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Abstract. Given a non-trivial family of pairs of faces of unital C*-algebras where each pair has a faithful state, it is proved that if the bi-free product state is faithful on the reduced bi-free product of this family of pairs of faces then each pair of faces arises as a minimal tensor product. A partial converse is also obtained.

1. introduction

The reduced free product was given independently by Avitzour [1] and Voiculescu [7] and it has been foundational in the development of free probability. Dykema proved in [2] that the free product state on the reduced free product of unital C*-algebras with faithful states is faithful. In consequence of this, if $\{\mathcal{A}_i\}_{i\in\mathcal{I}}$ is a free family of unital C*-algebras in the non-commutative C*-probability space $(\mathcal{A},\varphi)$ and if φ is faithful on $C^*(\{\mathcal{A}_i\}_{i\in\mathcal{I}})$ then

$$C^*(\{\mathcal{A}_i\}_{i\in\mathcal{I}}) \simeq *_{i\in\mathcal{I}}(\mathcal{A}_i, \varphi|_{\mathcal{A}_i}),$$

the reduced free product of the $\mathcal{A}_i$'s with respect to the given states. This can be deduced from a paper of Dykema and Rørdam, namely [3, Lemma 1.3].

The present paper is the result of the author's attempt to prove the same result in the new context of bi-free probability introduced by Voiculescu [8]. To this end, suppose $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}}$ is a non-trivial family of pairs of faces in the non-commutative C*-probability space $(\mathcal{A},\varphi)$. If $\varphi_i = \varphi|_{C^*(\mathcal{A}_l^{(i)},\mathcal{A}_r^{(i)})}$ is faithful on $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$, for all $i\in\mathcal{I}$, then it will be proven that if the bi-free product state $**_{i\in\mathcal{I}}\varphi_i$ is faithful on the reduced bi-free product $**_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ then $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_{min} \mathcal{A}_r^{(i)}, i\in\mathcal{I}$. A converse is shown with the added asumption that each φ_i is a product state. Moreover, in this case there is a commensurate result to that which follows from Dykema and Rørdam, mentioned above.

It should be mentioned that the failure in general of the faithfulness of the bi-free product state has been pointed out in [4] and this failure has been the cause of the introduction of weaker versions of faithfulness in the bi-free context [4, 5].

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2. Bi-free independence and the reduced bi-free product

We will first take some time to recall the definition of bi-free independence from [8] and then define the reduced bi-free product of C*-algebras and the bi-free product state.

Fix a non-commutative C*-probability space $(\mathcal{A}, \varphi)$, that is a unital C*-algebra and a state. Given a set $\mathcal{I}$, suppose that for each $i \in \mathcal{I}$ there is a pair of unital C*-subalgebras $\mathcal{A}_l^{(i)}$ and $\mathcal{A}_r^{(i)}$ of $\mathcal{A}$, a "left" algebra and a "right" algebra. We call the set $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}}$ a family of pairs of faces in $\mathcal{A}$. Such a family will be called *non-trivial* if $|\mathcal{I}| \geq 2$ and $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \neq \mathbb{C}$ for all $i \in \mathcal{I}$. That is, there are at least two pairs of faces and there are no trivial pairs of faces.

Let $(\pi_i, \mathcal{H}_i, \xi_i)$ be the GNS construction for $(C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}), \varphi_i)$ where $\varphi_i = \varphi|_{C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})}$. Voiculescu [8] (and even way back in [7]) observed that there are two natural representations of $B(\mathcal{H}_i)$ on the free product Hilbert space, which we will now introduce. The free product Hilbert space,

$$(\mathcal{H}, \xi) = *_{i\in\mathcal{I}}(\mathcal{H}_i, \xi_i),$$

is given by associating all of the distinguished vectors and then forming a Fock space like structure. Namely, if $\mathring{\mathcal{H}}_j = \mathcal{H}_j \ominus \mathbb{C}\xi_j$, then

$$\mathcal{H} := \mathbb{C}\xi \oplus \bigoplus_{\substack{n\in\mathbb{N} \\ i_1,\cdots,i_n\in\mathcal{I} \\ i_1\neq\cdots\neq i_n}} \mathring{\mathcal{H}}_{i_1} \otimes \cdots \otimes \mathring{\mathcal{H}}_{i_n}.$$

To define these representations we need to first build some Hilbert spaces and some unitaries. To this end, define

$$\mathcal{H}(l,i) := \mathbb{C}\xi \oplus \bigoplus_{\substack{n\in\mathbb{N} \\ i_1,\cdots,i_n\in\mathcal{I} \\ i\neq i_1\neq\cdots\neq i_n}} \mathring{\mathcal{H}}_{i_1} \otimes \cdots \otimes \mathring{\mathcal{H}}_{i_n} \qquad \text{and}$$

$$\mathcal{H}(r,i) := \mathbb{C}\xi \oplus \bigoplus_{\substack{n\in\mathbb{N} \\ i_1,\cdots,i_n\in\mathcal{I} \\ i_1\neq\cdots\neq i_n\neq i}} \mathring{\mathcal{H}}_{i_1} \otimes \cdots \otimes \mathring{\mathcal{H}}_{i_n}.$$

Then there are unitaries $V_i : \mathcal{H}_i \otimes \mathcal{H}(l,i) \to \mathcal{H}$ and $W_i : \mathcal{H}(r,i) \otimes \mathcal{H}_i$ given by concatenation (with appropriate handling of ξ_i and ξ). Finally, the two natural representations are the left representation $\lambda_i : B(\mathcal{H}_i) \to B(\mathcal{H})$ which is defined as

$$\lambda_i(T) = V_i(T \otimes I_{\mathcal{H}(l,i)})V_i^*$$

and the right representation $\rho_i : B(\mathcal{H}_i) \to B(\mathcal{H})$ which is defined as

$$\rho_i(T) = W_i(I_{\mathcal{H}(r,i)} \otimes T)W_i^*.$$

With all of this groundwork established we can finally define bi-free independence. Note that $\check{*}$ below refers to the full (or universal) free product of C*-algebras.

Definition 2.1 (Voiculescu [8])**.** The family of pairs of faces $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}}$ in the non-commutative probability space $(\mathcal{A}, \varphi)$ is said to be *bi-freely independent* with respect to φ if the following diagram commutes

$$\begin{array}{ccccc}
\check{*}_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)} \check{*} \mathcal{A}_r^{(i)}) & \xrightarrow{\quad\iota\quad} & \mathcal{A} & \xrightarrow{\ \varphi\ } & \mathbb{C} \\
{\scriptstyle *_{i\in\mathcal{I}}(\pi_i * \pi_i)}\big\downarrow & & & & \big\| \\
\check{*}_{i\in\mathcal{I}}(B(\mathcal{H}_i)\check{*}B(\mathcal{H}_i)) & \xrightarrow{*_{i\in I}(\lambda_i * \rho_i)} & B(\mathcal{H}) & \xrightarrow{\langle\cdot\xi,\xi\rangle} & \mathbb{C}
\end{array}$$

where ι is the unique $*$-homomorphism extending the identity on each $\mathcal{A}_\chi^{(i)}$, for all $\chi \in \{l, r\}$ and $i \in \mathcal{I}$.

From this we can now define the main objects of this paper.

Definition 2.2. Let $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}}$ be a family of pairs of faces in the non-commutative C*-probability space $(\mathcal{A}, \varphi)$. As before, denote φ_i to be the restriction of φ to $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ and let $(\pi_i, \mathcal{H}_i, \xi_i)$ be the GNS construction of $(C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}), \varphi_i)$.

The *reduced bi-free product* of $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}}$ with respect to the states φ_i is

$$(**_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}), **_{i\in\mathcal{I}}\varphi_i) = **_{i\in\mathcal{I}}((\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}), \varphi_i)$$

which is made up of the unital C*-subalgebra of $B(\mathcal{H})$, called the *reduced bi-free product of C*-algebras*,

$$**_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) := C^*((\lambda_i \circ \pi_i(\mathcal{A}_l^{(i)}), \rho_i \circ \pi_i(\mathcal{A}_r^{(i)}))_{i\in\mathcal{I}}) \subset B(\mathcal{H})$$

and the *bi-free product state*

$$**_{i\in\mathcal{I}}\varphi_i(\cdot) := \langle\cdot\xi, \xi\rangle.$$

It is an immediate fact that the family of pairs of faces $(\lambda_i \circ \pi_i(\mathcal{A}_l^{(i)}), \rho_i \circ \pi_i(\mathcal{A}_r^{(i)}))_{i\in\mathcal{I}}$ is bi-freely independent with respect to the bi-free product state.

It should be noted that we are working within the framework of the original non-commutative C*-probability space $(\mathcal{A}, \varphi)$. This means that the reduced bi-free product is taking into account the behaviour of φ not just on the left and right faces but on the C*-algebra they generate, $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$. Since bi-free independence is a statement about the behaviour in the original C*-probability space this definition makes sense.

That being said, one can create the reduced bi-free product as an external product. Start with pairs of faces in different C*-probability spaces and simply create a new C*-probability space by taking the full free product of the C*-algebras and their associated states and then proceed with the above reduced bi-free product construction.

3. Faithfulness of bi-free product states

We first establish what happens when the bi-free product state is faithful.

Theorem 3.1. *Let $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}}$ be a non-trivial family of pairs of faces in the non-commutative C^*-probability space $(\mathcal{A}, \varphi)$ such that $\varphi_i = \varphi|_{C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})}$ is faithful on $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ for each $i \in \mathcal{I}$. If $**_{i\in\mathcal{I}}\varphi_i$ is faithful on the reduced bi-free product $**_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ then*

$$C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_{min} \mathcal{A}_r^{(i)}.$$

Proof. First we will establish that $\mathcal{A}_l^{(i)}$ and $\mathcal{A}_r^{(i)}$ commute in $\mathcal{A}$, then we will show that they induce a C*-norm on the algebraic tensor product $\mathcal{A}_l^{(i)} \odot \mathcal{A}_r^{(i)}$ and finally that this is in fact the minimal tensor norm.

We will be using the notation from Section 2. To simplify things a little bit, because the φ_i are assumed to be faithful, consider $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ as already a subalgebra of $B(\mathcal{H}_i)$ and so $\varphi_i(\cdot) = \langle \cdot\xi_i, \xi_i\rangle$. That is, we are suppressing the π_i notation from the GNS construction. Moreover, we will be using the convention that $\lambda_i(x), \rho_i(x), \lambda_i * \rho_i(x)$ all are living in $B(\mathcal{H})$.

Suppose $a_\chi \in \mathcal{A}_\chi^{(i)}$ such that $\varphi_i(a_\chi) = 0$, $\chi \in \{l, r\}$ and $0 \neq b \in \mathcal{A}_l^{(j)} \cup \mathcal{A}_r^{(j)}$ for $j \neq i$ such that $\varphi_j(b) = 0$. Such a b exists by the non-triviality of the family of pairs of faces. This gives that $\langle b^*\xi_j, \xi_j\rangle = \varphi_j(b) = 0$ and so $b^*\xi_j \in \mathring{\mathcal{H}}_j$ while $\langle b(b^*\xi_j), \xi_j\rangle = \varphi_j(bb^*) \neq 0$ by the faithfulness of φ_j.

Now, [8, Section 1.5] establishes that $[\lambda_i(\mathcal{A}_l^{(i)}), \rho_i(\mathcal{A}_r^{(i)})](\mathcal{H} \ominus \mathcal{H}_i) = 0$ which gives that

$$(\lambda_i(a_l)\rho_i(a_r)\lambda_j * \rho_j(b) - \rho_i(a_r)\lambda_i(a_l)\lambda_j * \rho_j(b))\xi = 0$$

since $b\xi \in \mathring{\mathcal{H}}_j \subset \mathcal{H}$. The faithfulness of $**_{i\in\mathcal{I}}\varphi_i$ implies that ξ is a separating vector for the reduced bi-free product and thus

$$\lambda_i(a_l)\rho_i(a_r)\lambda_j * \rho_j(b) - \rho_i(a_r)\lambda_i(a_l)\lambda_j * \rho_j(b) = 0$$

which gives that

$$\begin{aligned} 0 &= P_{\mathcal{H}_i}(\lambda_i(a_l)\rho_i(a_r)\lambda_j * \rho_j(b) - \rho_i(a_r)\lambda_i(a_l)\lambda_j * \rho_j(b))b^*\xi_j \\ &= (\lambda_i(a_l)\rho_i(a_r) - \rho_i(a_r)\lambda_i(a_l))\langle bb^*\xi_j, \xi_j\rangle\xi \\ &= \langle bb^*\xi_j, \xi_j\rangle(a_l a_r - a_r a_l)\xi_i. \end{aligned}$$

Since ξ_i is separating for $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ this implies that a_l and a_r commute. Thus, $\mathcal{A}_l^{(i)}$ and $\mathcal{A}_r^{(i)}$ commute in $\mathcal{A}$ for every $i \in \mathcal{I}$.

Claim: The canonical map from $\mathcal{A}_l^{(i)} \odot \mathcal{A}_r^{(i)}$ to $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ is injective.

Since $\mathcal{A}_l^{(i)}$ and $\mathcal{A}_r^{(i)}$ commute, the universal property of $\mathcal{A}_l^{(i)} \odot \mathcal{A}_r^{(i)}$ gives that there exists a $*$-homomorphism

$$\sum_{k=1}^{m} a_{k,l} \odot a_{k,r} \mapsto \sum_{k=1}^{m} a_{k,l} a_{k,r}.$$

We need to establish its injectivity. To this end, consider $h \in \mathring{\mathcal{H}}_j$, $\|h\| = 1$ where $j \neq i$ and the isometric map

$$V_h : \mathcal{H}_i \otimes \mathcal{H}_i \to \mathcal{H}_i \otimes h \otimes \mathcal{H}_i$$

defined by $V_h(h_l \otimes h_r) = h_l \otimes h \otimes h_r$ for $h_l, h_r \in \mathcal{H}_i$. This map is inspired by Dykema's proof of the faithfulness of the free product state [2, Theorem 1.1]. Note that in $\mathcal{H}$ we really have that

$$\mathcal{H}_i \otimes h \otimes \mathcal{H}_i = \mathbb{C}h \oplus (\mathring{\mathcal{H}}_i \otimes h) \oplus (h \otimes \mathring{\mathcal{H}}_i) \oplus (\mathring{\mathcal{H}}_i \otimes h \otimes \mathring{\mathcal{H}}_i)$$

but hopefully the reader will pardon the simplified notation.

Now $\mathcal{H}_i \otimes h \otimes \mathcal{H}_i$ is a reducing subspace of $C^*(\lambda_i(\mathcal{A}_l^{(i)}), \rho_i(\mathcal{A}_r^{(i)}))$ since for all $a \in \mathcal{A}_l^{(i)}, b \in \mathcal{A}_r^{(i)}$ and $\eta_1, \eta_2 \in \mathcal{H}_i$ we have that

$$\begin{aligned} V_h^* \lambda_i(a)\rho_i(b)V_h(\eta_1 \otimes \eta_2) &= V_h^* \lambda_i(a)\rho_i(b)(\eta_1 \otimes h \otimes \eta_2) \\ &= a\eta_1 \otimes b\eta_2. \end{aligned}$$

Thus, compressing to $\mathcal{H}_i \otimes h \otimes \mathcal{H}_i$ gives

$$V_h^* C^*(\lambda_i(\mathcal{A}_l^{(i)}), \rho_i(\mathcal{A}_r^{(i)}))V_h = \mathcal{A}_l^{(i)} \otimes_{min} \mathcal{A}_r^{(i)}.$$

So, if $\sum_{k=1}^m a_{k,l} \odot a_{k,r} \neq 0 \in \mathcal{A}_l^{(i)} \odot \mathcal{A}_r^{(i)}$ then $\sum_{k=1}^m a_{k,l} \otimes a_{k,r} \neq 0 \in \mathcal{A}_l^{(i)} \otimes_{min} \mathcal{A}_r^{(i)}$ which implies that

$$\begin{aligned} 0 &\neq \sum_{k=1}^{m} a_{k,l} \otimes a_{k,r}(\xi_i \otimes \xi_i) \\ &= V_h^* \sum_{k=1}^{m} \lambda_i(a_{k,l})\rho_i(a_{k,r})V_h(\xi_i \otimes \xi_i) \\ &= \sum_{k=1}^{m} \lambda_i(a_{k,l})\rho_i(a_{k,r})h \end{aligned}$$

since the state $\langle \cdot \xi_i \otimes \xi_i, \xi_i \otimes \xi_i \rangle$ is faithful on the min tensor product. But then $\sum_{k=1}^m \lambda_i(a_{k,l})\rho_i(a_{k,r}) \neq 0 \in C^*(\lambda_i(\mathcal{A}_l^{(i)}), \rho_i(\mathcal{A}_r^{(i)}))$ which by the faithfulness of $**_{i\in\mathcal{I}}\varphi_i$ gives that $\sum_{k=1}^m \lambda_i(a_{k,l})\rho_i(a_{k,r})\xi \neq 0$. Finally,

$$\begin{aligned}0 &\neq \left\langle \sum_{k=1}^m \lambda_i(a_{k,l})\rho_i(a_{k,r})\xi, \sum_{k=1}^m \lambda_i(a_{k,l})\rho_i(a_{k,r})\xi \right\rangle \\ &= \left\langle \left(\sum_{k=1}^m \lambda_i(a_{k,l})\rho_i(a_{k,r})\right)^* \left(\sum_{k=1}^m \lambda_i(a_{k,l})\rho_i(a_{k,r})\right)\xi, \xi \right\rangle \\ &= \varphi_i\left(\left(\sum_{k=1}^m a_{k,l}a_{k,r}\right)^* \left(\sum_{k=1}^m a_{k,l}a_{k,r}\right)\right)\end{aligned}$$

which gives by the faithfulness of φ_i that $\sum_{k=1}^m a_{k,l}a_{k,r} \neq 0$. Therefore, the claim is verified.

Now, this implies that $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_\alpha \mathcal{A}_r^{(i)}$ where $\|\cdot\|_\alpha$ is a C*-norm on $\mathcal{A}_l^{(i)} \odot \mathcal{A}_r^{(i)}$. So by Takesaki's Theorem [6] we have that there exists a surjective $*$-homomorphism

$$q : C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \to \mathcal{A}_l^{(i)} \otimes_{min} \mathcal{A}_r^{(i)}.$$

To finish the proof all we need to do is show that q is injective.

To this end, let $a \in C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ such that $q(a) = 0$. Again as in the first part of this proof, find $0 \neq b \in \mathcal{A}_l^{(j)} \cup \mathcal{A}_r^{(j)}$ for $j \neq i$ such that $\varphi_j(b) = 0$ and $h \in \mathring{\mathcal{H}}_j$ such that $\langle bh, \xi_j\rangle \neq 0$. Additionally, assume that $\|b\xi_j\| = 1$.

In the second part of this proof we saw that compressing to $\mathcal{H}_i \otimes b\xi_j \otimes \mathcal{H}_i$ is tantamount to this quotient homomorphism q. Namely, suppose

$$\iota_i : \mathcal{A}_l^{(i)} \check{*} \mathcal{A}_r^{(i)} \to C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \quad (\subseteq B(\mathcal{H}_i) \text{ by assumption})$$

is the unique $*$-homomorphism extending the identity in each component. There then exists $\tilde{a} \in \mathcal{A}_l^{(i)} \check{*} \mathcal{A}_r^{(i)}$ such that $\iota_i(\tilde{a}) = a$. An important fact to record is that, by uniqueness,

$$\lambda_i * \rho_i(\cdot)|_{\mathcal{H}_i} = \iota_i(\cdot),$$

remembering that we have that $\lambda_i * \rho_i(\cdot) \in B(\mathcal{H})$. Thus,

$$V_{b\xi_j}^* \lambda_i * \rho_i(\tilde{a}) V_{b\xi_j} = q(a) = 0,$$

which implies, by the fact that $V_{b\xi_j}(\mathcal{H}_i \otimes \mathcal{H}_i)$ is reducing for $\lambda_i * \rho_i(\mathcal{A}_l^{(i)} \check{*} \mathcal{A}_r^{(i)})$, that

$$\begin{aligned}0 &= \lambda_i * \rho_i(\tilde{a}) V_{b\xi_j}(\xi_i \otimes \xi_i) \\ &= \lambda_i * \rho_i(\tilde{a})(b\xi_j) \\ &= \lambda_i * \rho_i(\tilde{a})\lambda_j * \rho_j(b)\xi.\end{aligned}$$

By the faithfulness of the bi-free product state $\lambda_i * \rho_i(\tilde{a})\lambda_j * \rho_j(b) = 0$ and so

$$\begin{aligned} 0 &= P_{\mathcal{H}_i}\lambda_i * \rho_i(\tilde{a})\lambda_j * \rho_j(b)h \\ &= \lambda_i * \rho_i(\tilde{a})\langle bh, \xi_j\rangle\xi \\ &= \langle bh, \xi_j\rangle\iota_i(\tilde{a})\xi \\ &= \langle bh, \xi_j\rangle a\xi_i. \end{aligned}$$

Hence, by the faithfulness of φ_i we have that $a = 0$. Therefore, for all $i \in \mathcal{I}$, $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_{min} \mathcal{A}_r^{(i)}$. □

We turn now to a partial converse of the previous theorem. This is probably known among the experts in bi-free probability but we could not find a published proof. The following proof may be a tad clunky but we find it the clearest from a non-expert perspective.

Theorem 3.2. *Let* $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}}$ *be a family of pairs of faces in the non-commutative* C^**-probability space* $(\mathcal{A}, \varphi)$*. If* $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_{min} \mathcal{A}_r^{(i)}$ *and* $\varphi_i = \varphi_i|_{\mathcal{A}_l^{(i)}} \otimes \varphi_i|_{\mathcal{A}_r^{(i)}}$ *is a faithful product state on* $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$*, for all* $i \in \mathcal{I}$*, then* $**_{i\in\mathcal{I}}\varphi_i$ *is faithful on the reduced bi-free product and*

$$**_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}} \ \simeq\ *_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \varphi) \otimes_{min} *_{i\in\mathcal{I}}(\mathcal{A}_r^{(i)}, \varphi).$$

Proof. As before, we will be using the notation of Section 2.

For each $i \in \mathcal{I}$, since $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_{min} \mathcal{A}_r^{(i)}$ and φ_i is a product state we can a priori choose $\mathcal{H}_i = \mathcal{H}_{i,l} \otimes \mathcal{H}_{i,r}$, unit vectors $\xi_{i,l} \in \mathcal{H}_{i,l}, \xi_{i,r} \in \mathcal{H}_{i,r}$ such that $\xi_i = \xi_{i,l} \otimes \xi_{i,r}$ and $*$-homomorphisms $\pi_{i,\chi} : \mathcal{A}_\chi^{(i)} \to B(\mathcal{H}_{i,\chi})$ such that $\pi_i = \pi_{i,l} \otimes \pi_{i,r}$. This will give for $a_\chi \in \mathcal{A}_\chi^{(i)}$, $\chi \in \{l, r\}$, that

$$\begin{aligned} \varphi_i(a_l a_r) &= \langle \pi_i(a_l a_r)\xi_i, \xi_i\rangle \\ &= \langle \pi_{i,l}(a_l)\xi_{i,l}, \xi_{i,l}\rangle\langle \pi_{i,r}(a_r)\xi_{i,r}, \xi_{i,r}\rangle. \end{aligned}$$

Along with the free product Hilbert space

$$(\mathcal{H}, \xi) = *_{i\in\mathcal{I}}(\mathcal{H}_i, \xi_i)$$

we need to also define, for $\chi \in \{l, r\}$, the free product Hilbert spaces

$$(\mathcal{H}_\chi, \xi_\chi) = *_{i\in\mathcal{I}}(\mathcal{H}_{i,\chi}, \xi_{i,\chi}).$$

Since there are multiple free product Hilbert spaces we will use subscripts to denote the different left and right representations, namely,

$$\lambda_{\mathcal{H}_i} : B(\mathcal{H}_i) \to B(\mathcal{H}) \quad \text{and} \quad \lambda_{\mathcal{H}_{i,l}} : B(\mathcal{H}_{i,l}) \to B(\mathcal{H}_l)$$

for the left representations and

$$\rho_{\mathcal{H}_i} : B(\mathcal{H}_i) \to B(\mathcal{H}) \quad \text{and} \quad \rho_{\mathcal{H}_{i,r}} : B(\mathcal{H}_{i,r}) \to B(\mathcal{H}_r)$$

for the right representations.

Dykema's original result [2] proves that $\langle \cdot \xi_\chi, \xi_\chi\rangle$ is faithful on $*_{i\in\mathcal{I}}(\mathcal{A}_\chi^{(i)}, \varphi)$ for $\chi \in \{l, r\}$ and it is a folklore result that the minimal tensor product of faithful states is faithful. Thus, $\langle \cdot\ \xi_l \otimes \xi_r, \xi_l \otimes \xi_r\rangle$ is faithful on $*_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \varphi)\ \otimes_{min}\ *_{i\in\mathcal{I}}(\mathcal{A}_r^{(i)}, \varphi)$.

Fix $k \geq 1$ and $j_1, \cdots, j_k \in \mathcal{I}$ such that $j_i \neq j_{i+1}, 1 \leq i \leq k-1$. Now fix a unit vector

$$\begin{aligned} h &= (\xi_{j_1,l} \otimes h_{j_1,r}) \otimes h_{j_2} \otimes \cdots \otimes h_{j_{k-1}} \otimes (h_{j_k,l} \otimes \xi_{j_k,r}) \\ &\in (\xi_{j_1,l} \otimes \mathring{\mathcal{H}}_{j_1,r}) \otimes \mathring{\mathcal{H}}_{j_2} \otimes \cdots \otimes \mathring{\mathcal{H}}_{j_{k-1}} \otimes (\mathring{\mathcal{H}}_{j_k,l} \otimes \xi_{j_k,r}). \end{aligned}$$

If $k = 1$ the only possible h is $\xi = \xi_{j_1} = \xi_{j_1,l} \otimes \xi_{j_1,r}$. Call the collection of such h, as k and the indices vary, $\mathcal{S} \subset \mathcal{H}$.

As will be shown below, this set of unit vectors $\mathcal{S}$ plays an important role in decomposing simple tensors in $\mathcal{H}$, in particular for every simple tensor $\eta \in \mathcal{H}$ that is also a simple tensor in each component there exists a unique $h \in \mathcal{S}$ such that $\eta \in \mathcal{H}_l \otimes h \otimes \mathcal{H}_r$. By abuse of tensor notation this is not very hard to see in one's mind but the reality of proving this carefully needs plenty of indices.

To this end, for $m \geq 1$ suppose $s_1, \ldots, s_m \in \mathcal{I}$ such that $s_t \neq s_{t+1}$ for $1 \leq t \leq m-1$, and $\eta_{t,l} \in \mathcal{H}_{s_t,l}$, $\eta_{t,r} \in \mathcal{H}_{s_t,r}$ such that $\eta_{t,l} \otimes \eta_{t,r} \in \mathring{\mathcal{H}}_{s_t}$ for $1 \leq t \leq m$. This last condition implies that $\eta_{t,\chi} = \|\eta_{t,\chi}\|\xi_{t,\chi}$ cannot hold for both $\chi = l$ and $\chi = r$. In summary,

$$\eta := (\eta_{1,l} \otimes \eta_{1,r}) \otimes \cdots \otimes (\eta_{m,l} \otimes \eta_{m,r}) \in \mathring{\mathcal{H}}_{s_1} \otimes \cdots \otimes \mathring{\mathcal{H}}_{s_m}.$$

Note that the conditions imposed on the $\eta_{t,\chi}$ in the above paragraph imply that the form of η above is as reduced as it can be.

As mentioned above, it will be established that there exists $h \in \mathcal{S}$ such that

$$\eta \in \mathcal{H}_l \otimes h \otimes \mathcal{H}_r.$$

To prove the required decomposition, let

$$v = \max\{0 \leq t \leq m\ :\ \eta_{j,r} = \|\eta_{j,r}\|\xi_{s_j,r},\ 1 \leq j \leq t\}$$

and

$$w = \min\{1 \leq t \leq m+1\ :\ \eta_{j,l} = \|\eta_{j,l}\|\xi_{s_j,l},\ t \leq j \leq m\}.$$

This gives that v is the number of terms in a row from the left with trivial right tensor components and $m+1-w$ is the number of terms in a row from the right with trivial left tensor components.

By the fact that $\eta_{t,l} \otimes \eta_{t,r} \in \mathring{\mathcal{H}}_{s_t}$, that is $\eta_{t,\chi} = \|\eta_{t,\chi}\|\xi_{t,\chi}$ cannot hold for both $\chi = l$ and $\chi = r$, we have that $v < w$. If $v = m$ then $w = m+1$ and $\eta \in \mathcal{H}_l$, and if $w = 1$ then $v = 0$ and $\eta \in \mathcal{H}_r$. Otherwise, when $0 \leq v \leq m-1$ and $2 \leq w \leq m$, define

$$\begin{aligned} \eta_l &= (\eta_{1,l} \otimes \|\eta_{1,r}\|\xi_{s_1,r}) \otimes \cdots \otimes (\eta_{v,l} \otimes \|\eta_{v,r}\|\xi_{s_v,r}) \otimes (\eta_{v+1,l} \otimes \xi_{s_{v+1},r}), \\ \eta_{\mathcal{S}} &= (\xi_{v+1,l} \otimes \eta_{v+1,r}) \otimes \cdots \otimes (\eta_{w-1,l} \otimes \xi_{w-1,r}), \\ \eta_r &= (\xi_{w-1,l} \otimes \eta_{w-1,r}) \otimes (\|\eta_{w,l}\|\xi_{w,l} \otimes \eta_{w,r}) \otimes \cdots \otimes (\|\eta_{m,l}\|\xi_{m,l} \otimes \eta_{m,r}) \end{aligned}$$

with $\eta_{\mathcal{S}} = \xi$ if $v + 1 = w$. Hence, by the usual slight abuse of the tensor notation, $\eta = \eta_l \otimes \eta_{\mathcal{S}} \otimes \eta_r \in \mathcal{H}_l \otimes \eta_{\mathcal{S}} \otimes \mathcal{H}_r$ with $\frac{1}{\|\eta_{\mathcal{S}}\|}\eta_{\mathcal{S}} \in \mathcal{S}$. Therefore,

$$\overline{\operatorname{span}}\{\mathcal{H}_l \otimes h \otimes \mathcal{H}_r : h \in \mathcal{S}\} = \mathcal{H}.$$

For any $h \in \mathcal{S}$, which is a unit vector, there is a natural isometric map $S_h : \mathcal{H}_l \otimes \mathcal{H}_r \to \mathcal{H}$ given by the concatenation $\mathcal{H}_l \otimes \mathcal{H}_r \mapsto \mathcal{H}_l \otimes h \otimes \mathcal{H}_r$ with the appropriate simplification of tensors when needed. In particular, there exist $k \geq 1$ and $j_1, \cdots, j_k \in \mathcal{I}$ such that $j_i \neq j_{i+1}, 1 \leq i \leq k-1$ and then

$$\begin{aligned} h &= (\xi_{j_1,l} \otimes h_{j_1,r}) \otimes h_{j_2} \otimes \cdots \otimes h_{j_{k-1}} \otimes (h_{j_k,l} \otimes \xi_{j_k,r}) \\ &\in (\xi_{j_1,l} \otimes \mathring{\mathcal{H}}_{j_1,r}) \otimes \mathring{\mathcal{H}}_{j_2} \otimes \cdots \otimes \mathring{\mathcal{H}}_{j_{k-1}} \otimes (\mathring{\mathcal{H}}_{j_k,l} \otimes \xi_{j_k,r}). \end{aligned}$$

We can now carefully specify that the isometric map is given by

$$\xi_l \otimes \xi_r \mapsto h,$$

$$\xi_l \otimes (\mathring{\mathcal{H}}_{i_1,r} \otimes \cdots \otimes \mathring{\mathcal{H}}_{i_m,r}) \to$$
$$\begin{cases} h \otimes (\xi_{i_1,l} \otimes \mathring{\mathcal{H}}_{i_1,r}) \otimes \cdots \otimes (\xi_{i_m,l} \otimes \mathring{\mathcal{H}}_{i_m,r}), & i_1 \neq j_k \\ (\xi_{j_1,l} \otimes h_{j_1,r}) \otimes h_{j_2} \otimes \cdots \otimes h_{j_{k-1}} \otimes (h_{j_k,l} \otimes \mathring{\mathcal{H}}_{i_1,r}) \otimes & \\ \qquad\qquad \cdots \otimes (\xi_{i_m,l} \otimes \mathring{\mathcal{H}}_{i_m,r}), & i_1 = j_k \end{cases}$$

$$(\mathring{\mathcal{H}}_{i_1,l} \otimes \cdots \otimes \mathring{\mathcal{H}}_{i_m,l}) \otimes \xi_r \to$$
$$\begin{cases} (\mathring{\mathcal{H}}_{i_1,l} \otimes \xi_{i_1,r}) \otimes \cdots \otimes (\mathring{\mathcal{H}}_{i_m,l} \otimes \xi_{i_m,r}) \otimes h, & i_m \neq j_1 \\ (\mathring{\mathcal{H}}_{i_1,l} \otimes \xi_{i_1,r}) \otimes \cdots \otimes (\mathring{\mathcal{H}}_{i_m,l} \otimes h_{j_1,r}) \otimes h_{j_2} \otimes & \\ \qquad\qquad \cdots \otimes h_{j_{k-1}} \otimes (h_{j_k,l} \otimes \xi_{j_k,r}), & i_m = j_1 \end{cases}$$

and

$$(\mathring{\mathcal{H}}_{i_1,l} \otimes \cdots \otimes \mathring{\mathcal{H}}_{i_m,l}) \otimes (\mathring{\mathcal{H}}_{t_1,r} \otimes \cdots \otimes \mathring{\mathcal{H}}_{t_s,r}) \to$$
$$(\mathring{\mathcal{H}}_{i_1,l} \otimes \xi_{i_1,r}) \otimes \cdots \otimes (\mathring{\mathcal{H}}_{i_m,l} \otimes \xi_{i_m,r}) \otimes h \otimes (\xi_{t_1,l} \otimes \mathring{\mathcal{H}}_{t_1,r}) \otimes \cdots \otimes (\xi_{t_s,l} \otimes \mathring{\mathcal{H}}_{t_s,r})$$

if $i_m \neq j_1$ and $j_k \neq t_1$ with similar statements as the cases above when $i_m = j_1$ or $j_k = t_1$ or both happen. Perhaps the most natural case of S_h is when $h = \xi$. It certainly minimizes, but doesn't remove, the need for all of the cases above.

A careful examination of the S_h isometric map implies that for $a \in \mathcal{A}_l^{(i_1)}$, $b \in \mathcal{A}_r^{(i_2)}$ and $\eta_\chi \in \mathcal{H}_\chi$ for $\chi \in \{l, r\}$ we have that, by abuse of the tensor notation,

$$\begin{aligned} \lambda_{\mathcal{H}_{i_1}}&(\pi_{i_1}(a))\rho_{\mathcal{H}_{i_2}}(\pi_{i_2}(b))S_h(\eta_l \otimes \eta_r) \\ &= \lambda_{\mathcal{H}_{i_1}}(\pi_{i_1}(a))\rho_{\mathcal{H}_{i_2}}(\pi_{i_2}(b))(\eta_l \otimes h \otimes \eta_r) \\ &= \lambda_{\mathcal{H}_{i_1,l}}(\pi_{i_1,l}(a))\eta_l \otimes h \otimes \rho_{\mathcal{H}_{i_2,r}}(\pi_{i_2,r}(b))\eta_r \\ &= S_h(\lambda_{\mathcal{H}_{i_1,l}}(\pi_{i_1,l}(a))\eta_l \otimes \rho_{\mathcal{H}_{i_2,r}}(\pi_{i_2,r}(b))\eta_r) \end{aligned}$$

Hence, $S_h(\mathcal{H}_l \otimes \mathcal{H}_r)$ is a reducing subspace of the reduced bi-free product. Moreover,

$$S_h^* \lambda_{\mathcal{H}_i} \circ \pi_i(\cdot) S_h = (\lambda_{\mathcal{H}_{i,l}} \circ \pi_{i,l}(\cdot)) \otimes I_{\mathcal{H}_r} \qquad \text{on } \mathcal{A}_l^{(i)}$$

and

$$S_h^* \rho_{\mathcal{H}_i} \circ \pi_i(\cdot) S_h = I_{\mathcal{H}_l} \otimes (\rho_{\mathcal{H}_{i,r}} \circ \pi_{i,r}(\cdot)) \qquad \text{on } \mathcal{A}_r^{(i)}$$

Therefore, for any $h \in \mathcal{S}$,

$$S_h^* (**_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})) S_h = *_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \varphi) \ \otimes_{min} \ *_{i\in\mathcal{I}}(\mathcal{A}_r^{(i)}, \varphi)$$

and furthermore, by the identities involving S_h, λ and ρ, $S_h^* a S_h = S_\xi^*(a) S_\xi$ for all $h \in \mathcal{S}$ and $a \in **_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$.

Finally, we want to show that compression to $S_\xi(\mathcal{H}_l \otimes \mathcal{H}_r)$ is a $*$-isomorphism. Note that this is the same as compression to $S_h(\mathcal{H}_l \otimes \mathcal{H}_r)$ being injective for any $h \in \mathcal{S}$. This gives us a way forward. Suppose that $a \in **_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})$ such that $**_{i\in\mathcal{I}}\varphi_i(a^*a) = 0$. This implies that

$$\begin{aligned} 0 &= a\xi \\ &= S_\xi^* a S_\xi(\xi_l \otimes \xi_r). \end{aligned}$$

By the faithfulness of $\langle \cdot\, \xi_l \otimes \xi_r, \xi_l \otimes \xi_r \rangle$ this gives that $S_\xi^* a S_\xi = 0$ or rather a is 0 on the reducing subspace $S_\xi(\mathcal{H}_l \otimes \mathcal{H}_r)$. But then for all $h \in \mathcal{S}$ we have that

$$S_h^* a S_h = S_\xi^* a S_\xi = 0$$

and a is 0 on the reducing subspace $S_h(\mathcal{H}_l \otimes \mathcal{H}_r)$. By what we proved about the set $\mathcal{S}$, we have that a is 0 on

$$\overline{\text{span}}\{S_h(\mathcal{H}_l \otimes \mathcal{H}_r) : h \in \mathcal{S}\} = \overline{\text{span}}\{\mathcal{H}_l \otimes h \otimes \mathcal{H}_r : h \in \mathcal{S}\} = \mathcal{H}.$$

Therefore, $a = 0$ and thus $**_{i\in\mathcal{I}}\varphi_i$ is faithful. □

There may exist a full converse to Theorem 3.1 but the previous proof highly depends on the state φ_i arising as a tensor product of states. In general, φ_i need not be of this form. We should note here that if $\varphi_i|_{\mathcal{A}_l^{(i)}}$ or $\varphi_i|_{\mathcal{A}_l^{(i)}}$ is a pure state then φ_i will be a tensor product of states.

To end this paper, we summarize with the following corollary.

Corollary 3.3. *Let $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}}$ be a non-trivial family of pairs of faces in the non-commutative C^*-probability space $(\mathcal{A}, \varphi)$. If φ is faithful on $C^*((\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}})$, $C^*(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)}) \simeq \mathcal{A}_l^{(i)} \otimes_{min} \mathcal{A}_r^{(i)}$, $\varphi_i = \varphi_i|_{\mathcal{A}_l^{(i)}} \otimes \varphi_i|_{\mathcal{A}_r^{(i)}}$ and $(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}}$ is bi-freely independent with respect to φ, then*

$$\begin{aligned} C^*((\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}}) &\simeq **_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \mathcal{A}_r^{(i)})_{i\in\mathcal{I}} \\ &\simeq *_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}, \varphi) \otimes_{min} *_{i\in\mathcal{I}}(\mathcal{A}_r^{(i)}, \varphi). \end{aligned}$$

Proof. Recall, that by bi-free independence we know that the following diagram commutes

$$\begin{array}{ccccc}
\check{*}_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}\check{*}\mathcal{A}_r^{(i)}) & \xrightarrow{\iota} & C^*((\mathcal{A}_l^{(i)},\mathcal{A}_r^{(i)})_{i\in\mathcal{I}}) & \xrightarrow{\varphi} & \mathbb{C} \\
{\scriptstyle *_{i\in\mathcal{I}}(\pi_i*\pi_i)}\big\downarrow & & & & \big\| \\
\check{*}_{i\in\mathcal{I}}(B(\mathcal{H}_i)\check{*}B(\mathcal{H}_i)) & \xrightarrow{*_{i\in I}(\lambda_i*\rho_i)} & B(\mathcal{H}) & \xrightarrow{\langle\cdot\xi,\xi\rangle} & \mathbb{C}
\end{array}$$

Because both of the states are faithful on their algebras then for any $a^*a \in \check{*}_{i\in\mathcal{I}}(\mathcal{A}_l^{(i)}\check{*}\mathcal{A}_r^{(i)})$, a^*a is in the kernel of ι if and only if a^*a is in the kernel of $*_{i\in I}(\lambda_i * \rho_i)\circ *_{i\in\mathcal{I}}(\pi_i * \pi_i)$. Therefore, both quotients are $*$-isomorphic and Theorem 3.2 gives the final $*$-isomorphism. □

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