

PURE POINT SPECTRUM FOR DYNAMICAL SYSTEMS AND MEAN ALMOST PERIODICITY

DANIEL LENZ, TIMO SPINDELER, AND NICOLAE STRUNGARU

ABSTRACT. We consider metrizable ergodic topological dynamical systems over locally compact, σ -compact abelian groups. We study pure point spectrum via suitable notions of almost periodicity for the points of the dynamical system. More specifically, we characterize pure point spectrum via mean almost periodicity of generic points. We then go on and show how Besicovitch almost periodic points determine both eigenfunctions and the measure in this case. After this, we characterize those systems arising from Weyl almost periodic points and use this to characterize weak and Bohr almost periodic systems. Finally, we consider applications to aperiodic order.

CONTENTS

Introduction	1
1. Background on dynamical systems, pure point spectrum and the upper mean	3
2. Mean almost periodic points and pure point spectrum	5
3. Besicovitch almost periodic points and eigenfunctions	11
4. Weyl almost periodic points, unique ergodicity and continuity of eigenfunctions	17
5. Weakly and Bohr almost periodic dynamical systems	21
6. Application to measure dynamical systems	25
7. Abstract generalisations	28
Appendix A. Bohr almost periodic functions	31
Appendix B. Mean almost periodic functions	31
Appendix C. Besicovitch almost periodic functions and existence of means	34
Appendix D. Weyl almost periodic functions and uniform means	36
References	38

INTRODUCTION

This article is concerned with dynamical systems with pure point spectrum. The dynamical systems in question consist of a continuous action of a locally compact abelian group on a compact metric space together with an invariant probability measure on the space. Pure point spectrum means that there exists an orthonormal basis of eigenfunctions. In a sense, such systems are the simplest possible dynamical systems. Their study is a most basic ingredient in the conceptual theory of dynamical systems as witnessed by such fundamental results as the Halmos–von Neumann theorem or the Furstenberg structure theorem. Accordingly, a variety of characterizations for pure point spectrum has been established over the decades.

Recent years have brought two new lines of interest in such systems. One line is given by a series of works which analyze such systems via (weak) notions of equicontinuity [10, 15, 16, 17,

18]. The main thrust of these works is to characterize pure point spectrum as well as various strengthened versions thereof (see [14, 35] for related work as well). The other line comes from the investigation of aperiodic order. Aperiodic order, also known as mathematical theory of quasicrystals, has emerged as fruitful field of (not only) mathematics over the last three decades, see e.g. [2] for a recent monograph and [5, 21] for recent collections of surveys. A key feature of aperiodic order is the occurrence of (pure) point diffraction. A central result in the mathematical treatment of aperiodic order gives that pure point diffraction can be understood as pure point spectrum of suitable associated dynamical systems. In fact, this result is the outcome of a cumulative effort of various people over the last decades [11, 23, 20, 32, 3, 29, 27].

A common feature of all these works is that their considerations share a flavor of almost periodicity. On an intuitive level this is not surprising. After all, pure point spectrum means that all spectral measures are pure point measures. This in turn is equivalent to the Fourier transforms of these measures being almost periodic [31, 19]. In this way, almost periodicity properties of functions and their averages come into play in a very natural way.

However, what is lacking is a description of pure point spectrum via almost periodicity properties of the **points** of the dynamical system. The goal of this article is to provide such a characterization and to study some of its consequences.

In order to do so, we introduce for points in dynamical systems the concepts of mean almost periodicity, Besicovitch almost periodicity, Weyl almost periodicity, weak almost periodicity and Bohr almost periodicity, respectively.

After the discussion of the necessary background in Section 1, our first achievement is Theorem 2.8 in Section 2. This theorem says that a system has pure point spectrum if and only if every (or even just one) generic point is mean almost periodic. If the dynamical system is ergodic this is then equivalent to almost all points being mean almost periodic.

As for mean almost periodicity of points in turn various characterizations are discussed in Section 2. One of them proceeds via averages of distance between the orbit of the point and a shifted orbit. Another one characterizes mean almost periodicity of a point via almost periodicity properties of the sampling of continuous functions along the orbit of this point. In this way, we have a rather complete and clear picture of the meaning of mean almost periodic points for general dynamical systems.

This picture ties in with various earlier results. In the case where the group is just the integers a related characterization via sampling of bounded measurable function is given in [6] for measurable dynamical systems. In the more specific situation of subshifts over a finite alphabet there is also a characterization of pure point spectrum via a metric property of points in [35]. Moreover, we point out a companion article [28] dealing with fundamental issues in aperiodic order via almost periodicity of measures. As an application, it treats the situation of special dynamical system viz translation bounded measures dynamical systems. These systems are particularly relevant to aperiodic order. In these systems the points are measures and this allows one to work with almost periodicity properties of measures. Of course, this approach is not available for general dynamical systems.

In terms of methods it should be emphasized that our proof of Theorem 2.8 is completely different from the ones given in [28, 6, 35]. It relies on a recent characterization of pure point spectrum by Bohr almost periodicity of an averaged metric recently obtained by one of the authors [25].

In Section 3, we then introduce Besicovitch almost periodic points. While the condition of Besicovitch almost periodicity is strictly stronger than mean almost periodicity, we can still show that an ergodic dynamical system has pure point spectrum if and only if almost all

points are Besicovitch almost periodic and this holds if and only if there exists one generic Besicovitch almost periodic point (Theorem 3.7). In this respect the difference between mean and Besicovitch almost periodicity is not too big. The advantage of Besicovitch almost periodic points is that they allow for averaging with characters. In particular, it is possible to compute the eigenfunctions via the Besicovitch almost periodic points (Theorem 3.7). In fact, the complete spectral theory of the dynamical system can be directly computed from any single typical Besicovitch almost periodic point (Theorem 3.4).

In Section 4, we introduce Weyl almost periodic points. Weyl almost periodicity of a point is substantially stronger than Besicovitch almost periodicity. In fact, a point is Weyl almost periodic if and only if its orbit closure is uniquely ergodic and has pure point spectrum with continuous eigenfunctions (Theorem 4.5). In this case all points in the orbit closure are Weyl almost periodic (Lemma 4.3). To put these results in perspective we note that [10, 15] show that a dynamical system is mean equicontinuous if and only if it is uniquely ergodic with pure point spectrum and continuous eigenvalues. Hence, a point is Weyl almost periodic if and only if its orbit closure is mean equicontinuous. So, our results can be understood to provide a natural pointwise counterpart to the results of [15].

In Section 5, we investigate two special classes of Weyl almost periodic points viz Bohr and weakly almost periodic points respectively. This allows us to characterize Bohr and weakly almost periodic dynamical systems. In particular, we reprove a main result of [30]. An application of our results to aperiodic order is given in Section 6. This includes an alternative proof for the main results on measure dynamical systems contained in [28].

Our article give a comprehensive treatment of most relevant concepts of almost periodicity in the context of pure point spectrum. Some parts of our considerations allow for simple abstractions. As this may be of value for further investigations we include a brief discussion of some basic results in Section 7.

It seems that Besicovitch almost periodicity is not well known (for groups other than \mathbb{R}) and there is also some ambiguity in the way it is defined. For this reason we include some appendices discussing the almost periodicity properties needed in this article as well as basic properties of the class of continuous functions with these almost periodicity properties.

Acknowledgments. Parts of this work were done while D.L. was visiting the department of mathematics in Geneva in 2019. He would like to thank the department for hospitality. He would also like to acknowledge most inspiring discussions with Gerhard Keller and Michael Baake at the workshop on *Model sets and aperiodic order* in Durham in 2018. Part of the work was done when T.S. and N.S. visited Friedrich-Schiller-Universität Jena, and they would like to thank the department of mathematics for hospitality. The work was supported by DFG with grant 415818660(TS) and by NSERC with grants 03762-2014 and 2020-00038(NS), and the authors are grateful for the support.

1. BACKGROUND ON DYNAMICAL SYSTEMS, PURE POINT SPECTRUM AND THE UPPER MEAN

In this section we review the necessary concepts from dynamical systems and introduce the upper mean \overline{M} , which is crucial for our subsequent considerations.

Throughout the paper, we denote the set of continuous complex valued functions on the topological space Y by $C(Y)$.

We consider a compact metric space X equipped with a continuous action

$$\alpha : G \times X \longrightarrow X, \quad (t, x) \mapsto \alpha_t(x),$$

of a locally compact, σ -compact, abelian group G . We then call (X, G) a *dynamical system (over the space X)*. Often we will also be given a probability measure m on X , which is invariant under the action of G . We then call (X, G, m) a *dynamical system* as well. We will write tx instead of $\alpha_t(x)$ for $t \in G$ and $x \in X$. The composition on G itself is written additively and the neutral element of G is denoted as e . We fix a Haar measure on G . The Haar measure of a measurable subset $A \subset G$ is denoted by $|A|$ and the integral of an integrable function f on G by $\int f(t) dt$.

Whenever a dynamical system (X, G) is given we will furthermore make use of

- a metric d on X which generates the topology,
- a Følner sequence (B_n) in G , i.e. each B_n is an open relatively compact subset of G and

$$\frac{|(B_n \setminus (t + B_n)) \cap ((t + B_n) \setminus B_n)|}{|B_n|} \rightarrow 0, \quad n \rightarrow \infty,$$

for all $t \in G$.

Note that a Følner sequence exists in a locally compact abelian group G if and only if G is σ -compact [34, Prop. B6]. For this reason, we will always assume that G is σ -compact.

The *orbit* of x is given by $Gx := \{tx : t \in G\}$ and the *orbit closure* \overline{Gx} is the closure of the orbit. If the orbit closure of $x \in X$ agrees with X , the element x is called *transitive*. If every $x \in X$ is transitive the dynamical system is called *minimal*. The dynamical system (X, G) is called *uniquely ergodic* if there exists only one invariant probability measure on X . We then denote this measure by m and call (X, G, m) uniquely ergodic as well.

The dynamical system (X, G, m) is *ergodic* if any invariant measurable subset A of G satisfies $m(A) = 1$ or $m(A) = 0$. If (X, G, m) is ergodic any Følner sequence has a subsequence (B_n) for which *Birkhoff ergodic theorem holds* i.e.

$$\lim_{n \rightarrow \infty} \frac{1}{|B_n|} \int_{B_n} f(tx) dt = \int_X f dm$$

is valid for almost every $x \in X$ whenever $f : X \rightarrow \mathbb{C}$ integrable.

Whenever a dynamical system (X, G, m) and a Følner sequence (B_n) is given, a point $y \in X$ is called *m -generic* with respect to the Følner sequence if

$$\lim_{n \rightarrow \infty} \frac{1}{|B_n|} \int_{B_n} f(ty) dt = \int_X f(x) dm(x)$$

holds for any continuous $f : X \rightarrow \mathbb{C}$. If the measure m and the sequence (B_n) are clear from the context we just speak about generic points. Generic points play a key role in our subsequent considerations as they determine the measure and - in this sense - the whole dynamical system. As is well-known (and not hard to see) the set of generic points is measurable and invariant under the group action. Moreover, the set of generic points has full measure if (X, G, m) is ergodic and Birkhoff's ergodic theorem holds along the underlying Følner sequence. While we do not need it here, it is instructive for our subsequent considerations to note that a converse of sorts holds as well: If m is an invariant probability measure such that the set of m -generic points with respect to some Følner sequence has full measure, then (X, G, m) is ergodic. So, ergodicity is a necessary and sufficient condition for having an ample supply of generic points at ones disposal. This is ultimately the reason that most (but not all) of our theorems below deal with ergodic systems.

A dynamical system (X, G, m) is said to have *pure point spectrum* if there exists an orthonormal basis of $L^2(X, m)$ consisting of eigenfunctions. Here, an $f \in L^2(X, m)$ with $f \neq 0$

is called an *eigenfunction* if for any $t \in G$ there exist a $\xi(t) \in \mathbb{C}$ with

$$f(t \cdot) = \xi(t)f$$

in the sense of $L^2(X, m)$ functions. In this case, each $\xi(t)$ belongs to the group

$$\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$$

and the map

$$\xi : G \longrightarrow \mathbb{T}, \quad t \mapsto \xi(t),$$

can easily be seen to be a continuous group homomorphism. It is called *eigenvalue*. Clearly, eigenfunctions to different eigenvalues are orthogonal. As X is compact and metrizable, $L^2(X, m)$ is separable. Hence, the set of eigenvalues is (at most) countable. For $\xi \in \widehat{G}$ we denote by P_ξ the projection onto the eigenspace of ξ if ξ is an eigenvalue and set $P_\xi = 0$ if ξ is not an eigenvalue. The set of eigenvalues of (X, G, m) will be denoted by $\text{Eig}(X, G, m)$. As is well known the set of eigenvalues is a group if (X, G, m) is ergodic.

For our further discussion we will also rely on some concepts defined purely with respect to G (i.e. they do not need the dynamical system). Let (B_n) be a Følner sequence, and let $B(G)$ denote the set of bounded measurable real-valued functions on G . Then, we define the associated *upper mean* via

$$\overline{M}_{(B_n)} : B(G) \longrightarrow [0, \infty), \quad \overline{M}_{(B_n)}(h) := \limsup_{n \rightarrow \infty} \frac{1}{|B_n|} \int_{B_n} h(s) \, ds.$$

If the Følner sequence is clear from the context we drop the subscript (B_n) . Clearly, \overline{M} gives rise to seminorm on the space of bounded measurable functions on G via $f \mapsto \overline{M}(|f|)$.

A subset A of G is called *relatively dense* if there exists a compact set $K \subset G$ with

$$G = \bigcup_{a \in A} (a + K).$$

A continuous bounded $f : G \longrightarrow \mathbb{C}$ is called *Bohr almost periodic* if for any $\varepsilon > 0$ the set of $t \in G$ with

$$\|f - f(\cdot - t)\|_\infty < \varepsilon$$

is relatively dense. Here, the supremum norm $\|\cdot\|_\infty$ for bounded complex valued functions on G is defined via $\|f\|_\infty = \sup_{s \in G} |f(s)|$.

2. MEAN ALMOST PERIODIC POINTS AND PURE POINT SPECTRUM

In this section, we introduce and study mean almost periodic points. The main result of this section then provides a characterization of pure point spectrum via mean almost periodicity of points.

Whenever (X, G) is a dynamical system with metric d and (B_n) is a Følner sequence, we define

$$D = D_d^{(B_n)} : X \times X \longrightarrow [0, \infty), \quad D(x, y) := \overline{M}(s \mapsto d(sx, sy)).$$

Clearly, D is a pseudometric. Moreover, the Følner condition on (B_n) easily gives that D is invariant i.e. satisfies $D(tx, ty) = D(x, y)$ for all $x, y \in X$ and $t \in G$. Furthermore, for each $x \in X$ the function $G \rightarrow [0, \infty)$, $t \mapsto D(x, tx)$, is uniformly continuous. Indeed, continuity at $t = 0$ is easily seen from the definition. Moreover, due to the invariance and the pseudometric properties we infer

$$|D(x, tx) - D(x, sx)| \leq D(tx, sx) = D(x, (s - t)x). \quad (\clubsuit)$$

When combined with continuity at $t = 0$, this gives uniform continuity. We will refer to D as the *averaged metric on X associated to d and (B_n)* .

Definition 2.1 (Mean almost periodic points). *Let (X, G) be a dynamical system and d a metric on X generating the topology and (B_n) a Følner sequence and D the associated averaged metric. Then, a point $x \in X$ is called mean almost periodic with respect to d and (B_n) if for every $\varepsilon > 0$ the set*

$$\{t \in G : D(x, tx) < \varepsilon\}$$

is relatively dense.

Remark. Almost periodicity properties with respect to \overline{M} are often connected with the name of *Besicovitch*. We use this for a strengthened version to be introduced below. Here we stick to the term ‘mean’ as this seems to be the common term within the study of equicontinuity properties in recent years (see e.g. [10, 15, 17] as well as discussion in Appendix C).

By definition, mean almost periodicity depends on the chosen Følner sequence. In our subsequent discussion of mean almost periodic points, however, we will often refrain from explicitly referring to the Følner sequence (B_n) if it is clear from the context which sequence is involved.

Lemma 2.2. *Let (X, G) be a dynamical system and d a metric on X generating the topology. An $x \in X$ is mean almost periodic if and only if the function $G \ni t \mapsto D(x, tx) \in \mathbb{R}$ is Bohr almost periodic.*

Proof. Define f on G via $f(t) := D(x, tx)$. We have already noted that the function f is uniformly continuous. Clearly, Bohr almost periodicity of f implies that x is mean almost periodic (as $f(0) = 0$). Conversely, (♣) gives $|f(t+s) - f(s)| \leq f(t)$ for all $s, t \in G$ and mean almost periodicity of x implies Bohr almost periodicity of f . \square

Our next aim is to discuss independence of this definition from the metric and to provide an alternative way of defining mean almost periodicity via density of super level sets. The proofs of the corresponding statements are not difficult and rather close to each other. They rely on some simple facts stated in the next proposition. We denote the characteristic function of a set $A \subset G$ by 1_A (i.e. $1_A(x) = 1$ for $x \in A$ and $1_A(x) = 0$ for $x \notin A$).

Proposition 2.3. *Let $f : G \rightarrow [0, 1]$ be given. Then, for any $\delta > 0$ the following estimates hold for the set*

$$A(f, \delta) := \{s \in G : f(s) \geq \delta\} :$$

- (a) $\overline{M}(1_{A(f, \delta)}) \leq \frac{1}{\delta} \overline{M}(f)$.
- (b) $\overline{M}(f) \leq M(1_{A(f, \delta)}) + \delta$.

Proof. (a) We clearly have $1_{\{s \in G : f(s) \geq \delta\}} \leq \frac{1}{\delta} f$. This easily gives (a).

(b) We compute

$$\overline{M}(f) \leq \overline{M}(f \cdot 1_{\{s \in G : f(s) \geq \delta\}}) + \overline{M}(f \cdot 1_{\{s \in G : f(s) < \delta\}}) \leq \overline{M}(1_{\{s \in G : f(s) \geq \delta\}}) + \delta.$$

This finishes the proof. \square

Lemma 2.4 (Independence of the metric). *Let (X, G) be a dynamical system with metrizable X and (B_n) a Følner sequence on G . Then, mean almost periodicity of an $x \in X$ does not depend on the chosen metric (provided it generates the topology).*

Proof. Let e, d be two metrics on X , which generate the topology. For $x \in X$ and $t \in G$ we define the functions $d_{t,x}$ and $e_{t,x}$ on G via $d_{t,x}(s) = d(sx, tsx)$ and $e_{t,x}(s) = e(sx, tsx)$, respectively. We show that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $t \in G$ and $x \in X$ we have

$$\overline{M}(d_{t,x}) < \varepsilon$$

whenever $\overline{M}(e_{t,x}) < \delta$ holds. The statement with the roles of e and d reversed can be shown analogously and taken together these two statements prove the lemma.

Without loss of generality we assume $d, e \leq 1$.

Let $\varepsilon > 0$ be given. Choose $\delta' > 0$ with $d(z, y) < \frac{\varepsilon}{2}$ whenever $e(z, y) < \delta'$. Set

$$\delta := \delta' \cdot \frac{\varepsilon}{2}.$$

If $\overline{M}(e_{t,x}) < \delta$ then (a) of Proposition 2.3 gives

$$\overline{M}(1_{\{s:e_{t,x}(s) \geq \delta'\}}) \leq \frac{1}{\delta'} \overline{M}(e_{t,x}) < \frac{\delta}{\delta'} = \frac{\varepsilon}{2}.$$

Furthermore we note that, by definition of δ' , we have

$$\overline{M}(d_{t,x} 1_{\{s:e_{t,x}(s) < \delta'\}}) \leq \frac{\varepsilon}{2}.$$

Given this we can now estimate

$$\overline{M}(d_{t,x}) \leq \overline{M}(d_{t,x} 1_{\{s:e_{t,x}(s) \geq \delta'\}}) + \overline{M}(d_{t,x}(s) 1_{\{s:e_{t,x}(s) < \delta'\}}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This is the desired statement. □

By the previous lemma mean almost periodicity of a point is independent of the underlying metric. Hence, we can (and will) from now on refrain from specifying a metric when talking about mean almost periodicity.

We define the *upper density* of a subset $A \subset G$ via

$$\text{Dens}(A) := \overline{M}(1_A).$$

Lemma 2.5 (Mean almost periodicity via density of superlevel sets). *Let (X, G) be a dynamical system with metrizable X and (B_n) a Følner sequence on G . Then, the following assertions for $x \in X$ are equivalent:*

- (i) *The point x is mean almost periodic.*
- (ii) *For any $\delta > 0$ the set of $t \in G$ with*

$$\text{Dens}(\{s \in G : d(sx, tsx) \geq \delta\}) < \varepsilon$$

is relatively dense for any $\varepsilon > 0$.

Proof. We use the notation of the proof of the preceding lemma.

(i) \implies (ii): Let $\delta > 0$ and $\varepsilon > 0$ be arbitrary. By (i) the set of $t \in G$ with $\overline{M}(d_{t,x}) < \delta\varepsilon$ is relatively dense. Now, for any such $t \in G$ we obtain from (a) of Proposition 2.3

$$\overline{M}(1_{\{s:d_{t,x}(s) \geq \delta\}}) < \varepsilon.$$

(ii) \implies (i): Let $\varepsilon > 0$ be given. Assume without loss of generality that $d \leq 1$. Set $\delta = \frac{\varepsilon}{2}$. By (ii) the set of $t \in G$ with $\text{Dens}(\{s \in G : d(sx, tsx) \geq \delta\}) < \frac{\varepsilon}{2}$ is relatively dense. Choose such a $t \in G$. Then, (b) of Proposition 2.3 gives

$$\overline{M}(d_{t,x}) \leq \overline{M}(1_{\{s:d_{t,x}(s) \geq \delta\}}) + \delta = \text{Dens}(\{s \in G : d(sx, tsx) \geq \delta\}) + \delta < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This finishes the proof. \square

We finish this section with the discussion of a further characterization of mean almost periodicity via suitable functions. To state this characterization (and similar characterizations in subsequent sections) it is useful to define for $x \in X$ and $f \in C(X)$ the function

$$f_x : G \longrightarrow \mathbb{C}, \quad f_x(t) = f(tx).$$

Moreover, we set

$$\mathcal{A}_x := \{f_x : f \in C(X)\}.$$

Clearly, \mathcal{A}_x is an algebra.

Proposition 2.6 (Completeness of \mathcal{A}_x). *The algebra \mathcal{A}_x is complete with respect to $\|\cdot\|_\infty$.*

Proof. Consider a sequence $(f^{(n)})$ in $C(X)$ such that $(f_x^{(n)})$ is a Cauchy sequence with respect to $\|\cdot\|_\infty$. Then, a direct $\varepsilon/3$ argument shows that the restrictions of $f^{(n)}$ to the orbit closure of x converge uniformly to a continuous function on the orbit closure. Now, the desired statement follows from Tietze's extension theorem. \square

A bounded measurable function $f : G \longrightarrow \mathbb{C}$ is *mean almost periodic* with respect to (B_n) if for every $\varepsilon > 0$ the set

$$\{t \in G : \overline{M}(|f(\cdot) - f(\cdot - t)|) < \varepsilon\}$$

is relatively dense. The set of uniformly continuous mean almost periodic functions is an algebra and closed under complex conjugation and uniform convergence (see appendix B).

As a consequence of the previous considerations we can now characterize mean almost periodicity via functions.

Lemma 2.7 (Mean almost periodicity via functions). *Let (X, G) be a dynamical system and (B_n) a Følner sequence on G . For $x \in X$ the following assertions are equivalent:*

- (i) *The point x is mean almost periodic.*
- (ii) *Every element from \mathcal{A}_x is mean almost periodic.*
- (iii) *The set $\{f \in C(X) : f_x \text{ is mean almost periodic}\}$ separates the points of \overline{Gx} .*
- (iv) *For any $s \in G$ the function $d_x^{(s)}$ is mean almost periodic, where $d^{(s)} \in C(X)$ is defined via $d^{(s)}(y) := d(sx, y)$.*

Remark. As the proof shows condition (iii) could equivalently be formulated with \overline{Gx} replaced by the whole space X .

Proof. (iv) \implies (iii): This follows as the $d^{(s)}$, $s \in G$, clearly separate the points of \overline{Gx} .

(iii) \implies (ii): Invoking the corresponding properties of mean almost periodic functions, we can easily see that $\{f \in C(X) : f_x \text{ is mean almost periodic}\}$ is an algebra, which is closed under complex conjugation and uniform convergence. This algebra clearly contains the constant functions. Moreover, by assumption (iii) it separates the points of \overline{Gx} . Furthermore this algebra contains every function $f \in C(X)$, which vanishes on \overline{Gx} (as $f_x = 0$ for any such

functions). Thus, this algebra even separates the points of X . Now, (ii) follows from Stone–Weierstraß’ theorem.

(ii) \implies (i): Choose a countable set $\mathcal{C} \subset C(X)$ such that any $f \in \mathcal{C}$ satisfies $\|f\|_\infty \leq 1$ and such that the elements of \mathcal{C} separate the points of X . Let $c_f > 0$, $f \in \mathcal{C}$, with $\sum_{f \in \mathcal{C}} c_f < \infty$ be given. Then,

$$e(z, y) := \sum_{f \in \mathcal{C}} c_f |f(x) - f(y)|$$

is a metric on X , which generates the topology. Moreover, by assumption (ii) the function f_x is mean almost periodic for any $f \in C(X)$ and, hence, any $f \in \mathcal{C}$. This easily gives that the set of $t \in G$ with $\overline{M}(s \mapsto e(sx, (t + s)x)) < \varepsilon$ is relatively dense (compare Proposition B.4). Thus, x is mean almost periodic with respect to the metric e . As mean almost periodicity does not depend on the metric, we conclude (i).

(i) \implies (iv): Let $z \in X$ be arbitrary and define $d^{(z)} \in C(X)$ by $d^{(z)}(y) := d(z, y)$. The triangle inequality for d gives

$$\overline{M}(|d^{(z_x)}(\cdot - t) - d_x^{(z)}|) = \overline{M}(s \mapsto |d(z, (s-t)x) - d(z, sx)|) \leq \overline{M}(s \mapsto d(sx, (s-t)x)) = D(x, -tx)$$

for any $t \in G$ and $z \in X$. Now, mean almost periodicity of $d_x^z(\cdot)$ follows (for any $z \in X$) from (i). □

We now come to the main result of this section, which provides a characterization of pure point spectrum via mean almost periodic points.

Theorem 2.8. *Let (X, G, m) be a dynamical system and (B_n) a Følner sequence on G . Assume that there exists a generic point for (X, G, m) . Then, the following assertions are equivalent:*

- (i) *The dynamical system (X, G, m) has pure point spectrum.*
- (ii) *Every generic point of X is mean almost periodic.*
- (iii) *One generic point of X is mean almost periodic.*

If (X, G, m) is ergodic and the Birkhoff theorem holds along (B_n) , these statements are also equivalent to the following statement:

- (iv) *Almost every $x \in X$ is mean almost periodic.*

Remark. A related result for subshifts over a finite alphabet can be found in Lemma 5 of [35]. There, pure point spectrum (i) is characterized via a variant of (iv) given by a mean almost periodicity condition on points defined via a metric (close in spirit to what is discussed above in Lemma 2.5).

Proof. In the ergodic case almost every point is generic. Hence, (ii) \implies (iv) and (iv) \implies (iii) follow. So we now turn to showing equivalence between (i), (ii) and (iii) in the general case. We clearly have (ii) \implies (iii). To show (i) \implies (ii) and (iii) \implies (i) we define

$$\underline{d} : G \longrightarrow [0, \infty), \quad \underline{d}(t) = \int_X d(x, tx) \, dm(x).$$

The main result of [25] says that (i) is equivalent to \underline{d} being Bohr almost periodic. Thus, it remains to show that

- Bohr almost periodicity of \underline{d} implies (ii);
- (iii) implies Bohr almost periodicity of \underline{d} .

Now, for $t \in G$ we can consider $f_t : X \rightarrow [0, \infty)$, $f_t(x) = d(x, tx)$. Then, f_t is clearly continuous. Thus, whenever $y \in X$ is generic, we find

$$\underline{d}(t) = \int_X d(x, tx) \, dm(x) = \int_X f_t(x) \, dm(x) = \overline{M}(s \mapsto f_t(sy)) = D(y, ty)$$

for every $t \in G$. Moreover, the triangle inequality gives that \underline{d} is Bohr almost periodic if and only if the set

$$\{t \in G : \underline{d}(t) < \varepsilon\}$$

is relatively dense in G for all $\varepsilon > 0$. Putting this together we easily obtain that \underline{d} is Bohr almost periodic if and only if one (every) generic $y \in X$ is mean almost periodic. \square

We emphasize that the first part of the preceding theorem does not need an ergodicity assumption and illustrate this by the following example.

Example - pure point spectrum in non ergodic case. Consider $\{0, 1\}$ with discrete topology and $X = \{0, 1\}^{\mathbb{Z}}$ with product topology. Equip X with the shift action of \mathbb{Z} given by $\alpha_n(x) = x(\cdot - n)$ for $n \in \mathbb{Z}$. Let $\underline{1}$ and $\underline{0}$ be the elements of X which are constant equal to 1 and 0 respectively. Then, clearly each of these elements is invariant under the shift action and so are then the sets $\{\underline{0}\}$ and $\{\underline{1}\}$. Thus,

$$m := \frac{1}{2}(\delta_{\underline{0}} + \delta_{\underline{1}})$$

is an invariant probability measure (where δ_p denotes the unit point mass at p). Obviously, m is not ergodic. The space $L^2(X, m)$ is two-dimensional and $\sqrt{2} \cdot 1_{\{\underline{0}\}}, \sqrt{2} \cdot 1_{\{\underline{1}\}}$ is an orthogonal basis consisting of eigenfunctions (to the eigenvalue 1). In particular, (X, \mathbb{Z}, m) has pure point spectrum. Now, consider the point $x \in X$ with $x(-k)$ arbitrary for $k \geq 0$ and $x(k) = 1$ if $k \in \{2^n, \dots, 2^n + 2^{n-1} - 1\}$ for some $n \in \mathbb{N}$ and $x(k) = 0$ else. Then, it is not hard to see that x is generic for m with respect to the Følner sequence $B_n = \{1, \dots, 2^n\}$. So, the theorem gives that x is mean almost periodic. In this example neither $\underline{0}$ nor $\underline{1}$ are generic. Hence, m does not give mass to generic points and the set of generic points has measure zero. Note that the construction of x could easily be modified to yield a transitive generic point (by including suitable finite words of slowly increasing length between the blocks of 1's and 0's in x).

Combining the previous result, Theorem 2.8, with the characterization of mean almost periodicity via functions in Lemma 2.7 we obtain the following.

Corollary 2.9. *Let (X, G, m) be an ergodic dynamical system with metrizable X and assume that the Birkhoff ergodic theorem holds along (B_n) . Then, the following assertions are equivalent.*

- (i) *The dynamical system (X, G, m) has pure point spectrum.*
- (ii) *For almost every $x \in X$ the set $\{f \in C(X) : f_x \text{ is mean almost periodic}\}$ separates the points of X .*

Remark. A variant of this statement (with (ii) replaced by the stronger condition that \mathcal{A}_x consists only of mean almost periodic functions) is shown in [26] based on an earlier version of [28]. Our proof is different. Note also that for ergodic systems over $G = \mathbb{Z}$, it is known that pure point spectrum is equivalent to $\mathbb{Z} \ni n \mapsto f(nx)$ belonging to the Besicovitch class for almost every $x \in X$ whenever f is a bounded measurable function on X , see Theorem 3.22 in [6]. The condition of Besicovitch class is stronger than mean almost periodicity (see also next section).

If the system (X, G, m) is uniquely ergodic, then every $x \in X$ is generic irrespective of the underlying Følner sequence (Oxtoby's theorem). Thus, from the previous theorem we obtain immediately the following corollary.

Corollary 2.10. *Let (X, G, m) be a dynamical system and (B_n) a Følner sequence on G . Assume that (X, G, m) is uniquely ergodic. Then, the following assertions are equivalent:*

- (i) *The dynamical system (X, G, m) has pure point spectrum.*
- (ii) *Every $x \in X$ is mean almost periodic.*
- (iii) *One $x \in X$ is mean almost periodic.*

Remark. The concept of mean almost periodicity depends on the chosen Følner sequence. To see this, consider $X := \{0, 1\}^{\mathbb{Z}}$ with product topology and the shift action of \mathbb{Z} and the Bernoulli measure m (product measure of the measures giving equal weights $1/2$ to $\{0\}$ and $\{1\}$). This system is ergodic and m almost every $x \in X$ contains arbitrary long stretches of 0 's. For each of those x we can then choose a Følner sequence (B_n) with $\overline{M}(s \mapsto d(sx, (t+s)x)) = 0$ for all $t \in \mathbb{Z}$ (by each B_n being chosen 'within' a long stretch of 0 's with distance to the boundary of these stretches increasing in n). Hence, each of these x is mean almost periodic. On the other hand, as the system does not have pure point spectrum, we obtain from Theorem 2 that not every of these x will be almost periodic with respect to the standard Følner sequence $B_n = \{0, \dots, n\}$ along which Birkhoff's ergodic theorem holds.

3. BESICOVITCH ALMOST PERIODIC POINTS AND EIGENFUNCTIONS

In this section, we discuss a strengthened version of mean almost periodicity viz Besicovitch almost periodicity. We show that pure point spectrum can also be characterized via this strengthened version. In fact, our results can be understood as saying that in a dynamical system with pure point spectrum both eigenfunctions and eigenvalues can be read of from any of its (generic) Besicovitch almost periodic points.

We consider a σ -compact, locally compact abelian group together with a Følner sequence (B_n) . As usual the set of all continuous group homomorphisms $\xi : G \rightarrow \mathbb{T}$ is denoted as \widehat{G} and called the *dual group* of G . We say that a bounded function $f : G \rightarrow \mathbb{C}$ is *Besicovitch almost periodic* if for any $\varepsilon > 0$ there exist $k \in \mathbb{N}$, $\xi_1, \dots, \xi_k \in \widehat{G}$ and $c_1, \dots, c_k \in \mathbb{C}$ with

$$\overline{M}(|f - \sum_{j=1}^k c_j \xi_j|) < \varepsilon.$$

A discussion of basic properties of uniformly continuous Besicovitch almost periodic functions is given in Appendix C. This shows in particular that any uniformly continuous Besicovitch almost periodic function is also mean almost periodic and admits an average (see below as well). The discussion also shows that the set of these functions forms an algebra and is closed under uniform convergence.

Definition 3.1 (Besicovitch almost periodic points). *Let (X, G) be a metrizable dynamical system and (B_n) a Følner sequence. Then, a point $x \in X$ is called Besicovitch almost periodic with respect to (B_n) if \mathcal{A}_x consists only of Besicovitch almost periodic functions.*

As in the definition of mean almost periodicity also Besicovitch almost periodicity depends on the chosen Følner sequence. In our subsequent discussion, however, we will often refrain from explicitly referring to the Følner sequence (B_n) if it is clear from the context which sequence is involved.

Remark. (a) To set this definition in perspective we refer to Lemma 2.7. This lemma shows that a point is mean almost periodic if and only if \mathcal{A}_x consists of mean almost periodic functions only.

(b) Note also that the statements of Lemma 2.7 remain true (with essentially the same proof) after 'mean almost periodic' is replaced with 'Besicovitch almost periodic'.

(c) As Besicovitch almost periodic functions are mean almost periodic, any Besicovitch almost periodic point is mean almost periodic. The converse is not true. To see this consider $X := \{0, 1\}^{\mathbb{Z}}$ with product topology and the shift action of \mathbb{Z} . Set $B_n := \{0, \dots, n\}$ for n even and $B_n = \{-n, \dots, -1\}$ for n odd. Consider now $y \in X$ with $y(k) = 1$ for $k \geq 0$ and $y(k) = 0$ otherwise. Then, it is not hard to see that $D(y, ny) = 0$ for all $n \in \mathbb{Z}$. Hence, y is mean almost periodic. On the other hand, consider $f : X \rightarrow \{0, 1\}$ with $f(x) = x(0)$. Clearly f is continuous. Moreover,

$$a_n := \frac{1}{|B_n|} \sum_{k \in B_n} f(ky)$$

does not converge (as $a_{2n} = 1$ and $a_{2n+1} = 0$ for all $n \in \mathbb{N}$). By the discussion in the appendix (see subsequent proposition as well), this shows that f_y is not Besicovitch almost periodic. Hence, y is not Besicovitch almost periodic.

Here comes a characterization of Besicovitch almost periodic points via existence of means. To state it, we first introduce some notation. For a bounded measurable function $h : G \rightarrow \mathbb{C}$, we define the *mean* or *average* of h with respect to (B_n) by

$$A(h) := \lim_{n \rightarrow \infty} \frac{1}{|B_n|} \int_{B_n} h(t) dt$$

whenever the limit exists.

Proposition 3.2 (Averaging along orbits). *Let (X, G) be a dynamical system and (B_n) be a Følner sequence. Then, the following assertions are equivalent for $x \in X$:*

- (i) *The point x is Besicovitch almost periodic.*
- (ii) *For any $f \in C(X)$ there exists a countable set $F_f \subset \widehat{G}$ such that the limits*

$$A(|f_x|^2) = \lim_{n \rightarrow \infty} \frac{1}{|B_n|} \int_{B_n} |f(tx)|^2 dt \text{ and } A(f_x \bar{\xi}) = \lim_{n \rightarrow \infty} \frac{1}{|B_n|} \int_{B_n} f(tx) \overline{\xi(t)} dt$$

exist for all $\xi \in F_f$ and

$$A(|f_x|^2) = \sum_{\xi \in F_f} |A(f_x \bar{\xi})|^2$$

holds.

Moreover, in case that (i) and (ii) hold, $A(f \bar{\xi})$ exists and equals 0 for $f \in C(X)$ and $\xi \in \widehat{G} \setminus F_f$.

Proof. This is a direct consequence of the definition of Besicovitch almost periodicity of a point and Proposition C.4 in Appendix C. \square

Definition 3.3 (Frequency). *Let (X, G) be a dynamical system and $x \in X$ a Besicovitch almost periodic point. Then, every $\xi \in \widehat{G}$ with $A(f_x \bar{\xi}) \neq 0$ for some $f \in C(X)$ is called a frequency of x . The set of all frequencies of x is denoted by $\text{Freq}(x)$.*

Here is the first main result of this section. It shows that any Besicovitch almost periodic point completely determines a dynamical system with pure point spectrum.

Theorem 3.4. *Let (X, G) be a dynamical system, (B_n) a Følner sequence and $p \in X$ a Besicovitch almost periodic point. Then, there exists a (unique) ergodic probability measure m on X such that p is generic with respect to m . The dynamical system (X, G, m) has pure point spectrum and $\text{Eig}(X, G, m) = \text{Freq}(p)$ holds. To each eigenvalue $\xi \in \text{Eig}(X, G, m)$ there exists a (unique) eigenfunction $e_\xi \in L^2(X, m)$ with*

$$\int_X f \overline{e_\xi} \, dm = A(f_p \overline{\xi})$$

for all $f \in C(X)$.

Proof. Obviously, the map

$$\Phi : C(X) \longrightarrow \mathbb{C}, \quad f \mapsto A(f_p),$$

is linear and positive (i.e. $A(f_p) \geq 0$ for $f \geq 0$). Hence, there exist a unique measure m on X with $\Phi(f) = \int_X f \, dm$. Clearly, $m(X) = \int_X 1 \, dm = A(1) = 1$. By the Følner property of (B_n) , the mean A is invariant and so is then Φ . This easily gives that m is invariant. So, m is an invariant probability measure.

We now turn to the construction of the eigenfunctions. For $\xi \in \widehat{G}$ consider the map

$$\Phi_\xi : C(X) \longrightarrow \mathbb{C}, \quad f \mapsto A(f_p \overline{\xi}).$$

This map is obviously linear and defined on a dense subspace of $L^2(X, m)$. By Cauchy-Schwarz inequality, $A(1) = 1$ and the construction of m we find

$$|\Phi_\xi(f)|^2 = |A(f_p \overline{\xi})|^2 \leq A(|f_p|^2) A(1) = \int_X |f|^2 \, dm.$$

Hence, Φ_ξ can be extended to a linear continuous map, again denoted by Φ_ξ , on the whole $L^2(X, m)$. By Riesz Lemma there exists then an $e_\xi \in L^2(X, m)$ with $\|e_\xi\| \leq 1$ and

$$\Phi_\xi(f) = \int_X f \overline{e_\xi} \, dm$$

for all $f \in C(X)$. Define

$$E := \{\xi \in \widehat{G} : e_\xi \neq 0\}.$$

By construction $\xi \in \widehat{G}$ belongs to E if and only if there exists an $f \in C(X)$ with $A(f_p \overline{\xi}) \neq 0$. Hence, $E = \text{Freq}(p)$ holds. In particular, we have $A(f_p \overline{\varrho}) = 0$ for all $f \in C(X)$ and $\varrho \in \widehat{G} \setminus E$. A short computation shows for $t \in G$

$$\begin{aligned} \xi(-t) \int_X f \overline{e_\xi} \, dm &= \xi(-t) A(f_p \overline{\xi}) \\ &= A(f_p \overline{\xi(t + \cdot)}) \\ \text{(A invariant)} &= A(f_p(-t \cdot) \overline{\xi}) \\ \text{(construction of } m) &= \int_X f(-t \cdot) \overline{e_\xi} \, dm \\ \text{(} m \text{ invariant)} &= \int_X f \overline{e_\xi(t \cdot)} \, dm \end{aligned}$$

for all $f \in C(X)$. As these f are dense in $L^2(X, m)$ this gives

$$e_\xi(t \cdot) = \xi(t) e_\xi$$

for all $t \in G$. This shows that e_ξ is an eigenfunction (to ξ) for each $\xi \in E$. Clearly, eigenfunctions to different eigenvalues are orthogonal.

Next, we show that the e_ξ , $\xi \in E$, are normalized and form a basis. This gives that $E = \text{Eig}(X, G, m)$ and together with the already shown $E = \text{Freq}(p)$, this will then also imply $\text{Eig}(X, G, m) = \text{Freq}(p)$.

By Parseval inequality, the definition of m and Proposition 3.2, we have the following:

$$\begin{aligned}
\sum_{\xi \in E} \left| \int_X f \bar{e}_\xi \, dm \right|^2 &\leq \int_X |f|^2 \, dm \\
&= A(|f_p|^2) \\
\text{(Proposition 3.2)} &= \sum_{\xi \in \tilde{G}} |A(f_p \bar{\xi})|^2 \\
\text{(construction of } E) &= \sum_{\xi \in E} |A(f_p \bar{\xi})|^2 \\
\text{(construction of } e_\xi) &= \sum_{\xi \in E} \left| \int_X f \bar{e}_\xi \, dm \right|^2.
\end{aligned}$$

This shows

$$\sum_{\xi \in E} \left| \int_X f \bar{e}_\xi \, dm \right|^2 = \int_X |f|^2 \, dm$$

for all $f \in C(X)$. This is only possible if $\|e_\xi\| = 1$ for all $\xi \in E$ and (e_ξ) form an orthonormal basis of $L^2(X, m)$.

It remains to show ergodicity: For each eigenvalue $\xi \in E$ we have constructed an eigenfunction e_ξ and we have shown that these form a complete set (i.e. e_ξ , $\xi \in E$, is an orthonormal basis). Hence (as different of these eigenfunctions belong to different eigenspaces) each eigenspace is one-dimensional. In particular the eigenspace to the eigenvalue 1 is one dimensional and the system is ergodic. \square

Remark. Note that the proof relies (and only relies) on the characterizing properties of Besicovitch almost periodic points given in Proposition 3.2.

The previous theorem shows that any Besicovitch almost periodic point is generic with respect to a (uniquely determined) measure. It may well be that different Besicovitch almost periodic points are generic with respect to different measures. Consider e.g. the full shift $X = \{0, 1\}^{\mathbb{Z}}$ with $Tx(n) = x(n+1)$. Then, any periodic element of X is Besicovitch almost periodic. Clearly, elements with different periods will not be generic with respect to the same measure. This motivates the following definition.

Definition 3.5 (Generic Besicovitch almost periodic points). *Let (X, G) be a dynamical system and (B_n) a Følner sequence on G . Then, for any invariant probability measure m on (X, G) we denote by $\text{Bap}(X, G, m)$ the set of those Besicovitch almost periodic points which are generic with respect to m .*

We have the following consequence of the preceding theorem.

Proposition 3.6. *Let (X, G, m) be a dynamical system and (B_n) be a Følner sequence. Then, $\text{Bap}(X, G, m)$ is a measurable invariant set.*

Proof. Clearly, $\text{Bap}(X, G, m)$ is invariant as both the set of generic points and the set of Besicovitch almost periodic points are invariant. If $\text{Bap}(X, G, m)$ is empty there is nothing left to show. So, consider the case $\text{Bap}(X, G, m) \neq \emptyset$. By Theorem 3.4, the dynamical system (X, G, m) then has pure point spectrum and the set of its eigenvalues $\text{Eig}(X, G, m)$ equals $\text{Freq}(p)$ for any $p \in \text{Bap}(X, G, m)$. Set $E := \text{Eig}(X, G, m)$.

Claim. Let D be a dense subset of $C(X)$. Then, we have $p \in \text{Bap}(X, G, m)$ if and only if the following three points hold:

- $A(f_p)$ exists and equals $\int_X f \, dm$ for all $f \in D$.
- $A(f_p \bar{\xi})$ exists for all $\xi \in E$ and $f \in D$.
- $A(|f_p|^2) = \sum_{\xi \in E} |A(f_p \bar{\xi})|^2$ for all $f \in D$.

Proof of claim. Consider $p \in \text{Bap}(X, G, m)$. Then, p is generic and the first point holds (even for all $f \in C(X)$). In particular, $A(|f_p|^2)$ exists. Now, the second and third point follow from Proposition 3.2 as p is Besicovitch almost periodic with set of frequencies given by E .

Consider now a $p \in X$ satisfying the three points above. By density of D in $C(X)$ we then easily infer that $A(f_p) = \int_X f \, dm$ holds for all $f \in C(X)$ and $A(f_p \bar{\xi})$ exists for all $f \in C(X)$ and $\xi \in E$. In particular, we have $A(|f_p|^2) = \int |f|^2 \, dm$ for all $f \in C(X)$. Given this, we can now follow the proof of Theorem 3.4 to conclude the existence of (pairwise orthogonal) eigenfunctions e_ξ to $\xi \in \text{Eig}(X, G, m)$ with $\|e_\xi\| \leq 1$ and

$$\sum_{\xi \in E} \left| \int_X f \bar{e}_\xi \, dm \right|^2 = \int_X |f|^2 \, dm$$

for all $f \in D$. As D is dense, this is only possible if $\|e_\xi\| = 1$ holds for all $\xi \in E$ and the $e_\xi, \xi \in E$, are an orthonormal basis. This finishes the proof of the claim.

Given the claim, the desired measurability follows easily: By compactness and metrizablety of X we can choose a countable dense subset D of $C(X)$. Then, the claim gives that $p \in X$ belongs to $\text{Bap}(X, G, m)$ if countably many conditions are satisfied. Clearly, each of these conditions gives a measurable set. \square

The following theorem can be seen as both a converse to Theorem 3.4 and an analogue to Theorem 2.8.

Theorem 3.7 (Discrete spectrum via Besicovitch almost periodic points). *Let (X, G, m) be an ergodic dynamical system and (B_n) a Følner sequence along which Birkhoff's ergodic theorem holds. Then, the following assertions are equivalent:*

- (i) *The dynamical system (X, G, m) has pure point spectrum.*
- (ii) *$m(\text{Bap}(X, G, m)) = 1$.*
- (iii) *$\text{Bap}(X, G, m) \neq \emptyset$.*

If one of the equivalent conditions (i), (ii) and (iii) holds, then $\text{Eig}(X, G, m) = \text{Freq}(x)$ for every $x \in \text{Bap}(X, G, m)$. Moreover, in this case, for any $f \in C(X)$ and $\xi \in \widehat{G}$ the function

$$e_{f,\xi} : X \longrightarrow \mathbb{C}, \quad e_{f,\xi}(x) := \begin{cases} A(f_x \bar{\xi}) & : x \in \text{Bap}(X, G, m) \\ 0 & : \text{else} \end{cases},$$

satisfies $P_\xi f = e_{f,\xi}$ (in $L^2(X, m)$), $e_{f,\xi}(tx) = \xi(t)e_{f,\xi}(x)$ for all $t \in G$ and $x \in X$ and has constant modulus on $\text{Bap}(X, G, m)$.

Remark. By Theorem 3.4, the existence of a generic Besicovitch almost periodic point entails the ergodicity of m . For this reason the ergodicity assumption in the above theorem can not be dropped.

Proof. The implication (ii) \implies (iii) is clear. The implication (iii) \implies (i) follows from Theorem 3.4. We now show (i) \implies (ii). As the set of generic points has full measure, it suffices to show that almost every $x \in X$ is Besicovitch almost periodic. To do so, we denote the inner product on $L^2(X, m)$ by $\langle \cdot, \cdot \rangle$ and the associated norm by $\| \cdot \|_2$. Let $\xi_1, \xi_2, \xi_3, \dots$, be an enumeration of $\text{Eig}(X, G, m)$. Choose for any $\xi \in E$ a normalized eigenfunction $e_\xi : X \rightarrow \mathbb{C}$. Without loss of generality, we can assume

$$e_\xi(sx) = \xi(s) e_\xi(x)$$

for all $s \in G$ and $x \in X$. (Otherwise we could replace e_ξ by \tilde{e}_ξ defined by

$$\tilde{e}_\xi(x) := \lim_{n \rightarrow \infty} \frac{1}{|B_n|} \int_{B_n} e(sx) \bar{\xi}(s) \, ds$$

if the limit exists and $\tilde{e}_\xi(x) = 0$ else.) Moreover, for $\xi = 1$ we choose the constant function 1.

Consider now an arbitrary $g \in C(X)$. By Birkhoff's ergodic theorem, we can then find a subset X_g of X of full measure such that

$$\int_X \left| g - \sum_{j=1}^k \langle g, e_{\xi_j} \rangle e_{\xi_j} \right| \, dm(x) = \lim_{n \rightarrow \infty} \frac{1}{|B_n|} \int_{B_n} \left| g(sx) - \sum_{j=1}^k \langle g, e_{\xi_j} \rangle e_{\xi_j}(sx) \right| \, ds$$

for all $x \in X_g$ and $k \in \mathbb{N}$. Let $D \subset C(X)$ be a countable dense subset. Define

$$X' := \bigcap_{g \in D} X_g.$$

Then, X' has full measure and a short computation gives

$$\int_X \left| f - \sum_{j=1}^k \langle f, e_{\xi_j} \rangle e_{\xi_j} \right| \, dm(x) = \lim_{n \rightarrow \infty} \frac{1}{|B_n|} \int_{B_n} \left| f(sx) - \sum_{j=1}^k \langle f, e_{\xi_j} \rangle e_{\xi_j}(sx) \right| \, ds$$

for all $f \in C(X)$, $x \in X'$ and $k \in \mathbb{N}$. This, in turn implies that any $x \in X'$ is Besicovitch almost periodic: Indeed, a short calculation invoking Birkhoff's ergodic theorem and Cauchy–Schwarz' inequality shows

$$\begin{aligned}
 \overline{M}(|f(\cdot x) - \sum_{j=1}^k \langle f, e_{\xi_j} \rangle e_{\xi_j}(x) \xi_j(\cdot)|) &= \overline{M}(|f(\cdot x) - \sum_{j=1}^k \langle f, e_{\xi_j} \rangle e_{\xi_j}(\cdot x)|) \\
 \text{(Birkhoff's ergodic theorem)} &= \lim_{n \rightarrow \infty} \frac{1}{|B_n|} \int_{B_n} \left| f(sx) - \sum_{j=1}^k \langle f, e_{\xi_j} \rangle e_{\xi_j}(sx) \right| ds \\
 &= \int_X \left| f - \sum_{j=1}^k \langle f, e_{\xi_j} \rangle e_{\xi_j} \right| dm(x) \\
 \text{(Cauchy–Schwarz' inequality)} &\leq \|f - \sum_{j=1}^k \langle f, e_{\xi_j} \rangle e_{\xi_j}\|_2 \\
 &\rightarrow 0, \quad n \rightarrow \infty.
 \end{aligned}$$

This gives the desired claim.

We now turn to the remaining statements. The equality $\text{Freq}(x) = \text{Eig}(X, G, m)$ for an element $x \in \text{Bap}(X, G, m)$ directly follows from Theorem 3.4. As for $e_{f, \xi}$ we note that it is well-defined and invariant (as $\text{Bap}(X, G, m)$ is invariant). The equality $\text{Freq}(x) = \text{Eig}(X, G, m)$ for $x \in \text{Bap}(X, G, m)$ rather directly gives that $e_{f, \xi}$ vanishes identically for $\xi \in \widehat{G} \setminus \text{Eig}(X, G, m)$. In particular, it has constant modulus on $\text{Bap}(X, G, m)$. Now, by Theorem 3.4, for each $\xi \in E$ and $x \in \text{Bap}(X, G, m)$ there exists a normalized eigenfunction $e_{\xi}^{(x)}$ with

$$e_{f, \xi}(x) = A(f_x \bar{\xi}) = \langle f, e_{\xi}^{(x)} \rangle$$

As each eigenspace is one-dimensional the $e_{\xi}^{(x)}$ arising for different $x \in \text{Bap}(X, G, m)$ will only differ by a factor of modulus one. This gives the statement on constancy of the modulus.

That $e_{f, \xi}$ is the projection onto the eigenspace of ξ follows from standard theory, see e.g. [24] for recent discussion. \square

4. WEYL ALMOST PERIODIC POINTS, UNIQUE ERGODICITY AND CONTINUITY OF EIGENFUNCTIONS

In this section, we consider a strengthening of Besicovitch almost periodicity viz Weyl almost periodicity. We show that Weyl almost periodicity extends from one point to its orbit closure. This allows us to characterize transitive systems all of whose points are Weyl almost periodic. These are exactly the uniquely ergodic dynamical systems with pure point spectrum and continuous eigenfunctions. This ties in with various recent investigations (see below for details).

Let (X, G) be a dynamical system. Whenever d is a metric on X generating the topology and (B_n) is a Følner sequence, we define for each $n \in \mathbb{N}$ the map

$$\overline{M}_n : B(G) \longrightarrow \mathbb{R}, \quad \overline{M}_n(f) := \sup_{s \in G} \frac{1}{|B_n|} \int_{B_n + s} f(t) dt.$$

This gives then rise to the functions

$$D_n := D_{n,d} : X \times X \longrightarrow [0, \infty), \quad D_n(x, y) := \overline{M}_n(s \mapsto d(sx, sy)),$$

for each $n \in \mathbb{N}$. Then, each D_n can easily be seen to be an invariant pseudometric. Moreover, for each $x \in X$ the function $t \mapsto D_n(x, tx)$ is uniformly continuous (by the argument used in Section 1 to show uniform continuity of D). The function D_n is referred to as *averaged pseudometric* on level n .

A bounded measurable function $f : G \longrightarrow \mathbb{C}$ is *Weyl almost periodic* if for each $\varepsilon > 0$ there exist $k \in \mathbb{N}$, $\xi_1, \dots, \xi_k \in \widehat{G}$ and $c_1, \dots, c_k \in \mathbb{C}$ with

$$\limsup_{n \rightarrow \infty} \overline{M}_n(|f - \sum_{j=1}^k c_{\xi_j} \xi_j|) < \varepsilon.$$

As discussed in Appendix D, an equivalent alternative characterization is that for each $\varepsilon > 0$ there exist $N \in \mathbb{N}$ and a relatively dense set $R \subset G$ with

$$\overline{M}_N(|f - f(\cdot - t)|) < \varepsilon$$

for all $t \in R$. A crucial feature of Weyl almost periodic functions is the existence of the limits

$$\lim_{n \rightarrow \infty} \frac{1}{|B_n|} \int_{B_n + s_n} f(t) \xi(t) dt$$

irrespective of (and uniform in) the chosen sequence $(s_n) \in G$ for each $\xi \in \widehat{G}$, see Appendix D.

Definition 4.1 (Weyl almost periodic points). *Let (X, G) be a dynamical system and d a metric on X generating the topology and (B_n) a Følner sequence and D_n , $n \in \mathbb{N}$, the associated averaged metrics. Then, a point $x \in X$ is called *Weyl almost periodic with respect to d and (B_n)* if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that*

$$\{t \in G : D_N(x, tx) < \varepsilon\}$$

is relatively dense.

Remark. It follows from Proposition D.1 that an $x \in X$ is Weyl almost periodic if and only if for each $\varepsilon > 0$ there exists a relatively dense set $R \subset G$ and an $N_0 \in \mathbb{N}$ such that $D_N(x, tx) < \varepsilon$ for all $N \geq N_0$ and $t \in R$.

Arguing as in Section 2 with \overline{M} replaced by \overline{M}_n we see that Weyl almost periodicity is independent of the chosen metric and the following holds.

Lemma 4.2. *Let (X, G) be a dynamical system and d a metric on X generating the topology and (B_n) a Følner sequence and D_n , $n \in \mathbb{N}$, the associated averaged metrics. Then, the following assertions for $x \in X$ are equivalent:*

- (i) *The point x is Weyl almost periodic.*
- (ii) *The algebra \mathcal{A}_x consists of Weyl almost periodic functions.*
- (iii) *The set $\{f \in C(X) : f_x \text{ is Weyl almost periodic}\}$ separates the points of X .*
- (iv) *For any $s \in G$ the function $d_x^{(s)}$ is Weyl almost periodic, where $d^{(s)} \in C(X)$ is defined via $d^{(s)}(y) := d(sx, y)$.*

The previous lemma implies in particular that any Weyl almost periodic point is Besicovitch almost periodic. It is not hard to see by examples that the converse does not hold.

Weyl almost periodicity has a stability property.

Lemma 4.3 (Stability of Weyl almost periodicity along orbit closures). *Let (X, G) be a dynamical system. Assume that $x \in X$ is Weyl almost periodic. Then, any element in the orbit closure of x is Weyl almost periodic.*

Proof. The function D_N is lower semicontinuous for each $N \in \mathbb{N}$ as it is a supremum over continuous functions. From this and the invariance of D_N we find

$$D_N(y, ty) \leq \liminf_{n \rightarrow \infty} D_N(s_n x, ts_n x) = D_N(x, tx)$$

whenever $s_n x \rightarrow y$ for a sequence (s_n) in G . This easily gives the desired statement. \square

Proposition 4.4. *Let (X, G) be a dynamical system with transitive element $p \in X$. Let p be Weyl almost periodic. Then (X, G) is uniquely ergodic, has pure point spectrum, all eigenfunctions are continuous and $\text{Freq}(x) = \text{Eig}(X, G, m)$ holds for all $x \in X$. Moreover, for any $f \in C(X)$ and $\xi \in \widehat{G}$, the averages*

$$A_n(f_x \bar{\xi}) := \frac{1}{|B_n|} \int_{B_n} f(tx) \overline{\xi(t)} dt$$

converge (uniformly in x) towards the projection of f onto the eigenspace of ξ .

Proof. It is well-known that unique ergodicity is equivalent to uniform (in $y \in X$) convergence of the averages

$$\frac{1}{|B_n|} \int_{B_n} f(ty) dt$$

for each continuous $f : X \rightarrow \mathbb{C}$. Now, uniform existence of these averages on the orbit of x is a direct consequence of Weyl almost periodicity. This easily gives uniform existence on the orbit closure. As the orbit closure is X the desired statement on unique ergodicity follows. Denote the unique invariant probability measure by m .

By the previous lemma and the transitivity assumption on p every $x \in X$ is Weyl almost periodic. In particular, every element is Besicovitch almost periodic. As (X, G) is uniquely ergodic every $x \in X$ is also generic with respect to m . Hence, $X = \text{Bap}(X, G, m)$ follows. By Theorem 3.7, this implies pure point spectrum as well as pointwise convergence of the averages $A_n(f_x \bar{\xi})$ to the projection of f onto the eigenspace of ξ for each $f \in C(X)$ and $\xi \in \widehat{G}$. Now, by Weyl almost periodicity these averages converge uniformly in $x \in X$. Hence, their limit is continuous and continuity of the eigenfunctions follows. \square

Theorem 4.5. *Let (X, G) be a dynamical system with transitive point $p \in X$. Then, the following assertions are equivalent:*

- (i) *The dynamical system (X, G) is uniquely ergodic with pure point spectrum and continuous eigenfunctions.*
- (ii) *The point p is Weyl almost periodic.*
- (iii) *Every $x \in X$ is Weyl almost periodic.*

In this case, we have $\text{Freq}(x) = \text{Eig}(X, G, m)$ for all $x \in X$.

Proof. The implication (iii) \implies (ii) is obvious while (ii) \implies (iii) follows from Lemma 4.3.

The implication (ii) \implies (i) and the last part of the theorem were shown in Proposition 4.4. It remains to show the reverse implication (i) \implies (ii): This follows by a variant of the proof of the corresponding part in Theorem 3.7. We denote the unique invariant measure by m and use the notation introduced in the proof of Theorem 3.7. Thus, we denote the inner product on $L^2(X, m)$ by $\langle \cdot, \cdot \rangle$ and the associated norm by $\| \cdot \|_2$. As the spectrum is pure point, there

exists an orthonormal basis e_ξ , $\xi \in \text{Eig}(X, G, m)$, of $L^2(X, m)$ with e_ξ being an eigenfunction to the eigenvalue ξ for each $\xi \in \text{Eig}(X, G, m)$. By assumption each e_ξ , $\xi \in E$, can be chosen continuous. By unique ergodicity we then find for any finite subset $A \subset \text{Eig}(X, G, m)$ and any $y \in X$:

$$\begin{aligned} \limsup_{n \rightarrow \infty} \overline{M}_n(|f(\cdot y) - \sum_{\xi \in A} \langle e_\xi, f \rangle e_\xi(y) \xi(\cdot)|) &= \limsup_{n \rightarrow \infty} \overline{M}_n |f(\cdot y) - \sum_{\xi \in A} \langle e_\xi, f \rangle e_\xi(\cdot y)| \\ &= \lim_{n \rightarrow \infty} \frac{1}{|B_n|} \int_{B_n} |f(ty) - \sum_{\xi \in A} \langle e_\xi, f \rangle e_\xi(ty)| \, dt \\ &= \int_X |f(x) - \sum_{\xi \in A} \langle e_\xi, f \rangle e_\xi(x)| \, dm(x) \\ \text{(Cauchy-Schwarz' inequality)} &\leq \|f - \sum_{\xi \in A} \langle e_\xi, f \rangle e_\xi\|_2. \end{aligned}$$

As f_ξ , $\xi \in E$, is a basis of $L^2(X, m)$, the last term becomes arbitrarily small for suitable $A \subset E$. This shows that $t \mapsto f(ty)$ is Weyl almost periodic for any $y \in X$. \square

Remark. Recently, systems satisfying the equivalent conditions of the theorem have attracted substantial interest:

(a) For $G = \mathbb{Z}$ various equivalent characterizations of (i) have been investigated in [10]. In particular, it is shown there that (i) is equivalent to topological isomorphy of the system to its maximal equicontinuous factor. The case of general amenable groups G has been studied in [15]. In particular, it has been shown there that (i) is equivalent to the continuity of the averaged metric D on $X \times X$. This continuity is known as *mean equicontinuity* of the system. In this context our preceding result is remarkable as it does not assume control of D on the whole of $X \times X$ but just on $\{(x, tx) : t \in G\}$. In this sense, we have found a pointwise characterization of mean equicontinuity. Note, however, that we require a rather uniform control on the orbit of this one point.

(b) A large class of examples satisfying the conditions of the theorem are weakly almost periodic systems. A recent study of such systems is carried out in [29] to which we refer for a precise definition and further references, see also subsequent Section 5.

(c) A most important class of examples for the theorem are dynamical systems arising from regular cut and project schemes. Such systems are at the core of the study of aperiodic order (see [2]). They belong to the special class of dynamical systems known as translation bounded measure dynamical system (TMDS), see Section 6 for details. In fact, a huge bulk of material in the theory of aperiodic order deals with TMDS satisfying (i) of the theorem. A characterization of such systems via an almost periodicity property of its points had been missing for a long time. It was only given recently in [28]. The preceding theorem is a generalization of the corresponding result of [28] in that it is not restricted to TMDS but rather applies to general dynamical systems.

If a system is minimal every point is transitive and we can note the following immediate consequence of the preceding theorem.

Corollary 4.6. *Let (X, G) be a minimal dynamical system. Then, the following assertions are equivalent:*

- (i) *Every point in X is Weyl almost periodic.*

- (ii) *There exists a Weyl almost periodic point in X .*
- (iii) *The dynamical system (X, G) is uniquely ergodic with pure point spectrum and continuous eigenfunctions.*

Remark. A system can well be Weyl almost periodic without being minimal. Consider for example the orbit closure in the subshift $(\{0, 1\}^{\mathbb{Z}}, \mathbb{Z})$ of the element ω defined with $\omega_0 = 1$ and $\omega_n = 0$ for $n \neq 0$.

5. WEAKLY AND BOHR ALMOST PERIODIC DYNAMICAL SYSTEMS

In the preceding section, we have met Weyl almost periodic points. In this section, we introduce two special classes of Weyl almost periodic points, viz Bohr almost periodic points and weakly almost periodic points. This allows us to reanalyze and characterize weakly almost periodic dynamical system and Bohr almost periodic dynamical system via the new approach in this paper. Specifically, the main result of this section shows that a (transitive) dynamical system is weakly almost periodic (Bohr almost periodic) if and only if every of its points is weakly almost periodic (Bohr almost periodic). We refer to [30] for a recent discussion of Bohr and weakly almost periodic systems including relevance, background and further references.

Let G be a σ -compact locally compact abelian group and denote the set of uniformly continuous and bounded functions on G by $C_u(G)$. This space is a Banach space when equipped with the supremum norm $\|\cdot\|_\infty$. An $f \in C_u(G)$ is called *weakly almost periodic* if the set $\{f(\cdot - t) : t \in G\}$ is relatively compact in $C_u(G)$ with respect to the weak topology of the Banach space $(C_u(G), \|\cdot\|_\infty)$. Clearly, any Bohr almost periodic $f \in C_u(G)$ is weakly almost periodic (as Bohr almost periodicity means by definition that the set $\{f(\cdot - t) : t \in G\}$ is relatively compact in the original topology). In fact, a main result on weakly almost periodic functions, see e.g. [1] or [12, 30], gives that any weakly almost periodic f can be (uniquely) decomposed into $f = g + h$ with $g \in C_u(G)$ Bohr almost periodic and $h \in C_u(G)$ satisfying

$$\lim_{n \rightarrow \infty} \sup_{s \in G} \frac{1}{|B_n|} \int_{B_n} |h(s + t)| dt = 0$$

for any Følner sequence (B_n) . The existence of this decomposition is [31, Theorem 4.7.11], while the uniqueness follows immediately from [31, Lemma 4.6.8].

When combined with (♥) in Appendix A this easily gives that any weakly almost periodic function is Weyl almost periodic.

Definition 5.1 (Weakly and Bohr almost periodic points). *Let (X, G) be a dynamical system.*

(a) *A point $x \in X$ is called weakly almost periodic if \mathcal{A}_x consists only of weakly almost periodic functions.*

(b) *A point $x \in X$ is called Bohr almost periodic if \mathcal{A}_x consists only of Bohr almost periodic functions.*

Remark. By the discussion preceding the definition, any Bohr almost periodic point is weakly almost periodic and any weakly almost periodic point is Weyl almost periodic.

In the subsequent discussion, the weakly almost periodic case and the Bohr almost periodic case can be mostly treated in parallel. To facilitate the reading, we then give statements for the weakly almost periodic case and mention the Bohr almost periodic case in brackets only.

With essentially the same proof as Lemma 2.7, we obtain the following statement.

Lemma 5.2. *Let (X, G) be a dynamical system and let $x \in X$ be given. Then, the following assertions are equivalent:*

- (i) *The point x is weakly (Bohr) almost periodic.*
- (ii) *The weakly (Bohr) almost periodic functions in \mathcal{A}_x are dense in $(\mathcal{A}_x, \|\cdot\|_\infty)$.*
- (iii) *The set $\{f \in C(X) : f_x \text{ is weakly (Bohr) almost periodic}\}$ separates the points of X .*
- (iv) *For any $s \in G$ the function $d_x^{(s)}$ is weakly (Bohr) almost periodic, where $d^{(s)} \in C(X)$ is defined via $d^{(s)}(y) := d(sx, y)$.*

Whenever (X, G) is a dynamical system, any $f \in C(X)$ gives rise to a function on the product $G \times X$, viz

$$P_f : G \times X \longrightarrow \mathbb{C}, \quad (t, x) \mapsto f(tx).$$

Roughly speaking one can say that the preceding discussion was concerned with almost periodicity properties of the functions $f_x = P_f(\cdot, x)$ for $x \in X$. It is natural to consider almost periodicity properties of restrictions of the P_f to X as well for fixed $t \in G$. This leads to the notion of weakly (Bohr) almost periodic dynamical system. For $f \in C(X)$ and $t \in G$ we define

$$f_t : X \longrightarrow \mathbb{C}, \quad f_t(x) = f(tx) = P_f(t, \cdot).$$

Definition 5.3 (Weakly and Bohr almost periodic dynamical systems). *The dynamical system (X, G) is called weakly almost periodic and Bohr almost periodic¹ respectively if for any $f \in C(X)$ the family $\{f_t : t \in G\}$ has compact closure in the weak topology and the Banach space topology of $(C(X), \|\cdot\|_\infty)$ respectively.*

The next lemma relates weakly (Bohr) almost periodic dynamical systems to weakly (Bohr) almost periodic points.

Lemma 5.4. *Let (X, G) be a dynamical system.*

(a) *If (X, G) is weakly (Bohr) almost periodic then every $x \in X$ is a weakly (Bohr) almost periodic.*

(b) *If $x \in X$ is transitive and weakly (Bohr) almost periodic then (X, G) is a weakly (Bohr) almost periodic dynamical system.*

Proof. Fix an arbitrary $x \in X$. Define $F : C(X) \rightarrow C_u(G)$ via

$$F(f)(t) = f_x.$$

(a) It is easy to see that F is well defined and $\|F\| \leq 1$. It follows that F is continuous, and hence also weakly continuous [31, Lemma. 4.4.2]. Therefore, the image of a compact, respectively weak compact set, is compact, respectively weak compact. Since F commutes with the group action, the claim follows.

(b) We know that F is continuous. Moreover, since x has dense orbit, it follows immediately that for all $f \in C(X)$ we have

$$\|F(f)\|_\infty = \|f\|_\infty.$$

Therefore, F is an isometry and hence it induces an isomorphic isometry $F : C(X) \rightarrow \text{Im}(F)$. In particular, $\text{Im}(F)$ is closed in $C_u(G)$ and the mapping

$$F^{-1} : \text{Im}(F) \rightarrow C(X),$$

¹In some papers this is called almost periodic dynamical system, or strongly almost periodic dynamical system.

is a continuous operator, and hence is also weakly continuous [31, Lemma. 4.4.2]. Therefore, F^{-1} maps compact and weakly compact sets, respectively, into compact and weakly compact sets respectively. This easily gives the desired statement. \square

We obtain the following immediate consequence of the preceding lemma.

Theorem 5.5. *Let (X, G) be a dynamical system with transitive $p \in X$. Then, the following assertions are equivalent:*

- (i) (X, G) is weakly almost periodic.
- (ii) Every $x \in X$ is weakly almost periodic.
- (iii) The point $p \in X$ is weakly almost periodic.

We also note the following consequence of the theorem.

Corollary 5.6. *Let (X, G) be a dynamical system with transitive $p \in X$. For $s, t \in G$ define $g_{s,t} : X \rightarrow \mathbb{C}$ via*

$$g_{s,t}(x) = d(sp, tx).$$

Then, $p \in X$ is weakly almost periodic if and only if for each $s \in G$ the set $\{g_{s,t} : t \in G\}$ has compact closure in the weak topology of $(C(X), \|\cdot\|_\infty)$.

Proof. Indeed, the only if part of the statement is immediate from the definition of weak almost periodicity and the preceding theorem. To show the if part we will denote (in line with [30])

$$\text{WAP}(X) := \{f \in C(X) : \{f_t : t \in G\} \text{ has compact closure in the weak topology}\}.$$

Then, by [30, Prop. 3.3, Prop 3.4] $\text{WAP}(X)$ is a closed algebra of $(C(X), \|\cdot\|_\infty)$, which contains the constant function 1. Moreover, we have $g_{t,1} \in \text{WAP}(X)$ for all $t \in G$. We next show that the functions $g_{t,1}$ separate the points of X . Let $x \neq y \in X$ be arbitrary, and let $r = d(x, y)$.

Since p is a transitive point, there exists some $t \in G$ such that $d(tp, x) < \frac{r}{3}$. Then,

$$g_{t,1}(x) = d(tp, x) < \frac{r}{3} \quad \text{and} \quad g_{t,1}(y) = d(tp, y) \geq d(x, y) - d(tp, x) > \frac{2r}{3}.$$

This shows that $g_{t,1}(x) \neq g_{t,1}(y)$. Therefore, $\text{WAP}(X)$ is a closed algebra of $(C(X), \|\cdot\|_\infty)$ which is separating the points, and hence, by Stone–Weierstraß’ theorem, $\text{WAP}(X) = C(X)$. This gives the desired statement. \square

The preceding theorem allows us to recapture a main result of [30].

Corollary 5.7. *Let (X, G) be a transitive weakly almost periodic dynamical system. Then, (X, G) is uniquely ergodic with pure point spectrum and continuous eigenfunctions.*

Proof. Let p be a transitive point. Then, by Theorem 5.5, p is weakly almost periodic. Hence, p is Weyl almost periodic as well and the claim follows from Theorem 4.5. \square

As should be clear from the preceding discussion, the analogue of Theorem 5.5 with ‘weakly almost periodic’ replaced by ‘Bohr almost periodic’ holds as well (and similarly for Corollary 5.6). In fact, a somewhat stronger statement is true for Bohr almost periodic points. To state this properly, we introduce the following notation when dealing with an dynamical system (X, G) with metric d . For $f \in C(X)$ we consider the mapping

$$\pi_f : X \longrightarrow C_u(G), \quad \pi_f(x) := f_x,$$

and, when an $x \in X$ is fixed, we set

$$Y(f, x) := \overline{\{f_{tx} : t \in G\}},$$

where the closure is taken in $C_u(G)$ with the (usual) supremum norm. We also define

$$\bar{d} : X \times X \longrightarrow [0, \infty), \quad \bar{d}(x, y) := \sup_{s \in G} d(sx, sy).$$

Clearly, \bar{d} is a metric.

Theorem 5.8. *Let (X, G) be a dynamical system. For $x \in X$ the following assertions are equivalent:*

- (i) *The element $x \in X$ is Bohr almost periodic.*
- (ii) *For any $f \in C(X)$ the map $\pi_f : \overline{Gx} \longrightarrow C_u(G)$, $y \mapsto f_y$, is continuous with range given by $Y(f, x)$.*
- (iii) *There exists a G -invariant metric on \overline{Gx} generating the topology.*
- (iv) *The function \bar{d} is continuous on \overline{Gx} .*
- (v) *The orbit closure \overline{Gx} admits a structure of a locally compact group such that $G \longrightarrow \overline{Gx}$, $t \mapsto tx$, becomes a continuous group homomorphism.*

In particular, the orbit closure of any Bohr almost periodic point is minimal.

Proof. (i) \implies (ii): Clearly, $\pi_f(tx) = f_{tx} = f_x(\cdot + t)$ for any $t \in G$. Thus, it suffices to show that f_y belongs indeed to $Y(f, x)$ for any $y \in \overline{Gx}$ and π_f is continuous on $Y(f, x)$. It is enough to show that $\pi_f(y_n)$ converges to $\pi_f(y)$ whenever (y_n) is a sequence in \overline{Gx} converging to $y \in \overline{Gx}$ such that $\pi_f(y_n)$ belong to $Y(f, x)$. Now, it is not hard to see that the functions $\pi_f(y_n)$ converge pointwise to $\pi_f(y)$. Moreover, by (i), the set $Y(f, x)$ is compact and, hence, $\pi_f(y_n)$ has a uniform convergent subsequence. Now, any such subsequence must converge to $\pi_f(y)$ (as uniform convergence implies pointwise convergence). This gives the desired convergence statement.

(ii) \implies (iii): By (ii) the map \bar{d}_f with $\bar{d}_f(y, z) := \|\pi_f(y) - \pi_f(z)\|_\infty$ is a continuous pseudometric on \overline{Gx} . It is clearly invariant. Now, choose a countable dense subset $D \subset C(X)$ separating the points of \overline{Gx} . Assume without loss of generality that any element of D is normalized, and choose for any $f \in D$ a $c_f > 0$ with $\sum_{f \in D} c_f < \infty$. Then, $\sum_{f \in D} c_f \bar{d}_f$ is a continuous invariant metric on \overline{Gx} . As \overline{Gx} is compact any continuous metric determines its topology.

(iii) \implies (iv): Let \bar{d}' be a continuous invariant metric on \overline{Gx} . Let $\varepsilon > 0$ be arbitrary. As \bar{d}' is a continuous metric on \overline{Gx} there exists a $\delta > 0$ with $d(y_1, y_2) \leq \varepsilon$ for $y_1, y_2 \in \overline{Gx}$ whenever $\bar{d}'(y_1, y_2) \leq \delta$. As \bar{d}' is invariant, we obtain then $d(ty_1, ty_2) \leq \varepsilon$ for all $t \in G$ and, hence, $\bar{d}(y_1, y_2) \leq \varepsilon$ whenever $\bar{d}'(y_1, y_2) \leq \delta$ for $y_1, y_2 \in \overline{Gx}$. This is the desired statement.

(iv) \implies (v): Using the invariant metric \bar{d} it is not hard to see that there is a group structure on \overline{Gx} with $tx + sx = (t + s)x$. Here, we show only that this is well-defined. The remaining statements then follow easily. Assume $tx = t'x$ and $sx = s'x$. Then triangle inequality and invariance of the metric gives

$$\bar{d}((t+s)x, (t'+s')x) \leq \bar{d}((t+s)x, (t+s')x) + \bar{d}((t+s')x, (t'+s')x) \leq \bar{d}(sx, s'x) + \bar{d}(tx, t'x) = 0.$$

This shows well-definedness.

(v) \implies (i): This is standard. We include some details for convenience of the reader. Let f be a continuous function on X . We have to show that the set $S := \{f_x(t + \cdot) : t \in G\} =$

$\{f_{tx} : t \in G\}$ has compact closure in $C_u(G)$ with respect to the supremum norm. From (iv) we easily see that $\pi_f : \overline{Gx} \rightarrow C_u(G), y \mapsto f_y$, is continuous. Hence, $\pi_f(\overline{Gx})$ is compact and, as it clearly contains S , the desired statement follows.

The minimality statement follows directly from (iii). □

The preceding result shows that Bohr almost periodic points give rise to minimal orbit closures. In fact, within the weakly almost periodic points one can even characterize the Bohr almost periodic points by minimality of their orbit closures:

Proposition 5.9. *Let (X, G) be a dynamical system and $x \in X$ be weakly almost periodic. Then, x is Bohr almost periodic if and only if its orbit closure \overline{Gx} is minimal.*

Proof. We have just seen in Theorem 5.8 that the orbit closure of a Bohr almost periodic point is minimal. So, consider now a weakly almost periodic point $x \in X$ with minimal orbit closure \overline{Gx} . Then, clearly (\overline{Gx}, G) is weakly almost periodic by (b) of Lemma 5.4 and it is minimal (by assumption). Now, as is well known (see e.g. Prop. 3.7 of [30]) any minimal component of a weakly almost periodic system is Bohr almost periodic. Now, the desired claim follows from (a) of Lemma 5.4 □

Remark. We note that a Bohr almost periodic dynamical system does not need to be minimal as can easily be seen by considering the ‘disjoint union’ of two Bohr almost periodic systems.

It is possible to characterize Bohr almost periodic points by almost periodicity properties of the metric \bar{d} .

Theorem 5.10. *Let (X, G) be a dynamical system with metric d . Then the following assertions are equivalent for a point $p \in X$:*

- (i) *The point p is Bohr almost periodic.*
- (iii) *The function $G \mapsto [0, \infty), t \mapsto \bar{d}(p, tp)$, is Bohr almost periodic.*

Proof. We show that (i) implies (ii): By (i) and Theorem 5.8, the orbit closure of p is minimal and the function \bar{d} is a continuous metric on \overline{Gp} . Let $\varepsilon > 0$ be given. As \bar{d} is continuous and every Bohr almost periodic point has a minimal orbit, there exists a relatively dense set $R \subset G$ with $\bar{d}(sp, p) < \varepsilon$ for all $s \in R$. As \bar{d} is invariant, this gives

$$|\bar{d}(p, (t + s)p) - \bar{d}(p, tp)| \leq \bar{d}((t + s)p, tp) \leq \bar{d}(sp, p) < \varepsilon$$

for all $t \in G$ and $s \in R$. As $\varepsilon > 0$ was arbitrary this gives (ii).

We now show that (ii) implies that p is Bohr almost periodic: It is not hard to see that (ii) implies that $t \mapsto \bar{d}(sp, (t + s)p)$ is Bohr almost periodic for any $s \in G$. This easily implies that (iv) of Lemma 5.2 holds and (i) follows from that lemma. □

6. APPLICATION TO MEASURE DYNAMICAL SYSTEMS

In this section, we use our results to shed a light on recent investigations of aperiodic order. A fundamental issue in the study of aperiodic order is pure point diffraction, see e.g. [3] for a recent survey. Indeed, understanding of pure point diffraction has been a driving force in the field, see e.g. the survey article [22]. Recently, a complete understanding of pure point diffraction via mean almost periodicity has been provided in [28]. That article mostly deals with single measures. However, it also includes results on pure point spectrum of certain dynamical systems viz measure dynamical systems. Here, we discuss how our results allow one to provide a different approach to these results.

We start with a discussion of translation bounded measure dynamical systems. Such dynamical systems were brought forward in [4] to provide a systematic framework to study aperiodic order. In our exposition we follow [4] to which we refer for further details, proofs and references.

We denote by $C_c(G)$ the vector space of continuous complex valued functions on G with compact support. This space is equipped with the inductive limit topology of the injections

$$C_K(G) \longrightarrow C_c(G), \quad \varphi \mapsto \varphi,$$

for $K \subset G$ compact. Here, $C_K(G)$ denotes the subspace of $C_c(G)$ consisting of functions with support in K . The measures on G are the elements of the dual space of $C_c(G)$. The total variation $|\mu|$ of a measure μ is the smallest positive measure with

$$|\mu(\varphi)| \leq |\mu|(\varphi)$$

for all $\varphi \in C_c(G)$ with $\varphi \geq 0$. A measure μ on G is called *translation bounded* if its total variation $|\mu|$ satisfies

$$\|\mu\|_K := \sup |\mu|(t + U) < \infty$$

for one (all) relatively compact open U in G . We denote that set of all translation bounded measures by $M^\infty(G)$ and equip it with the vague topology. Then, G admits a natural action on $M^\infty(G)$ by translations. More specifically, for $t \in G$ and $\mu \in M^\infty(G)$ the measure $t\mu$ is defined by

$$t\mu(\varphi) = \mu(\varphi(\cdot + t))$$

for all $\varphi \in C_c(G)$.

A subset $\Omega \subset M^\infty(G)$ which is invariant under the translation action is compact if and only if it is vaguely closed and there exists a constant C such that [34, Thm. A.8]

$$\|\mu\|_U \leq C \quad \text{for all } \mu \in \Omega.$$

Whenever Ω is a compact subset of $M^\infty(G)$, which is invariant under the translation action to and m is an invariant probability measure on X , we call (X, G, m) a *dynamical system of translation bounded measures* or just TMDS for short. If G is second countable than any TMDS is metrizable. Hence, the theory developed above applies to TMDS whenever G is second countable. Consider now an arbitrary TMDS (Ω, G, m) and define for any $\varphi \in C_c(G)$ the function

$$N_\varphi : \Omega \longrightarrow \mathbb{C}, \quad N_\varphi(\omega) = \omega(\varphi).$$

Then, N_φ belongs to $C(\Omega)$. Also, there exists a unique translation bounded measure $\gamma = \gamma^m$ on (X, G, m) with

$$\gamma(\varphi * \tilde{\psi}) = \langle N_\varphi, N_\psi \rangle$$

for all $\varphi, \psi \in C_c(G)$ and all $t \in G$.² The measure γ is called the *autocorrelation* of the TMDS. This measure allows for a Fourier transform $\hat{\gamma}$ which is a (positive) measure on \hat{G} . It is known as *diffraction* of the TMDS. Of particular interest in this theory are now those TMDS whose diffraction is a pure point measure. By a main result of [4] (see references there as well for earlier results) the diffraction of a TMDS is pure point if and only if the TMDS has pure point spectrum. So, for this reason, TMDS with pure point spectrum are of utmost relevance in the

²Note that [4] uses a different sign in the definition of \mathcal{N} (called f there) as well as has the inner product linear in the second argument. This results in a different display of the formula for γ , viz. $(\gamma * \tilde{\varphi} * \psi)(0) = \langle f_\varphi, f_\psi \rangle$.

field of aperiodic order. One particular question is the calculation of the atoms of $\widehat{\gamma}$. Here, the basic idea is that

$$\widehat{\gamma}(\{\xi\}) = \lim_{n \rightarrow \infty} \left| \frac{1}{|B_n|} \int_{B_n} \xi(t) \, d\omega(t) \right|^2$$

(with (B_n) being a Følner sequence). Validity of this formula is often discussed under the heading of Bombieri–Taylor conjecture.

Having provided the framework of TMDS, we now discuss how the theory developed in the previous section can be used in the study of aperiodic order.

A translation bounded measure ω is called *mean almost periodic* (*Besicovitch almost periodic*, *Weyl almost periodic* respectively) if for any $\varphi \in C_c(G)$ the function

$$\omega * \varphi : G \longrightarrow \mathbb{C}, \quad (\omega * \varphi)(t) = \int \varphi(t - s) \, d\omega(s),$$

is mean almost periodic (Besicovitch almost periodic, Weyl almost periodic respectively).

Proposition 6.1. *Let (Ω, G, m) be a TMDS and (B_n) a Følner sequence. Then, for $\omega \in \Omega$ the following assertions are equivalent:*

- (i) *The measure ω is mean almost periodic (Besicovitch almost periodic, Weyl almost periodic, Weak almost periodic, Bohr almost periodic).*
- (ii) *The function $t \mapsto N_\varphi(t\omega)$ is mean almost periodic (Besicovitch almost periodic, Weyl almost periodic, Weak almost periodic, Bohr almost periodic) for any $\varphi \in C_c(G)$.*
- (iii) *The point $\omega \in \Omega$ is mean almost periodic (Besicovitch almost periodic, Weyl almost periodic, Weak almost periodic, Bohr almost periodic).*

Proof. We only discuss mean almost periodicity. The remaining statements follow analogously.

(i) \iff (ii): A short computation shows

$$N_\varphi(t\omega) = (\omega * \widetilde{\varphi})(t),$$

where $\widetilde{\varphi} : G \longrightarrow \mathbb{C}$, $t \mapsto \overline{\varphi(-t)}$. This gives that ω is mean almost periodic if and only if $t \mapsto N_\varphi(t\omega)$, $\varphi \in C_c(G)$, is mean almost periodic and the equivalence between (i) and (ii) follows.

(iii) \implies (ii): This follows easily as any N_φ , $\varphi \in C_c(G)$, is a continuous function on Ω .

(ii) \implies (iii): It is not hard to see that the set of N_φ , $\varphi \in C_c(G)$, separates the points of Ω and is closed under complex conjugation. Hence, the algebra generated by the N_φ is dense in the continuous functions on Ω with respect to the supremum norm (see e.g. [3] for further discussion of this type of argument). This gives that (ii) implies (iii). \square

When dealing with TMDS (Ω, G, m) we can now use the previous proposition to replace the assumption that $\omega \in \Omega$ is mean almost periodic as element of the dynamical system (Ω, G, m) by the assumption that ω is a mean almost periodic measure (and, similarly, with mean almost periodic replaced by Besicovitch almost periodic, Weyl almost periodic, weak almost periodic, Bohr almost periodic). This allows then for reformulations of our main results. We only state one of the applications and leave the remaining ones to the reader.

Theorem 6.2. *Let (Ω, G, m) be an ergodic TMDS. Assume that G is second countable and that (B_n) is a Følner sequence along which Birkhoff’s ergodic theorem holds. Then, the following assertions are equivalent:*

- (i) *(Ω, G, m) has pure point spectrum.*
- (ii) *Almost every $\omega \in \Omega$ is a mean almost periodic measure.*

Remark. The previous theorem was first shown in [28]. Here, we have provided a different proof.

7. ABSTRACT GENERALISATIONS

When closing the article, it may be instructive to stop a moment to have a look at the overall theme of this article from a more abstract point of view. The general approach in this article may be described as follows: We consider a dynamical system (X, G) and say that a point $x \in X$ is $(*)$ almost periodic if \mathcal{A}_x consists of $(*)$ almost periodic functions only. Then, the preceding sections have been devoted to a thorough study of consequences of existence of $(*)$ almost periodic points for $(*)$ being replaced by Bohr, weak, Weyl, Besicovitch and Mean (and in this order these are increasingly weaker notions of almost periodicity). Now, of course, any other concept of almost periodicity for a function could also be taken as the starting point of the theory. Then, some of our considerations will easily carry over. This is discussed in this section.

Let $C_u(G)$ be the set of uniformly continuous bounded functions on G and consider a dynamical system (X, G) . Whenever we are given an $\mathbb{A} \subset C_u(G)$ we can define for $p \in X$

$$\mathbb{A}_p := \{f \in C(X) : f_p \in \mathbb{A}\}$$

Then, \mathbb{A}_p inherits various properties of \mathbb{A} . In particular, if \mathbb{A} is an algebra, then so is \mathbb{A}_p and if $1 \in \mathbb{A}$ then \mathbb{A}_p contains the constant function 1. Moreover, if \mathbb{A} is closed in $(C_u(G), \|\cdot\|_\infty)$ then \mathbb{A}_p is closed in $(C(X), \|\cdot\|_\infty)$. Then, as abstraction of Lemma 2.7 (with the same proof), we obtain the following lemma.

Lemma 7.1. *Let \mathbb{A} be a closed subalgebra of $C_u(G)$ containing the constant function 1. Then, the following assertions are equivalent for $p \in X$:*

- (i) $\mathbb{A}_p \subseteq \mathbb{A}$.
- (ii) $\mathbb{A}_p = C(X)$.
- (iii) \mathbb{A}_p separates the points of \overline{Gp} .
- (iv) For each $s \in G$ the function $d^{(s)} \in C(X)$ with $d^{(s)}(y) := d(sp, y)$ belongs to \mathbb{A}_p .

A particular way to obtain a closed algebra \mathbb{A} is by suitable seminorms. This is discussed next: Call a seminorm N on $C_u(G)$ *admissible* if it is G -invariant and satisfies

- $N(f) \leq N(g)$ whenever $|f| \leq g$
- $N(1) = 1$

Note that any admissible seminorm N satisfies $N \leq \|\cdot\|_\infty$ (as $|f| \leq \|f\|_\infty \cdot 1$).

Remark (Examples). It is not hard to see that $\|\cdot\|_\infty$ and $\overline{M} \circ |\cdot|$ as well as

$$N(f) := \limsup_{n \rightarrow \infty} \sup_{t \in G} \frac{1}{|B_n|} \int_{t+B_n} |f(t)| dt$$

are admissible seminorms on $C_u(G)$.

Definition 7.2. (a) We say that an $f \in C_u(G)$ is N -almost periodic if for any $\varepsilon > 0$ the set

$$\{t \in G : N(|f - f(\cdot - t)|) < \varepsilon\}$$

is relatively dense in G .

(b) We say that an $f \in C_u(G)$ is N -trig almost periodic if for any $\varepsilon > 0$, there exists some trigonometric polynomial P such that $N(|f - P|) < \varepsilon$.

Lemma 7.3. *Let N be an admissible seminorm. Then,*

$$\mathbb{A}^N := \{f \in C_u(G) : f \text{ is } N\text{-almost periodic}\}$$

and

$$\mathbb{A}^T := \{f \in C_u(G) : f \text{ is } N\text{-trig almost periodic}\}$$

are closed subalgebras of $C_u(G)$. Both subalgebras contain the Bohr almost periodic functions and $\mathbb{A}^T \subseteq \mathbb{A}^N$ holds.

In particular $1 \in \mathbb{A}^N$.

Proof. The statement on \mathbb{A}^N is an abstraction of Theorem B.3 in our context. It can be shown by replacing $M \circ |\cdot|$ with N in the proof of this theorem. Similarly, the statement on \mathbb{A}^T is an abstraction of Lemma C.2 in our context and can be shown by mimicking the proof of that lemma. This also gives the last statement. \square

Definition 7.4 (N -almost periodic points). *Let (X, G) be a dynamical system and N an admissible seminorm on $C_u(G)$. We say, that an $x \in X$ is N -almost periodic if f_x is N -almost periodic for any $f \in C(X)$, i.e. if $\mathbb{A}_x^N = C(X)$ holds.*

For a continuous metric d on X and $x \in X$ we define the

$$d^{N,x} : G \longrightarrow [0, \infty), \quad d^{N,x}(t) := N((s \mapsto d(sx, (t+s)x)).$$

Reasoning as in Section 2, we can infer that for any continuous metric d the function $d^{N,x}$ is uniformly continuous. Having set up things, we can now discuss the following abstraction of the results in Section 2 (where we include some details for the convenience of the reader).

Analogously to Lemma 2.2 we find the following.

Lemma 7.5. *Let (X, G) be a dynamical system. Let N be admissible. Let d be a continuous metric on X . Then, the following assertions for $x \in X$ are equivalent:*

- (i) *For any $\varepsilon > 0$ the set*

$$\{t \in G : d^{N,x}(t) < \varepsilon\}$$

is relatively dense.

- (ii) *The function $d^{N,x}$ is Bohr almost periodic.*

The following is an abstraction of both the equivalence between (i) and (ii) in Lemma 2.7 (with $N = \overline{M} \circ |\cdot|$) and Theorem 5.10 (with $N = \|\cdot\|_\infty$).

Theorem 7.6. *Let (X, G) be a dynamical system. Let N be admissible. Then, for $x \in X$ the following assertions are equivalent:*

- (i) *The point x is N -almost periodic.*
- (ii) *There exists a continuous metric d on X such that $d^{N,x}$ is Bohr almost periodic.*
- (iii) *For every continuous metric d on X the function $d^{N,x}$ is Bohr almost periodic.*

Remark. As X is a compact metric space, a metric on X is continuous if and only if it generates the topology.

Proof. (iii) \implies (ii): This is clear.

(ii) \implies (iii): This is the analogue of Lemma 2.4 in our context. It can be shown by a variant of the proof of that lemma. Some extra effort is needed as N is not defined on measurable bounded functions but only on $C_u(G)$. This is tackled by means of Urysohn lemma. As every locally compact group is a normal space this lemma allows one to separate arbitrary closed disjoint sets by continuous functions and this is what we will use. Here are the details:

Let e be any metric on X such that $e^{N,x}$ is almost periodic and let d be any other metric on X , such that d, e generate the topology. Without loss of generality we can assume that $d, e \leq 1$. Let $\varepsilon > 0$ be arbitrary. By Lemma 7.5 it suffices to show that the set of $t \in G$ with $d^{N,x}(t) \leq \varepsilon$ is relatively dense.

Choose $\delta' > 0$ with $d(y, z) < \frac{\varepsilon}{2}$ whenever $e(y, z) < \delta'$. Set

$$\delta := \frac{\varepsilon}{4\delta'}.$$

Let $t \in G$ be so that $e^{N,x}(t) < \delta$. By Lemma 7.5 the set of such $t \in G$ is relatively dense. Thus, it remains to show $d^{N,x}(t) < \varepsilon$ for any such $t \in G$. Define $e_{t,x}$ and $d_{t,x}$ on G by $e_{t,x}(s) := e(sx, (t+s)x)$ and $d_{t,x}(s) := d(sx, (t+s)x)$. Set

$$A := \{s : e_{t,x} \geq \delta'\}; \quad B := \{s : e_{t,x} \leq \frac{\delta'}{2}\}.$$

Then, by Urysohn's Lemma, there exists some continuous function $f : G \rightarrow [0, 1]$ such that $f(x) = 1$ for all $x \in A$ and $f(x) = 0$ for all $x \in B$. Set $g := 1 - f$. Then, $e_{t,x} \geq \frac{\delta'}{2}f$ and hence, $\frac{\delta'}{2}N(f) \leq N(e_{t,x}) \leq \delta$, showing

$$N(f) \leq \frac{2\delta}{\delta'} = \frac{\varepsilon}{2}.$$

Moreover, $d_{t,x}(s)g(s) \leq \frac{\varepsilon}{2}$. Indeed, if $s \in A$ then $g(s) = 0$ by definition of g and if $s \notin A$ then $d_{t,x}(s) \leq \frac{\varepsilon}{2}$ by our choice of δ' . This shows

$$N(d_{t,x}) \leq \frac{\varepsilon}{2}.$$

Therefore, using $d \leq 1$ we obtain

$$d^{N,x}(t) = N(d_{t,x}) \leq N(d_{t,x}f) + N(d_{t,x}g) \leq N(f) + N(d_{t,x}g) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(i) \implies (ii): This can be shown as (ii) \implies (i) in Lemma 2.7.

(ii) \implies (i): As in the proof of (i) \implies (iv) of Lemma 2.7 we infer from (ii) that d_x^z is N -almost periodic for any $z \in X$. Now, (i) follows from Lemma 7.1. \square

Corollary 7.7. *Let (X, G, m) be a dynamical system with metric d . Let N be an admissible seminorm and $x \in X$ be N -almost periodic. If*

$$\int_X f(y) \, dm(y) \leq N(f_x)$$

holds for all $f \in C(X)$, then (X, G, m) has pure point spectrum.

Remark. Note that the assumption holds whenever there exists a Følner sequence (B_n) along which Birkhoff's ergodic theorem holds and N satisfies $\overline{M} \circ |\cdot| \leq N$ and x is generic with respect to m .

Proof. As x is N -almost periodic the preceding theorem (combined with Lemma 7.5) gives that for any $\varepsilon > 0$ the set of $t \in G$ with $d^{N,x}(t) < \varepsilon$ is relatively dense. By using the assumption on N with $f(y) = d(y, ty)$ (for $t \in G$ fixed), we furthermore find for the function

$$\underline{d} : G \longrightarrow [0, \infty), \quad \underline{d}(t) = \int_X d(y, ty) \, dm(y),$$

the inequality

$$\underline{d}(t) \leq d^{N,x}(t)$$

for all $t \in G$. Hence, for any $\varepsilon > 0$ the set of $t \in G$ with $\underline{d}(t) < \varepsilon$ is relatively dense as well. This gives that \underline{d} is Bohr almost periodic and the desired statement on pure point spectrum follows from the main result of [25] (compare proof of Theorem 2.8 as well). \square

APPENDIX A. BOHR ALMOST PERIODIC FUNCTIONS

In this section, we briefly present some background on Bohr almost periodic function. This is completely standard and can be found in many places. We include a discussion here in order to be self-contained and to set the perspective on the weaker (and less known) notions of almost periodicity underlying our considerations. For more details we refer the reader to [8, 31].

Let G be a locally compact abelian group. Recall from Section 1 that a continuous function $f : G \rightarrow \mathbb{C}$ is called *Bohr almost periodic* if for any $\varepsilon > 0$ the set of $t \in G$ with

$$\|f - f(\cdot - t)\|_\infty < \varepsilon \tag{\spadesuit}$$

is relatively dense. Any such $t \in G$ is then called an ε -almost *period* of f . It turns out that a continuous $f : G \rightarrow \mathbb{C}$ is Bohr almost periodic if and only if $\overline{\{f(\cdot - t) : t \in G\}}$ is compact, where the closure is taken with respect to the supremum norm. It is not hard to see that any Bohr almost periodic function is bounded and uniformly continuous. Moreover, the Bohr almost periodic functions form a closed subalgebra of the algebra of all uniformly continuous bounded functions on G . The main structural result on Bohr almost periodic functions is that a function f is Bohr almost periodic if and only if for any $\varepsilon > 0$ there exist $k \in \mathbb{N}$, $\xi_1, \dots, \xi_k \in \widehat{G}$ and $c_1, \dots, c_k \in \mathbb{C}$ with

$$\|f - \sum_{j=1}^k c_j \xi_j\|_\infty < \varepsilon. \tag{\heartsuit}$$

The basic idea of the weaker concepts of almost periodicity discussed in the subsequent sections is to replace the supremum norm $\|\cdot\|_\infty$ in (\spadesuit) and (\heartsuit) by suitable (semi)norms arising by averaging procedures.

APPENDIX B. MEAN ALMOST PERIODIC FUNCTIONS

The main result of this appendix shows that the bounded uniformly continuous mean almost periodic functions form a closed subalgebra of the algebra of bounded uniformly continuous functions on G . This is certainly well-known and a proof can be given by standard means. For the convenience of the reader and in order to keep this article self contained we include a discussion below. As our article deals with abelian groups we assume that the group G below is abelian. Note, however, that this is not used in the proofs.

We start with a general result on discrete geometry of groups.

Proposition B.1. *Let G be a locally compact abelian group. Let D and E be relatively dense subsets of G and V a relatively compact open neighborhood of the neutral element. Then, $((D - D) + V) \cap ((E - E) + V)$ is relatively dense.*

Proof. As both D and E are relatively dense, we can choose an open relatively compact set $U \subset G$ with the property that any translate of U intersects both D and E . As addition is continuous on G , we can choose furthermore a relatively compact open neighborhood W with

$W = -W$ and $W + W \subset V$. As U is relatively compact, there exist $N \in \mathbb{N}$ and $z_1, \dots, z_N \in G$ with

$$U = \bigcup_{n=1}^N (z_n + W) \cap U.$$

Consider now an arbitrary $t \in D$. Then, there exists an $s \in E$ with $t - s \in U$. Hence, for any $t \in D$ we can find $s_t \in E$ and $n_t \in \{1, \dots, N\}$ with

$$(t - s_t) \in z_{n_t} + W.$$

Fix now for each $n \in \{1, \dots, N\}$ elements $t_n \in D$ and $s_n \in E$ with

$$(t_n - s_n) \in z_n + W.$$

(If some of the n does not admit elements, we just remove this n from our list.) Then, for any $t \in D$ we can find $s_t \in E$ and $n \in \{1, \dots, N\}$ such that both $t - s_t$ and $t_n - s_n$ belong to $z_n + W$. Hence, we find

$$t - s_t + w = t_n - s_n + w'$$

for suitable $w, w' \in W$. This gives

$$t - t_n = s_t - s_n + v$$

with $v = w' - w \in W - W \subset V$. Now, we clearly have $t - t_n \in (D - D) + V$ and $(s_t - s_n) + v \in (E - E) + V$. Moreover, the set of all $t - t_n$ is relatively dense as t is an arbitrary element of the relatively dense D and there are only finitely many t_n . This finishes the proof. \square

Let G be a σ -compact locally compact abelian group and (B_n) a Følner sequence on G . Let f be a uniformly continuous bounded function on G . Let $\varepsilon > 0$ be given. As usual we say that a $t \in G$ is an ε -almost period of f if

$$\overline{M}(|f - f(\cdot - t)|) < \varepsilon.$$

Denote the set of all ε -almost periods of f by $\text{AP}(f, \varepsilon)$. Then, it is not hard to see that

$$\text{AP}(f, \varepsilon) - \text{AP}(f, \varepsilon) \subset \text{AP}(f, 2\varepsilon).$$

A uniformly continuous bounded $f : G \rightarrow \mathbb{C}$ is mean almost periodic if for any $\varepsilon > 0$ the set $\text{AP}(f, \varepsilon)$ is relatively dense.

Lemma B.2. *Let G be a σ -compact locally compact abelian group and (B_n) a Følner sequence on G . Let a natural number n and uniformly continuous bounded mean almost periodic functions f_1, \dots, f_n on G be given. Then, the set*

$$\bigcap_{k=1}^n \text{AP}(f_k, \varepsilon)$$

is relatively dense in G for any $\varepsilon > 0$.

Proof. This will be shown by induction in n . The case $n = 1$ is clear. So, assume now the statement holds for a chosen n . Let $\varepsilon > 0$ and uniformly continuous functions f_1, \dots, f_{n+1} be given. Then, the set $D := \bigcap_{k=1}^n \text{AP}(f_k, \varepsilon/3)$ is relatively dense by assumption. As the functions f_k , $k = 1, \dots, n+1$, are uniformly continuous we can find an open relatively compact neighborhood V of the neutral element such that

$$\|f_k - f_k(\cdot - s)\|_\infty < \frac{\varepsilon}{3}$$

for all $s \in V$ and $k = 1, \dots, n + 1$. Set $E := \text{AP}(f_{n+1}, \varepsilon/3)$. Then, the previous proposition gives that

$$((D - D) + V) \cap ((E - E) + V)$$

is relatively dense. On the other hand it is not hard to see that

$$(D - D) + V \subset \text{AP}(f_k, \varepsilon), k = 1, \dots, n \text{ and } (E - E) + V \subset \text{AP}(f_{n+1}, \varepsilon).$$

This finishes the proof. □

Theorem B.3. *Let G be a σ -compact locally compact abelian group and (B_n) a Følner sequence on G . Then, the set of all uniformly continuous bounded mean almost periodic functions is invariant under taking complex conjugates and a closed subalgebra of the uniformly continuous bounded functions on G equipped with $\|\cdot\|_\infty$.*

Proof. We have to show that the set in question is closed under complex conjugation, sums, products, multiplication by scalars and uniform convergence.

It is not hard to see that the set in question is closed under complex conjugation, uniform convergence and multiplication by scalars.

We next show that it is closed under sums: Let f, g be mean almost periodic uniformly continuous bounded functions. Then, the previous lemma easily gives that $f + g$ is also mean almost periodic.

Finally, we deal with products: Let f, g be mean almost periodic uniformly continuous bounded functions. Then, a short computation gives for any $t \in G$

$$\begin{aligned} |f(s)g(s) - f(s-t)g(s-t)| &\leq |f(s)g(s) - f(s)g(s-t)| + |f(s)g(s-t) - f(s-t)g(s-t)| \\ &\leq \|f\|_\infty |g(s) - g(s-t)| + \|g\|_\infty |f(s) - f(s-t)| \end{aligned}$$

From this we easily obtain

$$\overline{M}(|fg - (fg)(\cdot - t)|) \leq \|f\|_\infty \overline{M}(|g - g(\cdot - t)|) + \|g\|_\infty \overline{M}(|f - f(\cdot - t)|).$$

Now, the desired statement follows easily from the preceding lemma.

The last statement is clear. □

For later use we also note the following proposition.

Proposition B.4. *Let (f_n) be a sequence of uniformly continuous bounded mean almost periodic functions on G with $\|f_n\| \leq 1$ for all n . Let $c_n > 0$ with $\sum_{n=1}^\infty c_n < \infty$ be given. Then, there exists for any $\varepsilon > 0$ a relatively dense set D in G with*

$$\overline{M}\left(\sum_{n=1}^\infty c_n |f - f(\cdot - t)|\right) < \varepsilon$$

for all $t \in D$.

Proof. Choose n_0 large enough so that $\sum_{k=n_0+1}^\infty c_k < \varepsilon/4$. Then, $\sum_{n=n_0+1}^\infty c_n \|f - f(\cdot - t)\|_\infty < \varepsilon/2$ by assumption on the f_n and the c_n . By Lemma B.2 there exists a relatively dense set D in G with $\overline{M}(|f_k - f_k(\cdot - t)|) < \frac{\varepsilon}{2n_0}$ for $k = 1, \dots, n_0$. Now, the statement follows easily. □

Remark. The considerations of this section carry over when $\overline{M} \circ |\cdot|$ is replaced by any invariant seminorm N on the algebra of bounded uniformly continuous functions on G satisfying

- $N(f) \leq N(g)$ whenever $|f| \leq g$,
- $N(1) = 1$.

This point is taken up in Section 7.

APPENDIX C. BESICOVITCH ALMOST PERIODIC FUNCTIONS AND EXISTENCE OF MEANS

We consider a σ -compact locally compact abelian group G together with a Følner sequence (B_n) . Our aim is to study the set of uniformly continuous bounded functions $f : G \rightarrow \mathbb{C}$, which are Besicovitch almost periodic i.e. satisfy that for any $\varepsilon > 0$ there exist $k \in \mathbb{N}$, $\xi_1, \dots, \xi_k \in \widehat{G}$ and $c_1, \dots, c_k \in \mathbb{C}$ with

$$\overline{M}(|f - \sum_{j=1}^k c_{\xi_j} \xi_j|) < \varepsilon.$$

For further details and a more in depth study of these functions we refer to [28].

Proposition C.1. *Let G be a σ -compact locally compact abelian group and (B_n) a Følner sequence on G . Then, any uniformly continuous bounded Besicovitch almost periodic function is mean almost periodic.*

Proof. Let $\varepsilon > 0$ be given. Choose $c_1, \dots, c_k \in \mathbb{C}$ and $\xi_1, \dots, \xi_k \in \widehat{G}$ such that $P := \sum_{j=1}^k c_j \xi_j$ satisfies

$$\overline{M}(|f - P|) < \varepsilon.$$

As \overline{M} is invariant this inequality will then continue to hold if f is replaced by $f(\cdot - t)$ and P is replaced by $P(\cdot - t)$ for any $t \in G$. Now, clearly P is Bohr almost periodic. Hence, there exists a relatively dense set $R \subset G$ with $\|P - P(\cdot - t)\|_\infty < \varepsilon$ for all $t \in R$. This easily gives

$$\overline{M}(|f - f(\cdot - t)|) \leq \overline{M}(|f - P|) + \overline{M}(|P - P(\cdot - t)|) + \overline{M}(|P(\cdot - t) - f(\cdot - t)|) < 3\varepsilon$$

for all $t \in R$. □

From the definition and simple algebraic manipulations we infer the following.

Lemma C.2. *Let G be a σ -compact locally compact abelian group and (B_n) a Følner sequence on G . Then, the set of all uniformly continuous bounded Besicovitch almost periodic functions is invariant under taking complex conjugates and a closed subalgebra of the uniformly continuous bounded functions on G equipped with $\|\cdot\|_\infty$. It contains all Bohr almost periodic functions and is contained in the algebra of mean almost periodic functions.*

Proof. The set is clearly closed under taking limits with respect to $\|\cdot\|_\infty$ as well as under addition and taking complex conjugates. To show that it is closed under multiplication consider f, g in this set and let $\varepsilon > 0$ be arbitrary. Let P, Q be trigonometric polynomials so that

$$\overline{M}(|f - P|) < \frac{\varepsilon}{2\|g\|_\infty + 4} \quad \text{and} \quad \overline{M}(|g - Q|) < \frac{\varepsilon}{4\|f\|_\infty + 1}.$$

Define

$$Q'(x) := \begin{cases} Q(x) & \text{if } |Q(x)| \leq \|g\|_\infty + 1 \\ \frac{Q(x)}{|Q(x)|} (\|g\|_\infty + 1) & \text{otherwise} \end{cases}.$$

Then, Q' is a Bohr almost periodic function and $|g - Q'| \leq |g - Q|$, which gives $\overline{M}(|g - Q'|) \leq \overline{M}(|g - Q|)$.

Since Q' is Bohr almost periodic, there exists a trigonometric polynomial R such that $\|Q' - R\|_\infty < \min\{\frac{\varepsilon}{4\|f\|_\infty + 1}, 1\}$. In particular,

$$\|R\|_\infty \leq \|Q'\|_\infty + 1 \leq \|g\|_\infty + 2.$$

Then,

$$\begin{aligned} \overline{M}(|fg - PR|) &\leq \overline{M}(|fg - fR|) + \overline{M}(|fR - PR|) \\ &\leq \|f\|_\infty \overline{M}(|g - R|) + \|R\|_\infty \overline{M}(|f - P|) \\ &\leq \|f\|_\infty (\overline{M}(|g - Q'|) + \overline{M}(|Q' - R|)) + \frac{\varepsilon}{2} \\ &\leq \|f\|_\infty (\overline{M}(|g - Q'|) + \|Q' - R\|_\infty) + \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned}$$

This shows that the set in question is closed under multiplication. It clearly contains all Bohr almost periodic functions and is contained in the set of mean almost periodic functions. \square

Remark. The functions referred to as Besicovitch almost periodic were introduced by Besicovitch in [7] for $G = \mathbb{R}$. A corresponding class of functions was then studied by Følner for general locally compact abelian groups [13]. This class, however, does not coincide with the Besicovitch class for $G = \mathbb{R}$. Another approach to Besicovitch almost periodic functions is developed by Davis in [9]. An account of these developments with a focus on aperiodic order is given in Lagarias survey [22]. Here, we have taken a ‘shortcut’: We have not defined Besicovitch almost periodic functions by some intrinsic features. Instead we have defined them by what would be a main result in a proper theory starting with an intrinsic definition.

A crucial feature of Besicovitch almost periodic function is existence of means in the following sense.

Lemma C.3. *Let G be a σ -compact locally compact abelian group and (B_n) a Følner sequence on G . Let $f : G \rightarrow \mathbb{C}$ be a uniformly continuous, bounded Besicovitch almost periodic function. Then, the limit*

$$A(f) = \lim_{n \rightarrow \infty} \frac{1}{|B_n|} \int_{B_n} f(s) ds$$

exists.

Proof. The statement is well-known for functions of the form $f = \sum_{j=1}^k c_j \xi_j$ with $\xi_1, \dots, \xi_k \in \widehat{G}$, $c_1, \dots, c_k \in \mathbb{C}$. It then follows by a limiting procedure for Besicovitch almost periodic functions. \square

In fact, existence of means together with a Parseval type equality is a characterizing feature of Besicovitch almost periodic functions.

Proposition C.4. *Let G be a σ -compact locally compact abelian group and (B_n) a Følner sequence on G . Let $f : G \rightarrow \mathbb{C}$ be a uniformly continuous and bounded. Then, f is Besicovitch almost periodic if and only if there exists a countable set $F \subset \widehat{G}$ such that the following three statements hold:*

- *The limit $A(|f|^2) = \lim_{n \rightarrow \infty} \frac{1}{|B_n|} \int_{B_n} |f(t)|^2 dt$ exists.*
- *For any $\xi \in F$ the limit*

$$A(f\bar{\xi}) = \lim_{n \rightarrow \infty} \frac{1}{|B_n|} \int_{B_n} f(t) \overline{\xi(t)} dt$$

exists.

- *The equality*

$$A(|f|^2) = \sum_{\xi \in F} |A(f\bar{\xi})|^2$$

holds.

In this case $A(f\bar{\xi}) = 0$ holds for all $\xi \in \widehat{G} \setminus F$.

Proof. Let f be Besicovitch almost periodic. Clearly, any $\xi \in \widehat{G}$ is Bohr almost periodic and hence Besicovitch almost periodic. Thus, $f\bar{\xi}$ is Besicovitch almost periodic as product of Besicovitch almost periodic functions. Hence, the second point holds (even for all $\xi \in \widehat{G}$). Similarly, $|f|^2$ is Besicovitch almost periodic. Given this, the first point follows. The last point is contained in [28].

Now, consider f satisfying the three points above. Let ξ_1, ξ_2, \dots be an enumeration of the $\xi \in F$. A computation involving Cauchy–Schwarz’ inequality in the first step, $\overline{M}(1) = 1$ in the second step and existence of averages A in the third step gives

$$\begin{aligned} & \overline{M} \left(|f - \sum_{j=1}^N A(f\bar{\xi}_j)\xi_j|^2 \right) \\ & \leq \overline{M}(|f - \sum_{j=1}^N A(f\bar{\xi}_j)\xi_j|^2) \overline{M}(1) \\ & = \overline{M} \left(|f|^2 - f \overline{\sum_{j=1}^N A(f\bar{\xi}_j)\xi_j} - \overline{f \sum_{j=1}^N A(f\bar{\xi}_j)\xi_j} + \sum_{j,k=1}^N A(f\bar{\xi}_j) \overline{A(f\bar{\xi}_k)\xi_j\xi_k} \right) \\ & = A(|f|^2) - A(f \overline{\sum_{j=1}^N A(f\bar{\xi}_j)\xi_j}) - A(\overline{f \sum_{j=1}^N A(f\bar{\xi}_j)\xi_j}) + A(\sum_{j=1,k}^N A(f\bar{\xi}_j) \overline{A(f\bar{\xi}_k)\xi_j\xi_k}) \\ & = A(|f|^2) - \sum_{j=1}^N |A(f\bar{\xi}_j)|^2 \\ & \rightarrow 0 \end{aligned}$$

for $N \rightarrow \infty$. Here, the penultimate step is a direct computation invoking that A is linear with $A(\eta) = 0$ for $0 \neq \eta \in \widehat{G}$. Indeed, this shows that f can be approximated in mean by a trigonometric polynomial arbitrarily well. This finishes the proof. \square

Remark. The considerations of this appendix easily carry over to bounded measurable functions (instead of uniformly continuous bounded functions). As the article is concerned with continuous functions we do not elaborate on this but rather leave the details to the reader.

APPENDIX D. WEYL ALMOST PERIODIC FUNCTIONS AND UNIFORM MEANS

In this appendix, we consider a uniform type of mean.

Let (B_n) be a Følner sequence in G and define for bounded $f : G \rightarrow \mathbb{R}$ and $n \in \mathbb{N}$

$$\overline{M}_n(f) := \sup_{r \in G} \frac{1}{|B_n|} \int_{B_n+r} f(t) dt.$$

Proposition D.1. *For a bounded measurable function $f : G \rightarrow \mathbb{R}$ and $\varepsilon > 0$ the following assertions are equivalent:*

- (i) *There exists an $N \in \mathbb{N}$ with $\overline{M}_N(f) < \varepsilon$.*
- (ii) *There exists an $N_0 \in \mathbb{N}$ with $\overline{M}_N(f) < \varepsilon$ for all $N \geq N_0$.*

Proof. (ii) \implies (i): This is clear.

(i) \implies (ii): Consider $n \in \mathbb{N}$. Define for $r \in G$

$$C_r := \frac{1}{|B_N|} \int_{B_N} \left(\int \frac{1}{|B_n|} \int_{B_{n+r}} f(s+t) \, ds \right) dt.$$

Fubini's theorem and the assumption (i) directly give

$$C_r = \frac{1}{|B_n|} \int_{B_{n+r}} \left(\frac{1}{|B_N|} \int_{B_N} f(s+t) \, dt \right) ds \leq \frac{1}{|B_n|} \int_{B_{n+r}} \overline{M}_N(f) \, ds = \overline{M}_N(f).$$

On the other hand we can easily see that (for large n) the additional averaging over B_N does not play a role. More specifically, we can compute as follows:

$$\begin{aligned} \left| C_r - \frac{1}{|B_n|} \int_{B_{n+r}} f(s) \, ds \right| &= \left| C_r - \frac{1}{|B_N|} \int_{B_N} \left(\frac{1}{|B_n|} \int_{B_{n+r}} f(s) \, ds \right) dt \right| \\ &= \left| \frac{1}{|B_N|} \int_{B_N} \left(\frac{1}{|B_n|} \int_{B_{n+r}} (f(s+t) - f(s)) \, ds \right) dt \right| \\ &\leq \frac{1}{|B_N|} \int_{B_N} 2\|f\|_\infty \frac{|(B_n+t) \triangle B_n|}{|B_n|} dt. \end{aligned}$$

Now, due to the Følner condition the integrand in the last term can easily be seen to go pointwise to 0 for $n \rightarrow \infty$. As B_N is compact, we find convergence to zero of the integral for each fixed N . As there is no dependence on $r \in G$ this convergence is independent of $r \in G$. This easily gives the desired statement. \square

A bounded $f : G \rightarrow \mathbb{C}$ is Weyl almost periodic if for all $\varepsilon > 0$ there exist $k \in \mathbb{N}$, $\xi_1, \dots, \xi_k \in \widehat{G}$ and $c_1, \dots, c_k \in \mathbb{C}$ with

$$\limsup_{n \rightarrow \infty} \overline{M}_n(|f - \sum_{j=1}^k c_{\xi_j} \xi_j|) < \varepsilon.$$

By the preceding proposition it is possible to replace this condition by the requirement that for each $\varepsilon > 0$ there exists an $N \in \mathbb{N}$, $k \in \mathbb{N}$, $\xi_1, \dots, \xi_k \in \widehat{G}$ and $c_1, \dots, c_k \in \mathbb{C}$ with

$$\overline{M}_N(|f - \sum_{j=1}^k c_{\xi_j} \xi_j|) < \varepsilon.$$

As usual it is also possible to characterize this by relative denseness of ε -almost periods. More specifically, as discussed in [33] a measurable bounded $f : G \rightarrow \mathbb{C}$ is Weyl almost periodic if and only if for each $\varepsilon > 0$ there exists a relatively dense set $D \subset G$ and an $N_0 \in \mathbb{N}$ with

$$\overline{M}_n(|f - f(\cdot - t)|) < \varepsilon$$

for all $t \in D$ and $n \geq N_0$. By the preceding proposition validity for all $n \geq N_0$ can be replaced by validity for one n .

Clearly, the set of Weyl almost periodic functions forms an algebra, is closed under uniform limits and under multiplication with Bohr almost periodic functions. Moreover, a crucial feature of Weyl mean almost periodic functions is uniform existence of means.

Lemma D.2. *Let G be a locally compact abelian group and (B_n) a Følner sequence on G . Let $f : G \rightarrow \mathbb{C}$ be a bounded Weyl almost periodic function. Then, for any sequence (r_n) in G the limit*

$$\lim_{n \rightarrow \infty} \frac{1}{|B_n|} \int_{B_n + r_n} f(s) \, ds$$

exists and is independent of the sequence. In particular, the convergence is uniform in the chosen sequence.

The *proof* follows the same lines as the proof of the corresponding statement for Besicovitch almost periodic functions in the preceding section. For this reason we leave it to the reader.

REFERENCES

- [1] E. Akin and E. Glasner, *WAP Systems and Labeled Subshifts*, *Memoirs Amer. Math. Soc.* **262** (2019) no. 1265 (AMS, Providence, RI).
- [2] M. Baake and U. Grimm, *Aperiodic Order. Vol. 1: A Mathematical Invitation*, Cambridge Univ. Press, Cambridge (2013).
- [3] M. Baake and D. Lenz, *Spectral notions of aperiodic order*, *Discrete Contin. Dyn. Syst. Ser. S* **10** (2017), 161–190; [arXiv:1601.06629](#).
- [4] M. Baake and D. Lenz, *Dynamical systems on translation bounded measures: Pure point dynamical and diffraction spectra*, *Ergodic Th. & Dynam. Syst.* **24** (2004), 1867–1893; [arXiv:math.DS/0302231](#).
- [5] M. Baake and R.V. Moody (eds), *Directions in Mathematical Quasicrystals*, CRM Monograph Series, vol. 13, AMS, Providence, RI (2000).
- [6] A. Bellow and V. Losert, *The weighted pointwise ergodic theorem and the individual ergodic theorem along subsequences*, *Trans. Amer. Math. Soc.* **288** (1985), 307–345.
- [7] A. S. Besicovitch, *Almost periodic functions*, Dover Publications, Inc., New York, (1955).
- [8] C. Corduneanu, *Almost Periodic Functions*, 2nd English ed., Chelsea, New York (1989).
- [9] H. W. Davis, *Generalized almost periodicity in groups*, *Trans. Amer. Math. Soc.* **191** (1974), 329–352.
- [10] Downarowicz and E. Glasner, *Isomorphic extensions and applications*, *Topol. Methods Nonlinear Anal.* **48** (2016), 321–338.
- [11] S. Dworkin, *Spectral theory and X-ray diffraction*, *J. Math. Phys.* **34** (1993), 2965–2967.
- [12] R. Ellis and M. Nerurkar, *Weakly Almost Periodic Flows*, *Trans. Amer. Math. Soc.* **313** (1989), 103–119.
- [13] E. Følner, *Besicovitch almost periodic functions in arbitrary groups*. *Math., Scand.* **5** (1957), 47–53.
- [14] G. Fuhrmann, M. Gröger and T. Jaeger, *Amorphic complexity*, *Nonlinearity* **29** (2016), 528–565.
- [15] G. Fuhrmann, M. Gröger and D. Lenz, *The structure of mean equicontinuous systems*, preprint 2018, [arXiv:1812.10219](#).
- [16] F. Garcia-Ramos, *Weak forms of topological and measure theoretical equicontinuity: relationships with discrete spectrum and sequence entropy*, *Ergodic Th. & Dynam. Syst.* **37** (2017), 1211–1237; [arXiv:1402.7327](#).
- [17] F. Garcia-Ramos and B. Marcus, *Mean sensitive, mean equicontinuous and almost periodic functions for dynamical systems*, *Discrete Contin. Dyn. Syst.-A* **39(2)** (2019), 729–746; [arXiv:1509.05246](#).
- [18] F. Garcia-Ramos, T. Jaeger and X. Ye, *Mean equicontinuity, almost automorphy and regularity*, preprint 2019; [arXiv:1908.05207](#).
- [19] J. Gil de Lamadrid and L.N. Argabright, *Almost periodic measures*, *Memoirs Amer. Math. Soc.* **85** (1990) (AMS, Providence, RI).
- [20] J.-B. Gouéré, *Quasicrystals and almost periodicity*, *Commun. Math. Phys.* **255** (2005), 651–681; [arXiv:math-ph/0212012](#).
- [21] J. Kellendonk, D. Lenz and J. Savinien (eds), *Mathematics of aperiodic order*, *Progress in Mathematics* **309**. Birkhäuser/Springer, Basel, (2015).
- [22] J.C. Lagarias, *Mathematical quasicrystals and the problem of diffraction*, in: [5], 61–93.

- [23] J.-Y. Lee, R.V. Moody and B. Solomyak, *Pure point dynamical and diffraction spectra*, Ann. Henri Poincaré **3** (2002), 1003–1018; [arXiv:0910.4809](#)
- [24] D. Lenz, *Continuity of eigenfunctions of uniquely ergodic dynamical systems and intensity of Bragg peaks*, Commun. Math. Phys. **287** (2009), 225–258; [arXiv:math-ph/0608026](#).
- [25] D. Lenz, *An autocorrelation and discrete spectrum for dynamical systems on metric spaces* (2020), to appear in: Ergodic. Th. & Dynam. Syst.; [arXiv:1608.05636](#).
- [26] D. Lenz, *Spectral theory of dynamical systems as diffraction theory of sampling functions* (2020), to appear in: Monatshefte f. Mathematik; [arXiv:1809.07639](#).
- [27] D. Lenz and R.V. Moody, *Diffraction theory for stochastic processes*, Ergodic Th. & Dynam. Syst. **37** (2017), 2597–2642; [arXiv:1111.3617](#).
- [28] D. Lenz, T. Spindeler, N. Strungaru, *Mean almost periodicity and pure point diffraction*, preprint (2019).
- [29] D. Lenz and N. Strungaru, *Pure point spectrum for measure dynamical systems on locally compact Abelian groups*, J. Math. Pures Appl. **92** (2009), 323–341; [arXiv:0704.2498](#).
- [30] D. Lenz and N. Strungaru, *On weakly almost periodic measures*, Trans. Amer. Math. Soc **371** (2019), 6843–6881; [arXiv:1609.08219](#).
- [31] R.V. Moody, N. Strungaru, *Almost Periodic Measures and their Fourier Transforms*, in: *Aperiodic Order. Vol. 2. Crystallography and Almost Periodicity*, eds. M. Baake and U. Grimm, Cambridge University Press, Cambridge (2017), 173–270.
- [32] M. Schlottmann, *Generalised model sets and dynamical systems*, in: [5], 143–159.
- [33] T. Spindeler, *Stepanov and Weyl almost periodicity in LCAG*, preprint (2020).
- [34] T. Spindeler, N. Strungaru, *On the (dis)continuity of the Fourier transform of measures*, preprint (2020); [arXiv:2002.01544](#)
- [35] A. Vershik, *The Pascal automorphism has a continuous spectrum*, Functional Analysis and its Applications **45**, (2011), 173–186.

MATHEMATISCHES INSTITUT, FRIEDRICH SCHILLER UNIVERSITÄT JENA, 07743 JENA, GERMANY

E-mail address: daniel.lenz@uni-jena.de

URL: <http://www.analysis-lenz.uni-jena.de>

DEPARTMENT OF MATHEMATICAL AND STATISTICAL SCIENCES,

632 CAB, UNIVERSITY OF ALBERTA, EDMONTON, AB, T6G 2G1, CANADA

E-mail address: spindele@ualberta.ca

DEPARTMENT OF MATHEMATICAL SCIENCES, MACEWAN UNIVERSITY, 10700 – 104 AVENUE, EDMONTON, AB, T5J 4S2, CANADA, AND, INSTITUTE OF MATHEMATICS “SIMON STOILOW”, BUCHAREST, ROMANIA

E-mail address: strungarun@macewan.ca

URL: <http://academic.macewan.ca/strungarun/>