$$

Next, we find the derivative of L with respect to t:

$$\begin{aligned} \dot{L} &= \dot{y} + \frac{\partial(\mu u(t, y))}{\partial t} \\ &= \mu g(t, y) + \mu \frac{\partial(u(t, y))}{\partial t} \end{aligned} \tag{3.2.4}$$

Note that $y \in \mathbb{R}^n$. We get

$$\frac{\partial(u(t, y))}{\partial t} = \frac{\partial u(t, y)}{\partial t} + \sum_{i=1}^{n} \frac{\partial u(t, y_i)}{\partial y_i} \frac{\partial y_i}{\partial t}. \tag{3.2.5}$$

Where

$$\frac{\partial y_i}{\partial t} = \mu g_i(t, y). \tag{3.2.6}$$

Combining (3.2.5) and (3.2.6), we get

$$\begin{aligned} \frac{\partial(u(t, y))}{\partial t} &= \frac{\partial u(t, y)}{\partial t} + \sum_{i=1}^{n} \frac{\partial u(t, y_i)}{\partial y_i} \mu g_i(t, y) \\ &= \frac{\partial u(t, y)}{\partial t} + \mu \frac{\partial u(t, y)}{\partial y} \bullet g(t, y). \end{aligned} \tag{3.2.7}$$

Returning to (3.2.4), we get

$$\begin{aligned} \dot{L} &= \mu g(t, y) + \mu \left(\mu \frac{\partial u(t, y)}{\partial y} \bullet g(t, y) + \frac{\partial u(t, y)}{\partial t} \right) \\ &= \mu \left(g(t, y) + \frac{\partial u(t, y)}{\partial t} \right) + \mu^2 \frac{\partial u(t, y)}{\partial y} \bullet g(t, y). \end{aligned} \tag{3.2.8}$$

We choose the generating function u such that the contribution of $\frac{\partial u}{\partial t} + g(t, y)$ to be minimal with respect to the oscillations of g. In order to do so, we will choose u such that $\frac{\partial u}{\partial t} + g(t, y)$

is reduced to the mean value of g over its period, i.e.

$$u(t,y) = -\int_0^t (g(\tau,y) - \overline{g}(y))d\tau \tag{3.2.9}$$

where

$$\overline{g}(y) = \frac{1}{T}\int_0^T g(t,y)dt \tag{3.2.10}$$

is the mean value of g over its period.

By the choice of u

$$u(t,y) = -\int_0^t (g(\tau,y) - \overline{g}(y))d\tau, \tag{3.2.11}$$

and by $g \in C^2(S^1 \times D)$, we get that u is continuous on the compact set $(S^1 \times D)$, therefore u is bounded.

Using the equation (3.2.4) and the fact that u is bounded, we will show

$$\dot{L} = \mu\overline{g}(t,y) + O(\mu^2). \tag{3.2.12}$$

Applying (3.2.9) into (3.2.8), we get:

$$\begin{aligned}\dot{L} =& \mu\left(g(t,y) + \frac{\partial\left(-\int_0^t(g(\tau,y) - \overline{g}(y))d\tau\right)}{\partial t}\right)\\ &+ \mu^2\left(\frac{\partial\left(-\int_0^t(g(\tau,y) - \overline{g}(y))d\tau\right)}{\partial y} \bullet g(t,y)\right)\end{aligned} \tag{3.2.13}$$

Since both u and g are bounded, the second term in the equation (3.2.13) will be dominated by the magnitude of μ, therefore the term is of order $O(\mu^2)$. Then, the equation (3.2.13) becomes:

$$\dot{L} = \mu \left(g(t,y) + \frac{\partial \left(-\int_0^t (g(\tau,y) - \overline{g}(y)) d\tau \right)}{\partial t} \right) + O(\mu^2)$$
$$= \mu \left(g(t,y) - (g(t,y) - \overline{g}(y)) + O(\mu^2) \right.$$
$$= \mu \overline{g}(y) + O(\mu^2) \tag{3.2.14}$$

Combining the equations (3.2.14) and (3.2.3), we get:

$$\dot{L} = \mu \overline{g}(L) + O(\mu^2) \tag{3.2.15}$$

Note: From (3.2.3), v is bounded as y and L are bounded, therefore the term $O(\mu^2)$ is justified.

Let us show that under our choice of u and manipulating μ, L is one-to-one.

Let y_1, $y_2 \in \mathbb{R}^n$ such that $L(y_1) = L(y_2)$, then we have

$$L(y_1) = L(y_2)$$
$$y_1 + \mu u(t, y_1) = y_2 + \mu u(t, y_2)$$
$$y_1 - y_2 = \mu(u(t, y_2) - u(t, y_1))$$
$$\Downarrow$$
$$\|y_1 - y_2\| = \mu \|u(t, y_2) - u(t, y_1)\|. \tag{3.2.16}$$

We have as well that u is continuously differentiable on a compact set $S^1 \times D$, thus u is Lipschitz:

$$\|u(t, y(t_2)) - u(t, y(t_1))\| \leq c\|(y_2 - y_1)\|, \; c > 0 \textit{ positive constant} \tag{3.2.17}$$

Applying that u is Lipschitz, (3.2.16) becomes

$$\|y_2 - y_1\| \leq \mu c \|y_2 - y_1\|. \tag{3.2.18}$$

Pick μ such that $\mu c < 1$. From this choice of μ, the only way the inequality (3.2.18) can hold is if

$$\|y_2 - y_1\| = 0 \tag{3.2.19}$$

which is equivalent to

$$y_2 = y_1. \tag{3.2.20}$$

Therefore, L is one-to-one under our choice of μ.

Let us consider the averaged system

$$\dot{x} = \mu \overline{g}(x) \tag{3.2.21}$$

Let y and x be solutions to (3.2.1) and (3.2.21) respectively, satisfying the initial conditions $y(0) = y_0$ and $x(0) = x_0$ respectively, such that

$$\|y_0 - x_0\| < k_0\mu,\ k_0 > 0 \text{ positive constant, i.e. } \|y_0 - x_0\| = O(\mu) \tag{3.2.22}$$

Define

$$r = L - x \tag{3.2.23}$$

and differentiate with respect to time to get:

$$\begin{aligned} \dot{r} &= \dot{L} - \dot{x} \\ &= \mu \overline{g}(L) + O(\mu^2) - \mu \overline{g}(x) \\ &= \mu(\overline{g}(L) - \overline{g}(x)) + O(\mu^2) \end{aligned} \tag{3.2.24}$$

Now, we integrate (3.2.24) over $[0,t]$ where $0 \leq t \leq \frac{1}{\mu}$:

$$\begin{aligned}
\int_0^t \dot{r}(s)ds &= \int_0^t \left[\mu(\overline{g}(L(s)) - \overline{g}(x(s))) + O(\mu^2)\right] ds \\
r(s)\Big|_0^t &= \int_0^t \left[\mu(\overline{g}(L(s)) - \overline{g}(x(s))) + O(\mu^2)\right] ds \\
r(t) - r(0) &= \int_0^t \left[\mu(\overline{g}(L(s)) - \overline{g}(x(s))) + O(\mu^2)\right] ds \\
r(t) - (L(0) - x(0)) &= \int_0^t \left[\mu(\overline{g}(L(s)) - \overline{g}(x(s))) + O(\mu^2)\right] ds \qquad (3.2.25)
\end{aligned}$$

Where,

$$\begin{aligned}
L(0) &= y(0) + \mu u(0, y(0)) \\
&= y(0) - \mu \int_0^0 (g(\tau, y) - \overline{g}(y))d\tau \\
&= y(0) \\
&= y_0. \qquad (3.2.26)
\end{aligned}$$

Thus, (3.2.25) becomes:

$$\begin{aligned}
r(t) - y_0 + x_0 &= \int_0^t \left[\mu(\overline{g}(L(s)) - \overline{g}(x(s))) + O(\mu^2)\right] ds \\
r(t) &= y_0 - x_0 + \int_0^t \left[\mu(\overline{g}(L(s)) - \overline{g}(x(s))) + O(\mu^2)\right] ds \qquad (3.2.27)
\end{aligned}$$

Using (3.2.27), we can find a bound for the norm of r:

$$\begin{aligned}
\|r(t)\| &= \left\| y_0 - x_0 + \int_0^t \mu(\overline{g}(L(s)) - \overline{g}(x(s)))ds + \int_0^t O(\mu^2)ds \right\| \\
&\leq \|y_0 - x_0\| + \left\| \int_0^t \mu(\overline{g}(L(s)) - \overline{g}(x(s)))ds \right\| + \left\| \int_0^t O(\mu^2)ds \right\| \\
&\leq k_0\mu + \int_0^t \|\mu(\overline{g}(L(s)) - \overline{g}(x(s)))\| \, ds + \int_0^t \left\|O(\mu^2)\right\| ds \qquad (3.2.28)
\end{aligned}$$

Next, we apply the Lipschitz condition to $\overline{g}$ in the second term of (3.2.28), and we use as well the fact that all of the functions in $O(\mu^2)$ are bounded in (3.2.28), and therefore we have:

$$\|\overline{g}(L(s)) - \overline{g}(x(s))\| \leq k_1\|L(s) - x(s)\|,\ k_1 > 0 \text{ positive constant,} \tag{3.2.29}$$

and

$$\|O(\mu^2)\| \leq k_2\mu^2,\ k_2 > 0 \text{ positive constant.} \tag{3.2.30}$$

Applying (3.2.29) and (3.2.30) to (3.2.28), we get:

$$\begin{aligned}
\|r(t)\| &\leq k_0\mu + \mu\int_0^t k_1\|L(s) - x(s)\|ds + k_2\mu^2\Big|_0^t \\
&= k_0\mu + \mu\int_0^t k_1\|r(s)\|ds + k_2\mu^2 t \\
&= k_0\mu + k_2\mu^2 t + \int_0^t \mu k_1\|r(s)\|ds
\end{aligned} \tag{3.2.31}$$

In order to further bound the norm of r, we apply Gronwall's Lemma: If

$$a(t) \leq b(t) + \int_c^t c(s)\cdot a(s)ds,\ t \in [c, d] \tag{3.2.32}$$

then

$$a(t) \leq b(t) + \int_c^t b(s)\cdot c(s)\cdot e^{\int_s^t c(\xi)d\xi}ds \tag{3.2.33}$$

If we pick $a(t) = \|r(t)\|$, $b(t) = k_0\mu + k_2\mu^2 t$ and $c(t) = \mu k_1$, we get:

$$\begin{aligned}
\|r(t)\| &\leq k_0\mu + k_2\mu^2 t + \int_0^t \mu k_1\|r(s)\|ds \\
&\Downarrow \\
\|r(t)\| &\leq k_0\mu + k_2\mu^2 t + \int_0^t (k_0\mu + k_2\mu^2 s)(\mu k_1)\cdot e^{\int_s^t \mu k_1 d\xi}ds \\
&= k_0\mu + k_2\mu^2 t + \int_0^t (k_0\mu + k_2\mu^2 s)(\mu k_1)\cdot e^{\xi\mu k_1\big|_s^t}ds
\end{aligned}$$

$$
\begin{aligned}
&= k_0\mu + k_2\mu^2 t + \int_0^t (k_0\mu + k_2\mu^2 s)(\mu k_1)\cdot e^{t\mu k_1}e^{-s\mu k_1}ds \\
&= k_0\mu + k_2\mu^2 t + \int_0^t k_0\mu(\mu k_1)\cdot e^{t\mu k_1}e^{-s\mu k_1}ds \\
&\qquad\qquad + \int_0^t k_2\mu^2(\mu k_1)\cdot e^{t\mu k_1}e^{-s\mu k_1}sds \\
&= k_0\mu + k_2\mu^2 t - k_0\mu\cdot e^{t\mu k_1}e^{-s\mu k_1}\Big|_0^t \\
&\qquad\qquad + -\frac{k_2\mu\left(k_1\mu s+1\right)\mathrm{e}^{-k_1\mu(s-t)}}{k_1}\Big|_0^t \\
&= k_0\mu + k_2\mu^2 t - k_0\mu\cdot e^{t\mu k_1}(e^{-t\mu k_1} - e^{-0\mu k_1}) \\
&\qquad\qquad + -\frac{k_2\mu\left(k_1\mu t+1\right)\mathrm{e}^{-k_1\mu(t-t)}}{k_1}\Big|_0^t \\
&= k_0\mu + k_2\mu^2 t - k_0\mu\cdot e^{t\mu k_1}(e^{-t\mu k_1} - e^{-0\mu k_1}) \\
&\qquad\qquad + \frac{k_2\mu\left(\mathrm{e}^{k_1\mu t} - k_1\mu t - 1\right)}{k_1} \\
&= k_0\mu + k_2\mu^2 + k_0\mu(e^{t\mu k_1}-1) + \frac{k_2\mu\left(\mathrm{e}^{k_1\mu t} - k_1\mu t - 1\right)}{k_1} \\
&= k_0\mu + k_2\mu^2 + k_0\mu(e^{t\mu k_1}-1) + \frac{k_2\mu(e^{k_1\mu t}-1) - k_2k_1\mu^2 t}{k_1} \\
&= k_0\mu + k_2\mu^2 - k_2\mu^2 t + \frac{(e^{k_1\mu t}-1)(k_2\mu + k_1k_0)}{k_1} \\
&\le k_0\mu + k_2\mu^2 - k_2\mu^2 + \frac{k_0k_1+k_2}{k_1}\mu(e^{k_1\mu t}-1) \\
&= k_0\mu + \frac{k_0k_1+k_2}{k_1}\mu(e^{k_1\mu t}-1) \qquad (3.2.34)
\end{aligned}
$$

Thus, we arrive at

$$
\|r(t)\| \le k_0\mu + \frac{k_0k_1+k_2}{k_1}\mu(e^{\mu k_1 t}-1). \qquad (3.2.35)
$$

Since $t \le \frac{1}{\mu}$, (3.2.35) becomes:

$$
\begin{aligned}
\|r(t)\| &\le k_0\mu + \frac{k_0k_1+k_2}{k_1}\mu(e^{\mu k_1 t}-1) \\
&\le k_0\mu + \frac{k_0k_1+k_2}{k_1}\mu(e^{\mu k_1\frac{1}{\mu}}-1)
\end{aligned}
$$

$$= k_0\mu + \frac{k_0k_1 + k_2}{k_1}\mu(e^{k_1} - 1)$$
$$= \mu\left(k_0 + \frac{k_0k_1 + k_2}{k_1}(e^{k_1} - 1)\right) \tag{3.2.36}$$

As a conclusion, we get

$$\|L(t) - x(t)\| \leq \mu k \tag{3.2.37}$$

where

$$k = k_0 + \frac{k_0k_1 + k_2}{k_1}(e^{k_1} - 1). \tag{3.2.38}$$

Using (3.2.2) and (3.2.37), we will show

$$\|y - x\| = O(\mu) \tag{3.2.39}$$

We start by taking the norm of $y - x$, and use that L is injective, and thus invertable on some subset $B \subseteq R^n$ to get:

$$\begin{aligned}\|y(t) - x(t)\| &= \|L(t) - \mu v(t, L(t)) - x(t)\| \\ &= \|L(t) - x(t) - \mu v(t, L(t))\| \\ &\leq \|L(t) - x(t)\| + \mu\|v(t, L(t)\|\end{aligned} \tag{3.2.40}$$

Next we apply the boundedness of v to get

$$\|v(t, y(t))\| \leq M,\ M \in \mathbb{R} \tag{3.2.41}$$

Applying (3.2.41) to (3.2.40), we get:

$$\|y(t) - x(t)\| \leq \|L(t) - x(t)\| + \mu\|v(t, L(t))\| \leq \mu k + \mu M = \mu(k + M) \tag{3.2.42}$$

Thus, we get

$$\|y(t) - x(t)\| \leq \mu(k + M). \tag{3.2.43}$$

We showed that the initial system

$$\dot{y} = \mu g(t, y),\ 0 < \mu << 1 \tag{3.2.44}$$

and its associated averaged system

$$\dot{x} = \mu \bar{g}(x) \tag{3.2.45}$$

produce solutions that are close to each other to an order $O(\mu)$, when the Cauchy data prescribed on them is close to each other to an order $O(\mu)$.

That is why, in perturbation theory, the averaging methods are important in studying asymptotically a system of differential equations.

The system considered in (3.2.1) is said to be in Lagrange standard form.

3.3 Applying the First Order Averaging Method to the Van der Pol Oscillator

Consider the following system in Lagrange standard form

$$\dot{y} = \mu g(t, y),\ 0 < \mu << 1 \tag{3.3.1}$$

where $g \in C^2(S^1 \times D)$, $D \subseteq \mathbb{R}^2$ compact set, and g is periodic in its first argument with period 2π.

Consider the associated averaged system to the system (3.3.1):

$$\dot{x} = \mu \overline{g}(x) \tag{3.3.2}$$

where

$$\overline{g}(x) = \frac{1}{2\pi} \int_0^{2\pi} g(t, x) dt \tag{3.3.3}$$

In section 3.2, we proved that the system (3.3.2) is a good approximation of the system (3.3.1) within $0 < t < \mu$, i.e. on a time scale $t \sim \frac{1}{\mu}$. Therefore, we will study analytically, via perturbation theory, the Van der Pol oscillator for $0 < \mu << 1$. In this case, the time span in which the systems (3.3.1) and (3.3.2) have a similar behaviour, of order $O(\mu)$, is considerably improved, and therefore we will be able to extract valuable information about the Van der Pol oscillator in this regime, i.e. $0 < \mu << 1$.

Consider the Van der Pol equation

$$\ddot{z} - \mu(1 - z^2)\dot{z} + z = 0 \tag{3.3.4}$$

We will work with (3.3.4) in the form:

$$\ddot{z} + z = \mu f(t, z, \dot{z}) \tag{3.3.5}$$

where

$$f(t, z, \dot{z}) = (1 - z^2)\dot{z}. \tag{3.3.6}$$

In order to put the Van der Pol oscillator in Lagrange standard form, we will need to view it as an autonomous differential equation again.

Let

$$\begin{cases} I_1 = z \\ I_2 = \dot{I}_1 \end{cases} \tag{3.3.7}$$

Differentiating with respect to time both equations in (3.3.7), we obtain the following system

$$\begin{cases} \dot{I}_1 = I_2 \\ \dot{I}_2 = \ddot{I}_2 = -I_1 + \mu(1 - I_1^2)\dot{I}_1 \end{cases} \tag{3.3.8}$$

$$\Downarrow \tag{3.3.9}$$

$$\begin{cases} \dot{I}_1 = 0I_1 + I_2 + 0f(t, I_1, \dot{I}_1) \\ \dot{I}_2 = -I_1 + 0I_2 + \mu f(t, I_1, \dot{I}_1) = -I_1 + 0I_2 + \mu(1 - I_1^2)I_2 \end{cases} \tag{3.3.10}$$

This leads to the system

$$\dot{I} = AI + \mu f^*(t, I) \tag{3.3.11}$$

where

$$I = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{3.3.12}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{3.3.13}$$

$$f^*(t, I) = \begin{bmatrix} 0 \\ f(t, I_1, I_2) \end{bmatrix} \tag{3.3.14}$$

We are interested to bring the system (3.3.11) to Lagrange standard form.

We start by multiplying both sides of (3.3.11) by e^{-tA}:

$$\begin{aligned}
\dot{I}e^{-tA} &= AIe^{-tA} + \mu f^*(t, I)e^{-tA} \\
\dot{I}e^{-tA} - AIe^{-tA} &= \mu f^*(t, I)e^{-tA} \\
\frac{d(Ie^{-tA})}{dt} &= \mu f^*(t, I)e^{-tA}
\end{aligned} \tag{3.3.15}$$

Let

$$y = e^{-tA}I = e^{-tA}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} e^{-tA}I_1 \\ e^{-tA}I_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{3.3.16}$$

then we get

$$f^*(t, I) = f^*(t, e^{tA}y) \tag{3.3.17}$$

Thus, the equation (3.3.15) becomes

$$\dot{y} = \mu g(t, y), \tag{3.3.18}$$

where

$$g(t, y) = f^*(t, e^{tA}y)e^{-tA} \tag{3.3.19}$$

We can find the exact formula for (3.3.18) by finding the exact matrix corresponding to e^{tA} and e^{-tA}. To do this, we note that

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{3.3.20}$$

is the matrix representation of the complex number $-i$ using the mapping

$$a + ib \longleftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \; a, b \in \mathbb{R}. \tag{3.3.21}$$

Using this fact, the matrix representation of e^{tA} and e^{-tA} will be the images of e^{-it} and e^{it} respectively.

$$e^{it} = \cos(t) + i\sin(t) \tag{3.3.22}$$

$$e^{-it} = \cos(-t) + i\sin(-t) \tag{3.3.23}$$

Thus,

$$e^{it} \longleftrightarrow \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \tag{3.3.24}$$

$$e^{-it} \longleftrightarrow \begin{bmatrix} \cos(-t) & -\sin(-t) \\ \sin(-t) & \cos(-t) \end{bmatrix} \tag{3.3.25}$$

Therefore, we get

$$e^{-At} = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix} \tag{3.3.26}$$

$$e^{At} = \begin{bmatrix} \cos(-t) & -\sin(-t) \\ \sin(-t) & \cos(-t) \end{bmatrix} \tag{3.3.27}$$

Next, we find $e^{tA}y$

$$e^{tA}y = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1\cos(t) + y_2\sin(t) \\ -y_1\sin(t) + y_2\cos(t) \end{bmatrix} \tag{3.3.28}$$

Using (3.3.28), we can find $f^*(t, e^{tA}y)$:

$$f^*(t, e^{tA}y) = \begin{bmatrix} 0 \\ f(t, [e^{tA}y]_1, [e^{tA}y]_2) \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ (1 - (y_1 \cos(t) + y_2 \sin(t))^2)[-y_1 \sin(t) + y_2 \cos(t)] \end{bmatrix} \tag{3.3.29}$$

Using (3.3.29), we can find $g(t, y)$:

$$g(t, y) = f^*(t, e^{tA}y)e^{-tA}$$
$$= \begin{bmatrix} 0 \\ (1 - (y_1 \cos(t) + y_2 \sin(t))^2)[-y_1 \sin(t) + y_2 \cos(t)] \end{bmatrix} \cdot \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$
$$= \begin{bmatrix} (1 - (y_1 \cos(t) + y_2 \sin(t))^2)[-y_1 \sin(t) + y_2 \cos(t)](-\sin t) \\ (1 - (y_1 \cos(t) + y_2 \sin(t))^2)[-y_1 \sin(t) + y_2 \cos(t)](\cos t) \end{bmatrix} \tag{3.3.30}$$

Thus, the explicit formula for (3.3.18) is

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \mu \begin{bmatrix} (1 - (y_1 \cos(t) + y_2 \sin(t))^2)[-y_1 \sin(t) + y_2 \cos(t)](-\sin t) \\ (1 - (y_1 \cos(t) + y_2 \sin(t))^2)[-y_1 \sin(t) + y_2 \cos(t)](\cos t) \end{bmatrix} \tag{3.3.31}$$

We are interested to find the averaged system of the Van der Pol oscillator, starting from its Lagrange standard form (3.3.31).

The averaged system is given by (3.3.2), where

$$\overline{g}(y) = \frac{1}{2\pi} \int_0^{2\pi} g(t, y)dt \tag{3.3.32}$$

Next, we evaluate the integral in (3.3.32) by combining (3.3.32) with (3.3.30). We get:

$$\begin{aligned}
\overline{g}(y) &= \frac{1}{2\pi}\int_0^{2\pi} g(t,y)dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} \begin{bmatrix} (1-(y_1\cos(t)+y_2\sin(t))^2)[-y_1\sin(t)+y_2\cos(t)](-\sin t) \\ (1-(y_1\cos(t)+y_2\sin(t))^2)[-y_1\sin(t)+y_2\cos(t)](\cos t) \end{bmatrix} dt \\
&= \frac{1}{2\pi}\begin{bmatrix} \int_0^{2\pi}(1-(y_1\cos(t)+y_2\sin(t))^2)[-y_1\sin(t)+y_2\cos(t)](-\sin t)dt \\ \int_0^{2\pi}(1-(y_1\cos(t)+y_2\sin(t))^2)[-y_1\sin(t)+y_2\cos(t)](\cos t)dt \end{bmatrix} \\
&= \frac{1}{2\pi}\begin{bmatrix} \frac{\pi}{4}y_1[4-(y_1^2+y_2^2)] \\ \\ \frac{\pi}{4}y_2[4-(y_1^2+y_2^2)] \end{bmatrix}
\end{aligned} \tag{3.3.33}$$

Hence, the averaged system of the Van der Pol oscillator is

$$\begin{aligned}
\dot{y_1} &= \frac{\mu}{8}y_1[4-(y_1^2+y_2^2)] \\
\dot{y_2} &= \frac{\mu}{8}y_2[4-(y_1^2+y_2^2)]
\end{aligned} \tag{3.3.34}$$

Let us write the system (3.3.34) into polar coordinates. Let $y_1 = r\cos\theta$ and $y_2 = r\sin\theta$. Thus, we get

$$\begin{aligned}
\frac{\partial r}{\partial t} &= \frac{\partial r}{\partial y_1}\frac{\partial y_1}{\partial t} + \frac{\partial r}{\partial y_2}\frac{\partial y_2}{\partial t} \\
&= \frac{1}{\cos\theta}\frac{\mu}{8}r\cos\theta[4-r^2] + \frac{1}{\sin\theta}\frac{\mu}{8}r\sin\theta[4-r^2] \\
&= \frac{\mu}{4}r[4-r^2]
\end{aligned} \tag{3.3.35}$$

and

$$\begin{aligned}
\frac{\partial \theta}{\partial t} &= \frac{\partial \theta}{\partial y_1}\frac{\partial y_1}{\partial t} + \frac{\partial \theta}{\partial y_2}\frac{\partial y_2}{\partial t} \\
&= \frac{-1}{r\cos\theta}\frac{\mu}{8}r\cos\theta[4-r^2] + \frac{1}{r\sin\theta}\frac{\mu}{8}r\sin\theta[4-r^2]
\end{aligned}$$

$$= 0 \tag{3.3.36}$$

Therefore, the system (3.3.34) in polar coordinates is

$$\begin{aligned} \dot{r} &= \frac{\mu}{4} r[4 - r^2] \\ \dot{\theta} &= 0 \end{aligned} \tag{3.3.37}$$

Obviously, $r(t) = 2$, $\theta = c$, c=constant, is a periodic solution of the system (3.3.37), corresponding to the closed trajectory Γ: $y_1^2 + y_2^2 = 4$.

We want to show that the trajectory Γ is a limit cycle.To do this, we will see what happens to $\dot{r}$ inside of and outside of the trajectory Γ.

Inside of Γ, we have $r < 2$, thus $\dot{r} = \frac{\mu}{4} r[4 - r^2] > 0$. Therefore, the radius is increasing when inside of Γ.

Outside of Γ, we have $r > 2$, thus $\dot{r} = \frac{\mu}{4} r[4 - r^2] < 0$. Therefore, the radius is decreasing when outside of Γ.

We can see that particles inside of Γ will travel on trajectories away from the origin converging towards Γ, and particles outside of Γ will travel on trajectories towards the origin and converging towards Γ. We can see as well that Γ is the only closed trajectory for the system (3.3.37), since $r = 2, \theta = c$ are the only equilibrium points for (3.3.37). Thus, we can conclude that Γ is a limit cycle.

Using the first order averaging method presented in section 3.2, we showed that the Van der Pol oscillator has a unique limit cycle in the regime $0 < \mu << 1$. Due to the nature of the trajectories in the vicinity of Γ, the limit cycle is stable, i.e. the limit cycle is an attractor.

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