EBERLEIN DECOMPOSITION FOR PV INFLATION SYSTEMS

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Abstract. The Dirac combs of primitive Pisot–Vijayaraghavan (PV) inflations on the real line or, more generally, in $\mathbb{R}^d$ are analysed. We construct a mean-orthogonal splitting for such Dirac combs that leads to the classic Eberlein decomposition on the level of the pair correlation measures, and thus to the separation of pure point versus continuous spectral components in the corresponding diffraction measures. This is illustrated with two guiding examples, and an extension to more general systems with randomness is outlined.

1. Introduction

Symbolic Pisot–Vijayaraghavan (PV) substitutions induce a much-studied class of dynamical systems under the action of $\mathbb{Z}$. By means of suitable suspensions, they also define natural dynamical systems under the continuous translation action of $\mathbb{R}$. Of particular interest is the *self-similar* suspension, which turns the symbolic substitution system into a tiling inflation; see [7, Ch. 4] and references therein, as well as [18, 8], for general background. This setting is naturally connected with general inflation tilings in $\mathbb{R}^d$, which is our point of view here.

As the famous Pisot (or PV) substitution conjecture is still unresolved, despite great effort and progress (see [1] for a summary), it seems a good strategy to consider such systems in a wider setting, where one particularly takes mixed spectra more into focus. So, given a general PV inflation system, it is of considerable interest to decompose its spectrum in a constructive fashion. Here, we report on some progress in this direction, where we start from the Dirac comb of a PV inflation and split it into two parts, one of which leads to the pure point part of the diffraction measure and the other to the continuous part. Moreover, this splitting possesses an orthogonality relation in an averaged (or Eberlein) sense, which can also be established for more general systems.

An important predecessor of our approach is the work by Aujogue [2], where an Eberlein-type decomposition is established for measure-theoretic dynamical systems, hence in an almost sure sense. When dealing with the class of primitive inflation tilings (or a characteristic point set representing them), which define strictly ergodic Delone dynamical systems, one wants to achieve such a decomposition constructively, starting form a single member of the dynamical system, that is, from a single point set in $\mathbb{R}^d$, say. Below, in view of later extensions, we do not restrict our attention to Delone sets, but allow for more general point sets; see [29, 28] for some of the theory that will then become useful. Starting from the Dirac comb of such a point set, we construct a *splitting* into two measures that results in the Eberlein decomposition for the pair correlation measures, and do this in such a way that these two measures are mutually orthogonal in an averaged (or Eberlein) sense. For some systems, such a splitting has been used in the treatment of diffraction theory of systems with mixed spectrum, see [7] and references therein, both for deterministic and for stochastic systems.

The paper is organised as follows. First, in Section 2, we recall some of the necessary tools and results, which is followed by a guiding example with mixed spectrum, namely a twisted version of the Fibonacci tiling, in Section 3. Then, we state and prove the central orthogonality result, which takes most of Section 4, before we can formulate the decomposition theorem in Section 5. Here, we also show how it works for the Thue–Morse system, which has an inflation factor that is not a unit and serves as our second guiding example. Finally, in Section 6, we extend our splitting approach to two systems of stochastic nature, namely the interaction-free lattice gas and the Fibonacci random inflation system from [15].

2. Preliminaries

Due to our setting with Euclidean inflation tilings, we work with $\mathbb{R}^d$, though many steps can be generalised to any second countable, *locally compact Abelian group* (LCAG) as well. Let $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$ be an *averaging sequence* in $\mathbb{R}^d$, by which we denote a sequence of compact sets that are *nested*, meaning $A_n \subset A_{n+1}^{\circ}$ for all $n \in \mathbb{N}$, and *exhausting*, which refers to $\bigcup_n A_n = \mathbb{R}^d$. We call an averaging sequence *symmetric* when $A_n = -A_n$ holds for all $n \in \mathbb{N}$, and write this as $\mathcal{A} = -\mathcal{A}$. Later, we shall only consider averaging sequences that have the *van Hove* property, which is to say that, for any compact set $K \subset \mathbb{R}^d$, one has

$$\lim_{n\to\infty} \frac{\mathrm{vol}(\partial^K A_n)}{\mathrm{vol}(A_n)} \, = \, 0, \tag{1}$$

where $\partial^K C := \big((C+K)\setminus C^{\circ}\big) \cup \big((\overline{\mathbb{R}^d \setminus C} - K) \cap C\big)$ for K and C compact; compare [7, p. 29] and references given there for more. Symmetric van Hove averaging sequences in $\mathbb{R}^d$ that are widely used include cubes and balls, such as $\big([-n,n]^d\big)_{n\in\mathbb{N}}$ or $\big(\{\|x\| \leqslant n\}\big)_{n\in\mathbb{N}}$.

Recall that $C_{\mathsf{u}}(\mathbb{R}^d)$ denotes the complex vector space of bounded functions on $\mathbb{R}^d$ that are uniformly continuous.

Definition 2.1. Two functions $f,g \in C_{\mathsf{u}}(\mathbb{R}^d)$ are said to possess a well-defined *Eberlein convolution* with respect to, or along, a given averaging sequence $\mathcal{A}$ in $\mathbb{R}^d$ if

$$\big(f \overset{\mathcal{A}}{\circledast} g\big)(x) \, := \lim_{n\to\infty} \frac{1}{\mathrm{vol}(A_n)} \int_{A_n} f(x-t)\, g(t) \, \mathrm{d}t$$

exists for all $x \in \mathbb{R}^d$. When $\mathcal{A}$ is a fixed averaging sequence that has the van Hove property and is symmetric, so $\mathcal{A} = -\mathcal{A}$, we will usually write $f \circledast g$ instead of $f \overset{\mathcal{A}}{\circledast} g$.

At this point, it is relevant to ask whether or when $\circledast$ is commutative.

Lemma 2.2. *Let $\mathcal{A}$ be a van Hove averaging sequence in $\mathbb{R}^d$, let $f,g \in C_{\mathsf{u}}(\mathbb{R}^d)$, and assume that the Eberlein convolution of f and g exists along $\mathcal{A}$. Then, also the Eberlein convolution* *of g and f exists, this time along $-\mathcal{A}$, and one has*

$$g \overset{-\mathcal{A}}{\circledast} f \;=\; f \overset{\mathcal{A}}{\circledast} g.$$

In particular, if $\mathcal{A}$ is also symmetric, one has $f \circledast g = g \circledast f$.

Proof. This follows from the following (backwards) calculation,

$$\begin{aligned}
\big(f \overset{\mathcal{A}}{\circledast} g\big)(x) \;&=\; \lim_{n\to\infty} \frac{1}{\mathrm{vol}(A_n)} \int_{A_n} f(x-s)\, g(s) \,\mathrm{d}s \\
&=\; \lim_{n\to\infty} \frac{1}{\mathrm{vol}(A_n)} \int_{x-A_n} f(r)\, g(x-r) \,\mathrm{d}r \\
&=\; \lim_{n\to\infty} \frac{1}{\mathrm{vol}(A_n)} \int_{-A_n} g(x-r)\, f(r) \,\mathrm{d}r \;=\; \big(g \overset{-\mathcal{A}}{\circledast} f\big)(x),
\end{aligned}$$

where the first equality in the last line is a consequence of the van Hove property. $\square$

From now on, whenever we write $f \circledast g$, it is understood that the Eberlein convolution refers to a symmetric van Hove averaging sequence and is well defined, that is, all limits exist. For a generalisation of this notion to translation-bounded Radon measures on $\mathbb{R}^d$, which we denote by $\mathcal{M}^\infty(\mathbb{R}^d)$, we say that $\mu, \nu \in \mathcal{M}^\infty(\mathbb{R}^d)$ have a well-defined Eberlein convolution with respect to a symmetric van Hove averaging sequence $\mathcal{A}$ if the limit

$$\mu \circledast \nu \;=\; \lim_{n\to\infty} \frac{\mu|_{A_n} * \nu|_{A_n}}{\mathrm{vol}(A_n)}$$

exists in the vague topology, where $\mu|_K$ denotes the restriction of the measure μ to a compact set $K \subset \mathbb{R}^d$; see [20] for details.

Remark 2.3. For a general van Hove averaging sequence, one should define

$$\mu \overset{\mathcal{A}}{\circledast} \nu \;=\; \lim_{n\to\infty} \frac{\mu|_{-A_n} * \nu|_{A_n}}{\mathrm{vol}(A_n)},$$

which satisfies $\mu \overset{\mathcal{A}}{\circledast} \nu = \nu \overset{-\mathcal{A}}{\circledast} \mu$ in analogy to above. This definition is the consistent extension of Definition 2.1. Indeed, when μ and ν are absolutely continuous measures with Radon–Nikodym densities f and g, respectively, one finds

$$\mu \overset{\mathcal{A}}{\circledast} \nu \;=\; \big(f \overset{\mathcal{A}}{\circledast} g\big)\lambda_{\mathrm{L}} \quad\text{and}\quad \nu \overset{\mathcal{A}}{\circledast} \mu \;=\; \big(g \overset{\mathcal{A}}{\circledast} f\big)\lambda_{\mathrm{L}},$$

where λ_{L} denotes Lebesgue measure on $\mathbb{R}^d$. This observation follows from

$$\begin{aligned}
\big(\mu|_{-A_n} * \nu|_{A_n}\big)(h) \;&=\; \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(x+y)\, 1_{-A_n}(x)\, f(x)\, 1_{A_n}(y)\, g(y) \,\mathrm{d}x \,\mathrm{d}y \\
&=\; \int_{\mathbb{R}^d} h(s) \int_{A_n \cap (s+A_n)} f(s-y)\, g(y) \,\mathrm{d}y \,\mathrm{d}s,
\end{aligned}$$

where $h \in C_{\mathsf{c}}(\mathbb{R}^d)$ is arbitrary, after dividing by $\mathrm{vol}(A_n)$ and taking the limit as $n \to \infty$.

When $\mathcal{A}$ is also symmetric, these subtleties go away and $\circledast$ becomes commutative. Since this extra assumption poses no relevant restriction to any of our later arguments, we will usually make it, but always say so in our formal statements. $\Diamond$

Next, we need another notion, the *Fourier–Bohr* (FB) coefficients.

Definition 2.4. A function $g \in C_{\mathsf{u}}(\mathbb{R}^d)$ has a well-defined *FB coefficient* at $k \in \mathbb{R}^d$ with respect to $\mathcal{A}$ if

$$c_g(k) \, := \lim_{n\to\infty} \frac{1}{\mathrm{vol}(A_n)} \int_{A_n} \mathrm{e}^{-2\pi \mathrm{i} kx} g(x) \,\mathrm{d}x$$

exists. We say that g possesses FB coefficients relative to $\mathcal{A}$ when $c_g(k)$ exists for all $k \in \mathbb{R}^d$.

Likewise, a Radon measure $\mu \in \mathcal{M}^\infty(\mathbb{R}^d)$ has well-defined FB coefficients with respect to $\mathcal{A}$ on a set $S \subseteq \mathbb{R}^d$ if

$$c_\mu(k) \, := \lim_{n\to\infty} \frac{1}{\mathrm{vol}(A_n)} \int_{A_n} \mathrm{e}^{-2\pi \mathrm{i} kx} \,\mathrm{d}\mu(x)$$

exists for all $k \in S$.

From now on, whenever we write $c_g(k)$ or $c_\mu(k)$, it is understood that the corresponding limit exists.

The following result, which is a mild variant of [19, Lemma 1.9 and Cor. 1.20], see also [14, Prop. 8.2], provides the connection between the FB coefficients of a translation-bounded Radon measure μ and the convolutions $\mu * \varphi$ with arbitrary $\varphi \in C_{\mathsf{c}}(\mathbb{R}^d)$, where the latter denotes the space of continuous functions on $\mathbb{R}^d$ with compact support. Below, we use $\widehat{\varphi}$ for the Fourier transform of a function φ, employing the conventions of [7, Ch. 8].

Lemma 2.5. *Let $\mathcal{A}$ be a general van Hove averaging sequence in $\mathbb{R}^d$, and consider a Radon measure $\mu \in \mathcal{M}^\infty(\mathbb{R}^d)$. If $\varphi \in C_{\mathsf{c}}(\mathbb{R}^d)$ and $k \in \mathbb{R}^d$, one has*

$$\lim_{n\to\infty} \frac{1}{\mathrm{vol}(A_n)} \left| \int_{A_n} \mathrm{e}^{-2\pi \mathrm{i} kt} \big(\varphi * \mu\big)(t) \,\mathrm{d}t \, - \, \widehat{\varphi}(k) \int_{A_n} \mathrm{e}^{-2\pi \mathrm{i} kt} \,\mathrm{d}\mu(t) \right| \, = \, 0.$$

In particular, if $c_\mu(k)$ exists, then so does $c_{\varphi*\mu}(k)$, and one has

$$c_{\varphi*\mu}(k) \, = \, \widehat{\varphi}(k) \, c_\mu(k).$$

Conversely, if $c_{\varphi*\mu}(k)$ exists for some φ with $\widehat{\varphi}(k) \neq 0$, then $c_\mu(k)$ exists as well.

Proof. Let us first note that

$$\begin{aligned}
\widehat{\varphi}(k) \int_{A_n} \mathrm{e}^{-2\pi \mathrm{i} kt} \,\mathrm{d}\mu(t) \, &= \, \widehat{\varphi}(k) \int_{\mathbb{R}^d} 1_{A_n}(s) \, \mathrm{e}^{-2\pi \mathrm{i} ks} \,\mathrm{d}\mu(s) \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi \mathrm{i} k(r+s)} \varphi(r) \,\mathrm{d}r \; 1_{A_n}(s) \,\mathrm{d}\mu(s) \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{A_n}(s) \, \mathrm{e}^{-2\pi \mathrm{i} kt} \varphi(t-s) \,\mathrm{d}\mu(s) \,\mathrm{d}t,
\end{aligned}$$

where we used the substitution $t = r + s$ and Fubini's theorem for the last line.

Consequently, we have

$$\begin{aligned}
\Big| \int_{A_n} \mathrm{e}^{-2\pi\mathrm{i}kt} \big(\varphi * \mu\big)(t) \,\mathrm{d}t \, - \, \widehat{\varphi}(k) \int_{A_n} \mathrm{e}^{-2\pi\mathrm{i}kt} \,\mathrm{d}\mu(t) \Big| \\
= \Big| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathrm{e}^{-2\pi\mathrm{i}kt} \big(1_{A_n}(t) - 1_{A_n}(s)\big) \, \varphi(t-s) \,\mathrm{d}\mu(s) \,\mathrm{d}t \Big| \\
\leqslant \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \big|1_{A_n}(t) - 1_{A_n}(s)\big| \, |\varphi(t-s)| \,\mathrm{d}|\mu|(s) \,\mathrm{d}t.
\end{aligned}$$

With $K = \mathrm{supp}(\varphi)$, we have $\big|1_{A_n}(t) - 1_{A_n}(s)\big| \, |\varphi(t-s)| = 0$ for any $t \notin \partial^K A_n$, hence

$$\big|1_{A_n}(t) - 1_{A_n}(s)\big| \, |\varphi(t-s)| \, \leqslant \, 1_{\partial^K A_n}(t) \, |\varphi(t-s)|,$$

which then gives

$$\begin{aligned}
\frac{1}{\mathrm{vol}(A_n)} \Big| \int_{A_n} \mathrm{e}^{-2\pi\mathrm{i}kt} \big(\varphi * \mu\big)(t) \,\mathrm{d}t \, - \, \widehat{\varphi}(k) \int_{A_n} \mathrm{e}^{-2\pi\mathrm{i}kt} \,\mathrm{d}\mu(t) \Big| \\
\leqslant \frac{1}{\mathrm{vol}(A_n)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{\partial^K A_n}(t) \, |\varphi(t-s)| \,\mathrm{d}|\mu|(s) \,\mathrm{d}t \\
= \frac{1}{\mathrm{vol}(A_n)} \int_{\partial^K A_n} \int_{\mathbb{R}^d} |\varphi(t-s)| \,\mathrm{d}|\mu|(s) \,\mathrm{d}t \\
= \frac{1}{\mathrm{vol}(A_n)} \int_{\partial^K A_n} \big(|\varphi| * |\mu|\big)(t) \,\mathrm{d}t \, \leqslant \, \big\| |\varphi| * |\mu| \big\|_\infty \frac{\mathrm{vol}(\partial^K A_n)}{\mathrm{vol}(A_n)}.
\end{aligned}$$

As $n \to \infty$, the claim now follows from the translation-boundedness of μ and the van Hove property of $\mathcal{A}$ from Eq. (1). □

What we wrote down so far are special relations involving continuous *characters* on $\mathbb{R}^d$, which are the elements of the dual group. Since $\mathbb{R}^d$ is self-dual, it is common to employ an additive notation for the characters, and denote them by $\chi_k \colon \mathbb{R}^d \longrightarrow \mathbb{S}^1$, where $k \in \mathbb{R}^d$ is fixed and the mapping is given by

$$x \, \mapsto \, \chi_k(x) := \mathrm{e}^{2\pi\mathrm{i}kx}.$$

Clearly, one then has $\chi_k(x) \neq 0$ for all $x \in \mathbb{R}^d$, together with $\overline{\chi_k} = \chi_{-k}$.

Before we embark on the general result, let us discuss our first guiding example, which is built as a simple extension [4] of the classic Fibonacci tiling of the real line.

3. A simple PV inflation with mixed spectrum

Consider the aperiodic, four-letter substitution rule

$$\varrho \colon \; a \mapsto ab, \quad \underline{a} \mapsto \underline{a}\underline{b}, \quad b \mapsto \underline{a}, \quad \underline{b} \mapsto a,$$

which has the substitution matrix

$$M_\varrho \, = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Since M_ϱ has Perron–Frobenius eigenvalue $\tau = \frac{1}{2}(1+\sqrt{5}\,)$, which is a PV unit, with corresponding left eigenvector $(\tau,\tau,1,1)$, one can turn ϱ into a PV inflation with prototiles (intervals) of length τ for a and $\underline{a}$, and of length 1 for b and $\underline{b}$; see [4, Sec. 3.2] and references therein for details. On identifying a with $\underline{a}$ and b with $\underline{b}$, one obtains the classic Fibonacci tilings of the real line [7, Ex. 4.6 and Sec. 9.4.1] with pure point spectrum, both in the diffraction and in the dynamical sense. This implies that the 'twisted' inflation systems induced by ϱ is an almost everywhere $2:1$ extension of the classic Fibonacci system, and thus has mixed spectrum, with a pure point and a continuous part. The latter is purely singular continuous by [4, Thm. 3.2]. Let us sketch how to arrive at this conclusion constructively.

Now, working with an inflation fixed point and the left endpoints of the intervals of type $\alpha \in \{a, \underline{a}, b, \underline{b}\}$, we always get $\varLambda_\alpha \in \mathbb{Z}[\tau]$, wherefore we can employ the *cut and project scheme* (CPS) of the Fibonacci system, abbreviated by $(\mathbb{R},\mathbb{R},\mathcal{L})$, which is given by

$$\begin{array}{ccccc}
\mathbb{R} & \xleftarrow{\;\pi\;} & \mathbb{R}\times\mathbb{R} & \xrightarrow{\;\pi_{\text{int}}\;} & \mathbb{R} \\
\cup & & \cup & & \cup \;\; \text{dense} \\
\pi(\mathcal{L}) & \xleftarrow{\;1-1\;} & \mathcal{L} & \longrightarrow & \pi_{\text{int}}(\mathcal{L}) \\
\| & & & & \| \\
\mathbb{Z}[\tau] & & \xrightarrow{\qquad\quad\star\quad\qquad} & & \mathbb{Z}[\tau]
\end{array}$$

where $\star$ denotes the star map of the CPS, which is the unique field automorphism of $\mathbb{Q}(\sqrt{5}\,)$ induced by $\sqrt{5} \mapsto -\sqrt{5}$, and $\mathcal{L} := \big\{(x,x^\star) : x \in \mathbb{Z}[\tau]\big\}$ is the Minkowski embedding of $\mathbb{Z}[\tau]$, which is a lattice in $\mathbb{R}^2 \simeq \mathbb{R}\times\mathbb{R}$ of density $1/\sqrt{5}$; see [7] for background and details. Setting $W_\alpha = \overline{\varLambda^\star_\alpha}$, one obtains [4] the compact intervals

$$W_a \, = \, W_{\underline{a}} \, = \, [\tau-2, \tau-1] \quad \text{and} \quad W_b \, = \, W_{\underline{b}} \, = \, [-1, \tau-2].$$

Now, each set $\curlywedge(W_\alpha) := \{x \in \mathbb{Z}[\tau] : x^\star \in W_\alpha\}$ is a regular model set with pure point diffraction [7, Thm. 9.4], while this is *not* true of the point sets $\varLambda_\alpha$.

One can check that each $\varLambda_\alpha$ has half the density of $\curlywedge(W_\alpha)$. By [9, Thm. 5.3], the point sets $\varLambda^\star_\alpha$ are uniformly distributed in W_α, and each of the measures

$$\nu_\alpha \, = \, \delta_{\varLambda_\alpha} - \frac{1}{2}\,\delta_{\curlywedge(W_\alpha)}$$

has vanishing FB coefficients, where we use $\delta_S := \sum_{x\in S} \delta_x$ to denote the *Dirac comb* of a point set S. So, with $\omega_\alpha = \frac{1}{2}\delta_{\curlywedge(W_\alpha)}$, one has a decomposition

$$\delta_{\varLambda_\alpha} \, = \, \omega_\alpha + \nu_\alpha$$

into two summands. The crucial observation now is that the first gives rise to the pure point part of the spectrum, and the second to the continuous part, while all cross terms under the Eberlein convolution with respect to any symmetric van Hove averaging sequence in $\mathbb{R}$ vanish. Further details, and a closely related example with a more complicated window structure, are discussed in [9, Sec. 7]. Our goal now is to substantiate the decomposition claim and prove it in sufficient generality.

4. Orthogonality for Eberlein convolution

Let us first state an elementary result on the connection between the FB coefficients of a function $f \in C_{\mathsf{u}}(\mathbb{R}^d)$ and the Eberlein convolution of f with the characters on $\mathbb{R}^d$.

Lemma 4.1. *Let $\mathcal{A}$ be a symmetric van Hove averaging sequence in $\mathbb{R}^d$, and consider a function $f \in C_{\mathsf{u}}(\mathbb{R}^d)$ together with a character χ_k. Then, with respect to $\mathcal{A}$, the following statements are equivalent.*

(1) *The FB coefficient $c_f(k)$ exists.*
(2) *The Eberlein convolution $\chi_k \circledast f$ exists.*
(3) *There is some $x \in \mathbb{R}^d$ such that $\lim_{n\to\infty} \frac{1}{\mathrm{vol}(A_n)} \int_{\mathbb{R}^d} \chi_k(x-t)\, f(t) \,\mathrm{d}t$ exists.*

Further, assume that $c_f(k)$ exists. Then, for all $x \in \mathbb{R}^d$, one has the relation

$$\bigl(\chi_k \circledast f\bigr)(x) \, = \, \chi_k(x)\, c_f(k).$$

Proof. For arbitrary $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$, we have

$$\int_{A_n} \chi_k(x-t)\, f(t) \,\mathrm{d}t \, = \int_{A_n} \chi_k(x)\, \overline{\chi_k(t)}\, f(t) \,\mathrm{d}t \, = \, \chi_k(x) \int_{A_n} \overline{\chi_k(t)}\, f(t) \,\mathrm{d}t.$$

Since $\chi_k(x) \neq 0$, the claimed equivalences, as well as the final identity, follow via dividing by $\mathrm{vol}(A_n)$ and taking the limit as $n \to \infty$. □

This has an immediate consequence as follows.

Corollary 4.2. *Let $\mathcal{A}$ be a symmetric van Hove averaging sequence in $\mathbb{R}^d$, and consider a function $f \in C_{\mathsf{u}}(\mathbb{R}^d)$ together with a trigonometric polynomial $P = \sum_{j=1}^{n} \alpha_j\, \chi_{k_j}$. If the FB coefficients $c_f(k_j)$ exist for all $1 \leqslant j \leqslant n$, the Eberlein convolution $P \circledast f$ exists, too, with*

$$P \circledast f \, = \sum_{j=1}^{n} \alpha_j\, c_f(k_j)\, \chi_{k_j}. \qquad \square$$

To continue, we need another notion as follows.

Definition 4.3. Let $\mathcal{A}$ be a van Hove averaging sequence in $\mathbb{R}^d$ and $\mu \in \mathcal{M}^{\infty}(\mathbb{R}^d)$. If the FB coefficients of μ with respect to $\mathcal{A}$ exist, the corresponding *FB spectrum* of μ is the set $\{k \in \mathbb{R}^d : c_\mu(k) \neq 0\}$. Further, we say that μ has *null FB spectrum* with respect to $\mathcal{A}$ if, for all $k \in \mathbb{R}^d$, one has

$$c_\mu(k) \, = \lim_{n\to\infty} \frac{1}{\mathrm{vol}(A_n)} \int_{A_n} \overline{\chi_k(t)} \,\mathrm{d}\mu(t) \, = \, 0,$$

which is to say that all FB coefficients of μ exist and vanish.

Likewise, $f \in C_{\mathsf{u}}(\mathbb{R}^d)$ has *null FB spectrum* with respect to $\mathcal{A}$ if, for all $k \in \mathbb{R}^d$, we have

$$c_f(k) \, = \lim_{n\to\infty} \frac{1}{\mathrm{vol}(A_n)} \int_{A_n} \overline{\chi_k(t)}\, f(t) \,\mathrm{d}t \, = \, 0 .$$

An immediate consequence of Lemma 4.1 and Lemma 2.5 is the following.

Corollary 4.4. *Let $\mathcal{A}$ be a symmetric van Hove averaging sequence in $\mathbb{R}^d$. Consider a measure $\mu \in \mathcal{M}^{\infty}(\mathbb{R}^d)$ and a function $f \in C_{\mathsf{u}}(\mathbb{R}^d)$. Then, the following two properties hold.*

(a) *If μ has null FB spectrum with respect to $\mathcal{A}$, the function $\varphi * \mu$, for any $\varphi \in C_{\mathsf{c}}(\mathbb{R}^d)$, has null FB spectrum with respect to $\mathcal{A}$ as well.*

(b) *If f has null FB spectrum with respect to $\mathcal{A}$, one has $P \circledast f = 0$ for all trigonometric polynomials P, where $\circledast$ is the Eberlein convolution along $\mathcal{A}$.* □

At this point, we can prove the central tool for our later computations, where δ_S is again the Dirac comb of a point set $S \subset \mathbb{R}^d$. For generalisations of this result, we refer to [37].

Theorem 4.5. *Let $\mathcal{A}$ be a symmetric van Hove averaging sequence in $\mathbb{R}^d$, consider a measure $\mu \in \mathcal{M}^{\infty}(\mathbb{R}^d)$, and assume that μ has null FB spectrum with respect to $\mathcal{A}$. Further, let $\varLambda$ be a regular model set in $\mathbb{R}^d$. Then, for the Eberlein convolution along $\mathcal{A}$, one has*

$$\mu \circledast \delta_{\varLambda} \, = \, 0 .$$

Proof. Let $(\mathbb{R}^d, H, \mathcal{L})$ be the CPS for the description of the model set, where H is a compactly generated LCAG and $\mathcal{L}$ is a lattice in $\mathbb{R}^d \times H$, that is, a co-compact discrete subgroup. Note that we need H in this generality because the internal space need not be Euclidean; we refer to [24, 25] and [7, Sec. 7.2] for general background, and to [12, 35] for a detailed description of the explicit construction of the CPS. Further, let $W \subseteq H$ be the window that gives

$$\varLambda \, = \, \curlywedge(W) \, = \, \{ x \in \pi(\mathcal{L}) : x^{\star} \in W \}$$

as a regular model set, where $\pi \colon \mathbb{R}^d \times H \longrightarrow \mathbb{R}^d$ is the canonical projection and $\star$ denotes the star map of the CPS.

Now, since μ and $\delta_{\varLambda}$ are translation-bounded measures by construction, the set

$$\left\{ \frac{\mu|_{A_n} * \delta_{\varLambda}|_{A_n}}{\mathrm{vol}(A_n)} : n \in \mathbb{N} \right\}$$

is pre-compact in the vague topology and metrisable [10]. Therefore, to prove that $\mu \circledast \delta_{\varLambda} = 0$, it suffices to show that 0 is the only cluster point of this set (or sequence).

Let η be any cluster point of this sequence, and let $\mathcal{B} = (B_n)_{n\in\mathbb{N}}$ be a subsequence of $\mathcal{A}$ with respect to which η is a limit, which means that $B_n = A_{\ell_n}$ with $\ell_{n+1} > \ell_n$ for all $n \in \mathbb{N}$. Clearly, $\mathcal{B}$ is again a symmetric van Hove averaging sequence, and one has

$$\eta \, = \lim_{n\to\infty} \frac{\mu|_{B_n} * \delta_{\varLambda}|_{B_n}}{\mathrm{vol}(B_n)} \, = \lim_{n\to\infty} \frac{\mu|_{B_n} * \delta_{\varLambda}}{\mathrm{vol}(B_n)} , \tag{2}$$

where the second equality follows from the van Hove property of $\mathcal{B}$ via Schlottmann's lemma [32, Lemma 1.2]; see also [10, 20].

Now, fix $\varphi, \psi \in C_{\mathsf{c}}(\mathbb{R}^d)$, set $K := \operatorname{supp}(\varphi)$, and note that the translation-boundedness of μ implies $\|\varphi * \mu\|_\infty < \infty$. Let $\varepsilon > 0$, and select a function $h \in C_{\mathsf{c}}(H)$ that satisfies $1_W \leqslant h$ together with

$$\int_H \bigl(h(t) - 1_W(t)\bigr)\,\mathrm{d}t \,<\, \frac{\varepsilon}{1 + 2\,\mathrm{dens}(\mathcal{L})\,\|\varphi * \mu\|_\infty \int_{\mathbb{R}^d} |\psi(t)|\,\mathrm{d}t}, \tag{3}$$

which is clearly possible. Further, with $L = \pi(\mathcal{L}) \subset \mathbb{R}^d$, let

$$\nu \,=\, \omega_h \,:= \sum_{x \in L} h(x^\star)\,\delta_x,$$

which is a strongly almost periodic measure, $\nu \in \mathcal{SAP}(\mathbb{R}^d)$, by [35, Thm. 5.5.2]. Consequently, $\psi * \nu$ is a Bohr (or uniformly) almost periodic function. This implies that there exists a (multivariate) trigonometric polynomial P such that

$$\|\psi * \nu - P\|_\infty \,<\, \frac{\varepsilon}{1 + 2\|\varphi * \mu\|_\infty}. \tag{4}$$

In view of Eq. (2), we now have

$$\begin{aligned}
\bigl(\varphi * \psi * \eta\bigr)(0) \,&=\, \lim_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \int_{\mathbb{R}^d} \bigl(\varphi * \psi\bigr)(-s)\,\mathrm{d}\bigl(\mu|_{B_n} * \delta_{\varLambda}\bigr)(s) \\
&=\, \lim_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \bigl(\varphi * \psi\bigr)(-x-y)\,\mathrm{d}\mu|_{B_n}(x)\,\mathrm{d}\delta_{\varLambda}(y) \\
&=\, \lim_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{B_n}(x)\,\varphi(-x-y-s)\,\psi(s)\,\mathrm{d}s\,\mathrm{d}\mu(x)\,\mathrm{d}\delta_{\varLambda}(y),
\end{aligned}$$

which can be continued via the substitution $r = y + s$ and Fubini's theorem as

$$\begin{aligned}
\bigl(\varphi * \psi * \eta\bigr)(0) \,&=\, \lim_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{B_n}(x)\,\varphi(-x-r)\,\psi(r-y)\,\mathrm{d}r\,\mathrm{d}\mu(x)\,\mathrm{d}\delta_{\varLambda}(y) \\
&=\, \lim_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{B_n}(x)\,\varphi(-x-r)\,\psi(r-y)\,\mathrm{d}\delta_{\varLambda}(y)\,\mathrm{d}r\,\mathrm{d}\mu(x) \\
&=\, \lim_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{B_n}(x)\,\varphi(-x-r)\bigl(\psi * \delta_{\varLambda}\bigr)(r)\,\mathrm{d}\mu(x)\,\mathrm{d}r.
\end{aligned}$$

Here, we observe that $1_{B_n}(x)\varphi(-x-r) = 1_{B_n}(r)\varphi(-x-r)$ holds for $x \notin \partial^K B_n$. Due to the van Hove property of $\mathcal{B}$, we have $\lim_{n\to\infty} \mathrm{vol}(\partial^K B_n)/\,\mathrm{vol}(B_n) = 0$, so we get

$$
\begin{aligned}
\bigl(\varphi * \psi * \eta\bigr)(0) \;&=\; \lim_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \int_{\mathbb{R}^d} 1_{B_n}(r)\, \bigl(\psi * \delta_{\varLambda}\bigr)(r) \int_{\mathbb{R}^d} \varphi(-x-r)\,\mathrm{d}\mu(x)\,\mathrm{d}r \\
&=\; \lim_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \int_{\mathbb{R}^d} 1_{B_n}(r)\, \bigl(\psi * \delta_{\varLambda}\bigr)(r)\, \bigl(\varphi * \mu\bigr)(-r)\,\mathrm{d}r \\
&=\; \lim_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \int_{B_n} \bigl(\psi * \delta_{\varLambda}\bigr)(r)\, \bigl(\varphi * \mu\bigr)(-r)\,\mathrm{d}r .
\end{aligned}
$$

Therefore, employing a standard 3ε-strategy, we find

$$
\begin{aligned}
\bigl|\bigl(\varphi * \psi * \eta\bigr)(0)\bigr| \;&=\; \lim_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \left| \int_{B_n} \bigl(\psi * \delta_{\varLambda}\bigr)(r)\, \bigl(\varphi * \mu\bigr)(-r)\,\mathrm{d}r \right| \\
&\leqslant\; \limsup_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \left| \int_{B_n} \bigl(\psi * \delta_{\varLambda} - \psi * \nu\bigr)(r)\, \bigl(\varphi * \mu\bigr)(-r)\,\mathrm{d}r \right| \\
&\quad + \limsup_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \left| \int_{B_n} \bigl(\psi * \nu - P\bigr)(r)\, \bigl(\varphi * \mu\bigr)(-r)\,\mathrm{d}r \right| \\
&\quad + \limsup_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \left| \int_{B_n} P(r)\, \bigl(\varphi * \mu\bigr)(-r)\,\mathrm{d}r \right| ,
\end{aligned} \tag{5}
$$

where P is the trigonometric polynomial from (4).

Now, we need to estimate the three terms in the last expression of (5), where we begin with the middle one. Here, for all $n \in \mathbb{N}$, we recall (4) and obtain

$$
\frac{1}{\mathrm{vol}(B_n)} \left| \int_{B_n} \bigl(\psi * \nu - P\bigr)(r)\, \bigl(\varphi * \mu\bigr)(-r)\,\mathrm{d}r \right| \;\leqslant\; \|\psi * \nu - P\|_{\infty}\, \|\varphi * \mu\|_{\infty} \;<\; \frac{\varepsilon}{2},
$$

which then gives

$$
\limsup_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \left| \int_{B_n} \bigl(\psi * \nu - P\bigr)(r)\, \bigl(\varphi * \mu\bigr)(-r)\,\mathrm{d}r \right| \;\leqslant\; \frac{\varepsilon}{2}.
$$

Next, since μ has null FB spectrum relative to $\mathcal{A}$ by assumption, hence clearly also relative to the subsequence $\mathcal{B}$, Corollary 4.4 implies

$$
\limsup_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \left| \int_{B_n} P(r)\, \bigl(\varphi * \mu\bigr)(-r)\,\mathrm{d}r \right| \;=\; \bigl|P \overset{\mathcal{B}}{\circledast} (\varphi * \mu)\bigr|(0) \;=\; 0.
$$

The remaining term is a little harder. Here, we have

$$
T_1 \;:=\; \limsup_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \left| \int_{B_n} \bigl(\psi * \delta_{\varLambda} - \psi * \nu\bigr)(r)\, \bigl(\varphi * \mu\bigr)(-r)\,\mathrm{d}r \right|
$$

$$
\begin{aligned}
&\leqslant \limsup_{n\to\infty} \frac{1}{\mathrm{vol}(B_n)} \int_{B_n} \big|\psi * \delta_{\varLambda} - \psi * \nu\big|(r)\, \|\varphi * \mu\|_{\infty} \,\mathrm{d}r \\
&\leqslant \limsup_{n\to\infty} \frac{\|\varphi * \mu\|_{\infty}}{\mathrm{vol}(B_n)} \int_{B_n} \big(|\psi| * |\delta_{\varLambda} - \nu|\big)(r)\,\mathrm{d}r \\
&= \limsup_{n\to\infty} \frac{\|\varphi * \mu\|_{\infty}}{\mathrm{vol}(B_n)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} 1_{B_n}(r+s)\,\mathrm{d}|\delta_{\varLambda} - \nu|(s)\,|\psi|(r)\,\mathrm{d}r \\
&= \limsup_{n\to\infty} \frac{\|\varphi * \mu\|_{\infty}}{\mathrm{vol}(B_n)} \int_{\mathbb{R}^d} \big(\nu - \delta_{\varLambda}\big)(B_n - r)\,|\psi|(r)\,\mathrm{d}r \\
&\leqslant \|\varphi * \mu\|_{\infty} \int_{\mathbb{R}^d} |\psi|(r)\,\mathrm{d}r \ \limsup_{n\to\infty} \frac{\sup_{t\in\mathbb{R}^d}\big(\nu - \delta_{\varLambda}\big)(B_n - t)}{\mathrm{vol}(B_n)},
\end{aligned}
$$

where we have used the fact that $\nu - \delta_{\varLambda}$ is a positive measure in the second-last line. The crucial observation now is that, due to the model set structure with its uniform distribution properties [26], the last term satisfies

$$
\limsup_{n\to\infty} \frac{\sup_{t\in\mathbb{R}^d}\big(\nu - \delta_{\varLambda}\big)(B_n - t)}{\mathrm{vol}(B_n)} \; = \; \mathrm{dens}(\mathcal{L}) \int_H \big(h(t) - 1_W(t)\big)\,\mathrm{d}t
$$

which implies $T_1 < \frac{\varepsilon}{2}$ via (3) as required.

It now follows from Eq. (5) that $\big|\big(\varphi * \psi * \eta\big)(0)\big| < \varepsilon$ holds for all $\varepsilon > 0$. This implies

$$
\eta\big((I.\varphi) * (I.\psi)\big) \; = \; \eta\big(I.(\varphi * \psi)\big) \; = \; \big(\varphi * \psi * \eta\big)(0) \; = \; 0,
$$

where $\big(I.g\big)(x) := g(-x)$ for any (continuous) function g. Since $\varphi, \psi \in C_{\mathsf{c}}(\mathbb{R}^d)$ were arbitrary, we see that $\eta = 0$ holds on the subset

$$
K_2(\mathbb{R}^d) \; := \; \mathrm{span}\big\{ f * g : f, g \in C_{\mathsf{c}}(\mathbb{R}^d)\big\},
$$

which is dense in $C_{\mathsf{c}}(\mathbb{R}^d)$ by a standard approximate identity argument. Consequently, $\eta = 0$, and no other cluster point can exist. $\square$

Note that Theorem 4.5 remains true if the symmetry assumption on $\mathcal{A}$ is lifted. The proof remains unchanged, except for replacing B_n by $-B_n$ from Eq. (2) onwards, but commutativity of $\circledast$ is no longer implied; compare Remark 2.3.

5. Eberlein splitting and decomposition for PV inflations

Let $\varLambda_1, \ldots, \varLambda_N$ denote pairwise disjoint point sets in $\mathbb{R}^d$ and consider $\varLambda := \dot{\bigcup}_i \varLambda_i$, which we call a *typed point set*. Let us assume that $\varLambda$ is a Delone set with nice averaging properties, inlcuding the types. In particular, given some symmetric van Hove averaging sequence $\mathcal{A}$, we are interested in the situation that the *pair correlation measures*

$$
\gamma_{ij} \; := \; \widetilde{\delta_{\varLambda_i}} \circledast \delta_{\varLambda_j}
$$

with respect to $\mathcal{A}$ exist for all $1 \leqslant i, j \leqslant N$, where $\widetilde{\mu}$ denotes the (possibly complex) Radon measure defined by $\widetilde{\mu}(g) = \overline{\mu(\widetilde{g})}$ with $g \in C_{\mathsf{c}}(\mathbb{R}^d)$ and $\widetilde{g}(x) = \overline{g(-x)}$. For instance, this property is guaranteed when the $\varLambda_i$ are regular model sets in the same CPS, or when they emerge as the control points of a primitive inflation rule with N prototiles; see [7, 9] for background and various details, and [28] for extensions. Below, we mainly consider the case $d = 1$, though the setting is general enough to cover higher dimensions as well.

Theorem 5.1. *Let $\varLambda = \dot{\bigcup}_{1 \leqslant i \leqslant N} \varLambda_i$ be a typed point set generated from of a primitive PV inflation rule in one dimension that is aperiodic and has a PV unit as inflation factor, and consider the corresponding natural CPS $(\mathbb{R}, \mathbb{R}^m, \mathcal{L})$ that emerges via the Minkowski embedding of the module spanned by the points. Let W_i be the attractors of the induced, contractive iterated function system for the windows in internal space, and set*

$$\alpha_i \, := \, \frac{\operatorname{dens}(\varLambda_i)}{\operatorname{dens}(\mathcal{L}) \operatorname{vol}(W_i)} \, , \quad \omega_i \, := \, \alpha_i \, \delta_{\curlywedge(W_i)} \, , \quad \textit{and} \quad \nu_i \, := \, \delta_{\varLambda_i} - \omega_i .$$

Then, for any symmetric van Hove averaging sequence $\mathcal{A}$ and all $1 \leqslant i, j \leqslant N$, one has the splitting $\delta_{\varLambda_i} = \omega_i + \nu_i$ together with the decomposition and orthogonality relations

$$\bigl(\gamma_{ij}\bigr)_{\mathsf{s}} \, = \, \widetilde{\omega_i} \circledast \omega_j \, , \quad \bigl(\gamma_{ij}\bigr)_{0} \, = \, \widetilde{\nu_i} \circledast \nu_j \, , \quad \textit{and} \quad \widetilde{\omega_i} \circledast \nu_j \, = \, \widetilde{\nu_i} \circledast \omega_j \, = \, 0,$$

where $\gamma_{ij} = \bigl(\gamma_{ij}\bigr)_{\mathsf{s}} + \bigl(\gamma_{ij}\bigr)_{0}$ is the unique Eberlein decomposition of the pair correlation measures into their strongly almost periodic and their null-weakly almost periodic components.

Proof. When the inflation factor is a unit, the induced CPS has a Euclidean internal space, that is, we have $H = \mathbb{R}^m$ with $m \geqslant 1$, where the latter is a consequence of the assumed aperiodicity. This is the situation fully analysed in [9], with $\operatorname{dens}\bigl(\curlywedge(W_i)\bigr) = \operatorname{dens}(\mathcal{L}) \operatorname{vol}(W_i)$.

Let $\mathcal{A}$ be arbitrary, but fixed. First, by [9, Thm. 5.3], the FB coefficients of ν_i satisfy

$$c_{\nu_i}(k) \, = \, 0$$

for all $k \in \mathbb{R}$. Then, by Theorem 4.5, we get $\widetilde{\omega_i} \circledast \nu_j = \widetilde{\nu_i} \circledast \omega_j = 0$ as claimed.

Next, since $\curlywedge(W_i)$ is a regular model set for each i, in the same CPS, the Eberlein convolutions $\widetilde{\omega_i} \circledast \omega_j$ along $\mathcal{A}$ exist and are strongly almost periodic measures, that is,

$$\widetilde{\omega_i} \circledast \omega_j \, \in \, \mathcal{SAP}(\mathbb{R}).$$

Likewise, as the $\varLambda_i$ emerge from a primitive inflation rule, the pair correlation measures γ_{ij} exist and are weakly almost periodic, compare [5, 21], so

$$\gamma_{ij} \, \in \, \mathcal{WAP}(\mathbb{R}).$$

Therefore, as all required limits exist, we obtain the pair correlation measures [5] as

$$\begin{aligned} \gamma_{ij} \, &= \, \widetilde{\delta_{\varLambda_i}} \circledast \delta_{\varLambda_j} \, = \, \bigl(\widetilde{\omega_i} + \widetilde{\nu_i}\bigr) \circledast \bigl(\omega_j + \nu_j\bigr) \\ &= \, \widetilde{\omega_i} \circledast \omega_j \, + \, \widetilde{\omega_i} \circledast \nu_j \, + \, \widetilde{\nu_i} \circledast \omega_j \, + \, \widetilde{\nu_i} \circledast \nu_j \\ &= \, \widetilde{\omega_i} \circledast \omega_j \, + \, \widetilde{\nu_i} \circledast \nu_j \, , \end{aligned} \tag{6}$$

which implies that every $\widetilde{\nu_i} \circledast \nu_j$ is a weakly almost periodic measure. Moreover, all these measures are Fourier transformable by [5, Lemma 2.1], and we obtain

$$\widehat{\gamma_{ij}} \, = \, \widehat{\widetilde{\omega_i} \circledast \omega_j} \, + \, \widehat{\widetilde{\nu_i} \circledast \nu_j} .$$

Via the results from [9] and [5], we find

$$\widehat{\gamma_{ij}}(\{k\}) \, = \, \overline{A_{\varLambda_i}(k)} A_{\varLambda_j}(k) \, = \, \widehat{\widetilde{\omega_i} \circledast \omega_j} \, (\{k\})$$

for all $k \in \mathbb{R}$, where the amplitudes satisfy $A_{\varLambda_j}(k) = c_k(\delta_{\varLambda_j})$. But this implies

$$\bigl(\widehat{\widetilde{\nu_i} \circledast \nu_j}\bigr)_{\mathsf{pp}} \, = \, 0,$$

which means that $\widetilde{\nu_i} \circledast \nu_j \in \mathcal{WAP}_0(\mathbb{R})$. Since $\widetilde{\omega_i} \circledast \omega_j \in \mathcal{SAP}(\mathbb{R})$ and $\widetilde{\nu_i} \circledast \nu_j \in \mathcal{WAP}_0(\mathbb{R})$, the claim follows from Eq. (6) and the uniqueness of the Eberlein decomposition [27]. $\square$

Once again, the result of Theorem 5.1 remains true without the symmetry requirement for $\mathcal{A}$. Indeed, observe that

$$\widetilde{\widetilde{\omega_i} \circledast \nu_j} \, = \, \widetilde{\nu_j} \circledast \omega_i ,$$

which is an easy consequence of $\widetilde{\mu|_K} \, = \, \widetilde{\mu}|_{-K}$. Then, Theorem 4.5 still provides the two relations we need to derive (6), though $\circledast$ is no longer implied to be commutative.

Remark 5.2. The orthogonality relations in Theorem 5.1, via inserting the definitions of the measures, also imply that

$$\alpha_i \, \widetilde{\delta_{\curlywedge(W_i)}} \circledast \delta_{\varLambda_j} \, = \, \alpha_j \, \widetilde{\delta_{\varLambda_i}} \circledast \delta_{\curlywedge(W_j)}$$

holds for all $1 \leqslant i, j \leqslant N$, where $\alpha_i = \operatorname{dens}(\varLambda_i)/\operatorname{dens}\bigl(\curlywedge(W_j)\bigr)$. This proportionality of measures matches nicely with the density formula

$$\widehat{\widetilde{\delta_P} \circledast \delta_Q} \, \bigl(\{0\}\bigr) \, = \, \operatorname{dens}(P) \, \operatorname{dens}(Q) \tag{7}$$

for Meyer sets P and Q that define uniquely ergodic Delone dynamical systems. With $\mu = \delta_P$ and $\nu = \delta_Q$, this relation follows from the complex polarisation identity

$$\widetilde{\mu} \circledast \nu \, = \, \frac{1}{4} \sum_{\ell=1}^{4} (-\mathrm{i})^{\ell} \bigl(\widetilde{\mu + \mathrm{i}^{\ell} \nu}\bigr) \circledast \bigl(\mu + \mathrm{i}^{\ell} \nu\bigr),$$

where the right-hand side is a complex linear combination of four positive-definite measures. Then, under Fourier transform, one gets

$$\widehat{\widetilde{\mu} \circledast \nu} \, \bigl(\{0\}\bigr) \, = \, \frac{1}{4} \sum_{\ell=1}^{4} \bigl| M(\mu) + \mathrm{i}^{\ell} M(\nu) \bigr|^2$$

from the known intensity formula at 0, where $M(.)$ is the *mean* of a measure; see [7, Prop 9.2] and [27, Lemma 4.10.7]. Since $M(\mu) = \operatorname{dens}(P)$ and $M(\nu) = \operatorname{dens}(Q)$, Eq. (7) follows by expanding the last sum. It seems interesting to further analyse these identities under the geometric and topological constraints imposed by the projection setting. $\Diamond$

The formulation of Theorem 5.1 refers to PV inflations with an inflation multiplier that is a *unit*, because only this case has been fully treated so far [9]. It is clear that the key result, namely [9, Thm. 5.3], can be generalised to the non-unit situation, and also to PV inflations in higher dimensions. Then, the decomposition remains the same. Let us illustrate this with our second guiding example as follows.

Example 5.3. The Thue–Morse (TM) substitution

$$\varrho\colon\; a \mapsto ab, \quad b \mapsto ba$$

has constant length 2, and gives rise to a partition of the integers, $\varLambda = \mathbb{Z} = \varLambda_a \dot{\cup} \varLambda_b$. Neither $\varLambda_a$ nor $\varLambda_b$ is a model set, but our above approach is fully applicable. Here, for the construction of the splitting, we need a CPS with the 2-adic integers as internal space, equipped with its normalised Haar measure that is to be used for the window volumes. Then, the analogue of the construction from Theorem 5.1, for instance with $\mathcal{A} = \bigl([-n,n]\bigr)_{n\in\mathbb{N}}$, leads to $\omega_a = \omega_b = \frac{1}{2}\delta_{\mathbb{Z}}$ and thus to the signed measures

$$\nu_a \,=\, \delta_{\varLambda_a} - \frac{1}{2}\,\delta_{\mathbb{Z}} \quad \text{and} \quad \nu_b \,=\, -\nu_a .$$

For $\alpha,\beta \in \{a,b\}$, the corresponding pair correlation measures are

$$\gamma_{\alpha\beta} \,=\, \frac{1}{4}\,\delta_{\mathbb{Z}} + \widetilde{\nu_\alpha} \circledast \nu_\beta \,=\, \frac{1}{4}\,\delta_{\mathbb{Z}} + \frac{1}{4}\,\epsilon_{\alpha\beta}\,\gamma_{\mathrm{TM}}\,, \tag{8}$$

with $\epsilon_{\alpha\beta} = 1$ or -1 depending on whether α equals β or not, and with γ_{TM} denoting the classic autocorrelation measure of the signed TM sequence due to Mahler and Kakutani; see [7, Sec. 10.1] and references therein for details.

Indeed, (8) is the required Eberlein decompostion. Here, one has $\widehat{\delta_{\mathbb{Z}}} = \delta_{\mathbb{Z}}$ by the Poisson summation formula [7, Prop. 9.4], see also [30], while $\widehat{\gamma_{\mathrm{TM}}}$ is a positive measure that is purely singular continuous. It has the Riesz product representation

$$\widehat{\gamma_{\mathrm{TM}}} \,=\, \prod_{\ell=0}^{\infty} \bigl(1 - \cos\bigl(2^{\ell+1}\pi(.)\bigr)\bigr),$$

to be interpreted as the vague limit of a sequence of absolutely continuous measures; see [6] for a recent detailed analysis of this measure. ◊

The explicit extension of this type of analysis to tiling models in higher dimensions, for instance in the spirit of [5, 4], remains an interesting task for the near future, as does a further splitting of the null-weakly almost periodic part, as in [36], according to the different continuous types after Fourier transform.

6. Further directions

It is natural to ask whether our constructive approach can be made to work also beyond the PV inflation case, and thus perhaps add extra insight into systems that are covered by [2]. In particular, it is of interest to see the orthogonality in the Eberlein sense in more generality, at least almost surely with respect to some given invariant measure in a dynamical systems context. This is indeed possible, and we outline this with two concrete cases.

6.1. **Eberlein splitting for a lattice gas.** Let $0 < p < 1$ be fixed, and consider the Bernoulli *lattice gas* on $\mathbb{Z}$ with independent occupation probability p for the single sites. A realisation of this process can thus either be seen as a configuration, meaning an element of $\{0,1\}^{\mathbb{Z}}$, or as a subset $\varLambda \subseteq \mathbb{Z}$. Let us take the latter view, and select a typical subset $\varLambda$, thus with density p. By [7, Ex. 11.2], the corresponding autocorrelation measure is

$$\gamma \, = \, p^2 \delta_{\mathbb{Z}} + p(1-p)\delta_0 \, = \, (\gamma)_{\mathsf{s}} + (\gamma)_0 ,$$

which applies to almost all realisations of the Bernoulli process. This can easily be proved with the strong law of large numbers, see [11] for a detailed exposition, and provides the Eberlein decomposition of γ into its strongly almost periodic and its null-weakly almost periodic part. Indeed, the diffraction measure then simply is $\widehat{\gamma} = p^2 \delta_{\mathbb{Z}} + p(1-p)\lambda_{\mathrm{L}}$, where λ_{L} denotes Lebesgue measure and $\widehat{\gamma}$ applies to almost all realisations.

Now, in analogy to our above splitting, we set

$$\delta_{\varLambda} \, = \, \omega + \nu \qquad \text{with } \; \omega \, = \, p\, \delta_{\mathbb{Z}} \; \text{ and } \; \nu \, = \, \delta_{\varLambda} - \omega . \tag{9}$$

Using $\mathcal{A} = \bigl([-n,n]\bigr)_{n\in\mathbb{N}}$ as before, we get $\delta_{\mathbb{Z}} \circledast \delta_{\mathbb{Z}} = \delta_{\mathbb{Z}}$ from [7, Ex. 8.10], hence

$$\omega \circledast \widetilde{\omega} \, = \, \omega \circledast \omega \, = \, p^2 \delta_{\mathbb{Z}} \, = \, (\gamma)_{\mathsf{s}}$$

and $\omega \circledast \delta_{\mathbb{Z}} = p\, \delta_{\mathbb{Z}}$. Furthermore, $\delta_{\varLambda} \circledast \delta_{\mathbb{Z}} = \delta_{-\varLambda} \circledast \delta_{\mathbb{Z}} = p \delta_{\mathbb{Z}}$ follows from a simple density calculation, where one observes that $-\varLambda$ is another typical realisation if $\varLambda$ is one. But this implies the orthogonality relations $\omega \circledast \widetilde{\nu} = \widetilde{\omega} \circledast \nu = 0$ together with

$$\nu \circledast \widetilde{\nu} \, = \, \delta_{\varLambda} \circledast \delta_{-\varLambda} - p^2 \delta_{\mathbb{Z}} \, = \, \gamma - p^2 \delta_{\mathbb{Z}} \, = \, p(1-p)\, \delta_0 \, = \, (\gamma)_0 .$$

We thus see that (9) provides the splitting of $\delta_{\varLambda}$ in complete analogy to Theorem 5.1, and applies to almost all realisations of the lattice gas.

Clearly, this can be extended to higher dimensions, for instance via replacing $\mathbb{Z}$ by $\mathbb{Z}^d$ in the above example, and to many other stochastic systems, as treated in [7, Ch. 11] or in [3]. The constructive splitting approach gives a slightly different interpretation to the method put forward in [22, 3], where the splitting is done on the autocorrelation level by separating the mean from the fluctuations. This certainly deserves further clarification in the setting of the Bartlett spectrum from the theory of stochastic processes.

6.2. **Eberlein splitting for a random inflation.** Here, we return to the classic Fibonacci inflation that underlies Section 3 and turn it into a *random inflation* by setting

$$\varrho \colon \; b \mapsto a \, , \quad a \mapsto \begin{cases} ab, & \text{with probability } p, \\ ba, & \text{with probability } 1-p, \end{cases}$$

where $p \in [0,1]$ is fixed, and the rule is applied *locally*. This defines a system that was first analysed in [15], and has recently turned into an interesting paradigm for a random system with both long-range order and some form of disorder [31, 16, 17]. Here, we only consider the geometric setting where a and b stand for intervals of length τ and 1, respectively, with normalised Dirac measures on their left endpoints.

From [13, Thm. 3.19], we know that the diffraction measure almost surely satisfies

$$\widehat{\gamma} \, = \, \bigl(\widehat{\gamma}\bigr)_{\mathsf{pp}} + \bigl(\widehat{\gamma}\bigr)_{\mathsf{ac}}, \tag{10}$$

where explicit expressions can be given. Clearly, $(\widehat{\gamma})_{\mathsf{ac}} = 0$ for $p = 0$ or $p = 1$. To set this into our above scheme, let $\varLambda = \varLambda_a \,\dot{\cup}\, \varLambda_b$ be the (typed) control points of a typical realisation of the random Fibonacci inflation, say with $0 < p < 1$ to avoid the deterministic limiting cases. Then, as shown in [13], there are weighted Dirac combs ω_α, with $\alpha \in \{a, b\}$, of the form

$$\omega_\alpha \, = \sum_{x \in \mathbb{Z}[\tau]} h_\alpha(x^\star) \, \delta_x$$

with functions h_α that are continuous and supported on the window $[-\tau, \tau]$ of the covering model set. In particular, these Dirac combs are discretely supported in the regular model set $\curlywedge\bigl([-\tau, \tau]\bigr)$, and both have pure point spectrum [35]. What is more, as follows from [13], the pure point part of the diffraction of $\varLambda_\alpha$, agrees with the diffraction of ω_α.

Thus, using ω_α and defining $\nu_\alpha = \delta_{\varLambda_\alpha} - \omega_\alpha$, where the latter is a random measures for each $\alpha \in \{a, b\}$, we almost surely (in the sense of the underlying process) are in a situation analogous to the one from Theorem 5.1. In particular, for fixed $u_a, u_b \in \mathbb{C}$, the random weighted Dirac comb

$$\mu \, = \, u_a \, \delta_{\varLambda_a} + u_b \, \delta_{\varLambda_b}$$

almost surely has the autocorrelation $\gamma = (\gamma)_{\mathsf{s}} + (\gamma)_0$ where $(\gamma)_{\mathsf{s}}$ agrees with the autocorrelation of $u_a \omega_a + u_b \omega_b$ and $(\gamma)_0$ is the autocorrelation of $u_a \nu_a + u_b \nu_b$. Taking Fourier transforms brings us back to (10), where we refer to [23, 33, 13] for explicit formulas. Once again, this gives a constructive variant of the decomposition advocated in [2], which is fully compatible with the statistical separation of mean and variance from [22, 3].

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