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Exponential Bounds for the Density of the Law of the Solution of an SDE with Locally Lipschitz Coefficients

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Abstract: Under the uniform Hörmander hypothesis, we study the smoothness and exponential bounds of the density of the law of the solution of a stochastic differential equation (SDE) with locally Lipschitz drift that satisfies a monotonicity condition. We extend the approach used for SDEs with globally Lipschitz coefficients and obtain estimates for the Malliavin covariance matrix and its inverse. Based on these estimates and using the Malliavin differentiability of any order of the solution of the SDE, we prove exponential bounds of the solution's density law. These results can be used to study the convergence of implicit numerical schemes for SDEs.

Keywords: Malliavin covariance matrix; Hörmander's condition; exponential bounds for density; monotone growth; stochastic differential equation

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1. Introduction

We use Malliavin calculus to study the smoothness and exponential bounds for the density of the law of the solution of a stochastic differential equation (SDE) with a locally Lipschitz drift that satisfies a monotonicity condition. These exponential bounds are important, for example, to study the convergence rate of numerical schemes [1] for approximating the solutions of the SDE. SDEs with non-globally Lipschitz coefficients appear in models for financial securities and various models for dynamical systems [2–4].

We consider the SDE

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t), \quad X(0) = x \in \mathbb{R}^d, t \in [0,T], T > 0, \tag{1}$$

where $W(t)$ is an m-dimensional Brownian motion defined on the filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$. We make the following assumptions for the coefficients b and σ:

C: b and σ have derivatives of any order k with polynomial growth. More precisely, for any order $k \geq 1$ and any multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_d = k$, there exist $L_{b,k} > 0$, $L_{\sigma,k} > 0$ and $N_{b,k} \geq 0$, $N_{\sigma,k} \geq 0$, with $N_{\sigma,1} = 0$, such that for any $x \in \mathbb{R}^d$ we have

$$|\partial_\alpha b(x)| \leq L_{b,k}\left(1 + |x|^{N_{b,k}}\right), \quad |\partial_\alpha \sigma(x)| \leq L_{\sigma,k}\left(1 + |x|^{N_{\sigma,k}}\right). \tag{2}$$

M: There exists $L_b > 0$ such that for any $x_1, x_2 \in \mathbb{R}^d$ we have

$$< x_1 - x_2, b(x_1) - b(x_2) > \leq L_b |x_1 - x_2|^2 \tag{3}$$

P: There exist $L_{b,0} \geq 0$ and $N \geq 1$ such that for any $x_1, x_2 \in \mathbb{R}^d$ we have

$$|b(x_1) - b(x_2)|^2 \leq L_{b,0}(1 + |x_1|^{2N-2} + |x_2|^{2N-2})|x_1 - x_2|^2 \tag{4}$$

Supposing that b and σ are globally Lipschitz, C^∞, all their derivatives have polynomial growth, and Hörmander's hypothesis holds; in [5], it is shown that the strong solution of (1) is Malliavin-differentiable for any order and it is non-degenerate at any fixed positive time. Furthermore, an estimate for the Malliavin covariance matrix (Theorem 2.17 [5]) is used to show that the law of the solution of the SDE is absolutely continuous with respect to the Lebesgue measure, its density is infinity differentiable, and exponential bounds are proven under the uniform Hörmander hypothesis.

There are several approaches to extend these results for SDEs with non-globally Lipschitz coefficients. In [6], assuming that the coefficients of the SDE are smooth and non-degenerate on an open domain D, estimations of the Fourier transform are used to show that the law of the solution has a smooth density and upper bounds for this density are given. In [7], the Sobolev regularity of strong solutions with respect to the initial value is established for SDEs with local Sobolev and super-linear growth coefficients. For SDEs driven by fractional Brownian motions, in [8] it is shown that the density of the law of the solution is smooth and admits an upper sub-Gaussian bound in the rough case.

For SDEs with random coefficients with drifts satisfying locally Lipschitz and monotonicity conditions, in [9] the concepts of ray absolute continuity and stochastic Gâteaux differentiability are used to prove Malliavin differentiability and absolute continuity of the solution's law. In [10], we extend this result and under assumptions **C**, **M**, and **P** we show Malliavin differentiability of any order. Here, under assumptions **C**, **M**, and **P** we use the results in [9,10] to obtain an estimate for the Malliavin covariance matrix similar to the one in Theorem 2.17 in [5]. If, in addition, the uniform Hörmander hypothesis holds, we prove that the solution of the SDE is non-degenerate and we obtain exponential bounds for the density of the law of the solution of the SDE.

Recently, Malliavin calculus was used to study the convergence of various numerical schemes for SDEs with non-globally Lipschitz coefficients [11,12]. Without the global Lipschitz assumption, the Euler numerical scheme is no longer convergent [2], but under assumptions **C**, **M**, and **P**, the mean square convergence of a class of fully implicit methods is proven in [13]. In [1], the exponential bounds of the density obtained in [5] for SDEs with globally Lipschitz coefficients are used to find an expansion of the error for the explicit Euler scheme. An application of the results presented in this paper is to extend these results for fully implicit symplectic methods for stochastic Hamiltonian systems with coefficients satisfying assumptions **C**, **M**, and **P**.

The paper is organized as follows. In the next section, we present some results regarding the Malliavin differentiability of the solution of the SDE. Section 3 includes preliminary results about the Malliavin matrix and the statement of the main result. In Section 4, we include estimates for the Malliavin matrix, and based on these estimates in Section 5 we prove the exponential bounds for the density of the law of the solution of the SDE (1).

2. Notation and Results About Malliavin Differentiability

We denote by ∇f the gradient of a differentiable function $f : \mathbb{R}^d \to \mathbb{R}$, and for a vector-valued function $v : \mathbb{R}^d \to \mathbb{R}^d$ let ∇v denote the matrix with components $\partial v_{i,j}(x) = \frac{\partial v^i(x)}{\partial x_j}$, $i, j = 1, \ldots, d$. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$ with length $|\alpha| = \alpha_1 + \cdots + \alpha_d$, let ∂_α denote the partial derivative of order $|\alpha|$. If ϕ is a smooth function, we denote by $\partial_x^\alpha \phi(t, x, y)$ the derivation with respect to the coordinates of x, where t and y are fixed.

For a vector $v \in \mathbb{R}^l$, we denote by $|v| := \left(\sum_{i=1}^{l} v_i^2\right)^{\frac{1}{2}}$ the Euclidean norm, and if $A = (a_{ij})$ is an $l_1 \times l_2$ matrix we denote by $|A| := \left(\sum_{i=1}^{l_1} \sum_{j=1}^{l_2} a_{ij}^2\right)^{\frac{1}{2}}$ the Frobenius norm. For two vectors $u, v \in \mathbb{R}^l$, we denote $< u, v >= \sum_{i=1}^{l} u_i v_i$, and for two $l_1 \times l_2$ matrices A, B, $< A, B >= \sum_{i=1}^{l_1} \sum_{j=1}^{l_2} a_{ij} b_{ij}$ denotes the Frobenius inner product.

We consider the Banach space $(C([0,T]), \|\cdot\|_\infty)$, where $C([0,T]) := \{\phi : [0,T] \to \mathbb{R},\ \phi$ uniformly continuous and bounded$\}$, $\|\phi\|_\infty = \sup_{x\in[0,T]} |\phi(x)|$, and we denote $C_0([0,T]) := \{\phi \in C([0,T]), \phi(0) = 0\}$. We define $\mathcal{C}_p^\infty(\mathbb{R}^l) := \{\phi \in \mathcal{C}^\infty(\mathbb{R}^l)$, ϕ and all its derivatives are functions with polynomial growth $\}$.

For any open set $E \subseteq \mathbb{R}^d$ and $n \in \mathbb{N}$, we denote $C_b^n(E, \mathbb{R}^d) = \{f \in C^n(E, \mathbb{R}^d), f$ and all its derivatives of order at most n are bounded $\}$ with the norm $\|f\|_{C_b^n(E,\mathbb{R}^d)} = \max_{0\le i\le n} \sup_{x\in E} |\partial_x^i f(x)|$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ be a filtered probability space. For any separable Banach space $(E, \|\cdot\|)$, we denote $L^p(\Omega; E) = \{X : \Omega \to E$, X is $\mathcal{F}$-measurable and $\|X\|_p = E[\|X\|^p]^{1/p} < \infty\}$. Let L^∞ be the subset of bounded random variables with norm $\|X\|_{L^\infty} = ess\sup_{\omega\in\Omega} |X(\omega)|$.

Let $S^p([0,T], \mathbb{R}^d) = \{(Y_t)_{t\in[0,T]}$ stochastic processes, $Y_t \in \mathbb{R}^d$, that are $\{\mathcal{F}_t\}_{t\in[0,T]}$, adapted, and $\|Y\|_{S^p} = E[|Y\|_\infty^p]^{1/p} = E[\sup_{t\in[0,T]} |Y(t)|^p]^{1/p} < \infty\}$. Let $S^\infty([0,T], \mathbb{R}^d) = \cap_{p\ge 1} S^p([0,T], \mathbb{R}^d)$.

2.1. Malliavin Calculus

Let $\Omega = C_0([0,T], \mathbb{R}^m) = \{\omega : [0,T] \to \mathbb{R}^m, \omega = (\omega_1, \ldots, \omega_m)^\top$, $\omega_i \in C_0([0,T])$, $i = 1, \ldots, m\}$ be the canonical Wiener space, and $W := (W_t^1, \ldots, W_t^m)_{t\in[0,T]}^\top$ be the canonical Wiener process defined as $W_t^i(\omega) := \omega_t^i$ for any $\omega \in \Omega$, $i = 1, \ldots, m$. We set $\mathcal{F}^0$ as the natural filtration of W, $\mathbb{P}$ the Wiener measure, and $\mathcal{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ the usual augmentation (which is right-continuous and complete) of $\mathcal{F}^0$. In this setting, W is a standard Brownian motion.

We denote by $\mathcal{H} := L^2([0,T]; \mathbb{R}^m) = \{f : [0,T] \to \mathbb{R}^m$ Borel-measurable and $\int_0^T |f(s)|^2 ds < \infty\}$, and the canonical inner product is

$$< f, g >_{\mathcal{H}} := \int_0^T < f(s), g(s) > ds = \sum_{i=1}^{m} \int_0^T f^i(s) g^i(s) ds,\ f, g \in \mathcal{H}.$$

Let H be the Cameron–Martin space:

$$H := \left\{ h : [0,T] \to \mathbb{R}^m, h \in \Omega, \text{there exists } \dot{h} \in \mathcal{H}, \text{ such that } h(t) = \int_0^t \dot{h}(s) ds, t \in [0,T] \right\}$$

For $h \in H$, we denote by $\dot{h}$ a version of its Radon–Nykodym density with respect to the Lebesgue measure. For any Hilbert space K we define $L^p(K) = \{f : \Omega \to K$, f is $\mathcal{F}_T$-measurable and $\|f\|_{L^p(K)}^p := (E[\|f\|_K^p)^{1/p} < \infty\}$. Let

$$W(h) := \int_0^T \dot{h}_s dW_s := \sum_{i=1}^{m} \int_0^T \dot{h}_s^i dW_s^i,\ h \in H.$$

Following [14], we set

$$\mathcal{S} := \left\{ F : \Omega \to \mathbb{R}, F = f(W(h_1), \ldots, W(h_n)), f \in C_b^\infty(\mathbb{R}^n), \right.$$
$$\left. h_i \in H, i = 1, \ldots, n, \text{ for some } n \in \mathbb{N}, n \ge 1 \right\}$$

For any $F \in \mathcal{S}$ we define the Malliavin derivative $\mathcal{D}F : \Omega \to H$ by

$$\mathcal{D}F := \sum_{i=1}^{n} \partial_i f(W(h_1), \ldots, W(h_n))h_i.$$

We identify $\mathcal{D}F$ with the stochastic process $\{\mathcal{D}_t F\}_{t\in[0,T]}$, where $\mathcal{D}_t F \in \mathbb{R}^m$ and

$$\mathcal{D}_t F(\omega) = \sum_{i=1}^{n} \partial_i f(W(h_1)(\omega), \ldots, W(h_n)(\omega))h_i(t),\ (t,\omega) \in [0,T] \times \Omega.$$

$\mathcal{D}_t^j F$ denotes the jth component of $\mathcal{D}_t F$. We denote by $\mathbb{D}^{1,p}$, $p \geq 1$ the closure of $\mathcal{S}$ with respect to the semi-norm

$$\|F\|_{1,p} := \Big(E[|F|^p] + E[\|\mathcal{D}F\|_H^p\Big)^{1/p},$$

and we set $\mathbb{D}^{1,\infty} = \cap_{p\geq 2}\mathbb{D}^{1,p}$.

The kth order Malliavin derivative $\mathcal{D}^k F : \Omega \to H^k$ is defined iteratively and its components are $\mathcal{D}^{j_1,\ldots,j_k}_{t_1,\ldots,t_k} F := \mathcal{D}^{j_k}_{t_k} \ldots \mathcal{D}^{j_1}_{t_1} F$, with $(t_1, \ldots, t_k)^\top \in [0,T]^k$, $j_1, \ldots, j_k \in \{1, \ldots, m\}$. For the Nth order Malliavin derivative, $\mathbb{D}^{N,p}$, $p \geq 1$ is the closure of $\mathcal{S}$ with the semi-norm

$$\|F\|_{N,p} := \left(E[|F|^p] + \sum_{i=1}^{N} E\Big[\|\mathcal{D}^i F\|_{H^i}^p\Big]\right)^{1/p} = \left(\|F\|^p_{L^p(\Omega)} + \sum_{i=1}^{N} \|\mathcal{D}^i F\|^p_{L^p\left(\Omega; L^2([0,T]^i, \mathbb{R}^m)\right)}\right)^{\frac{1}{p}}.$$

We set $\mathbb{D}^\infty = \cap_{p\geq 2} \cap_{i\geq 1} \mathbb{D}^{i,p}$

The definition of Malliavin derivative can be extended to mappings $G : \Omega \to E$, where $(E, \|\cdot\|_E)$ is a separable Banach space ([9]). We consider the family

$$\mathcal{S}_E := \left\{G : \Omega \to E, G = \sum_{j=1}^{k} F_j e_j, F_j \in S,\ e_j \in E \text{ for some } k \in \mathbb{N}, k \geq 1\right\}$$

$\mathcal{S}_E$ is dense in $L^p(\mathcal{F}; E; \mathbb{P})$ [9]. For any $G \in \mathcal{S}_E$ we define the Malliavin derivative $\mathcal{D}G$: $\Omega \to H \otimes E$ by

$$\mathcal{D}G := \sum_{j=1}^{k} \mathcal{D}F_j \otimes e_j$$

We denote by $\mathbb{D}^{1,p}(E)$, $p \geq 1$ the closure of $\mathcal{S}_E$ with respect to the semi-norm

$$\|G\|_{1,p,E} := \Big(E[\|G\|_E^p] + E[\|\|\mathcal{D}G\|_H\|_E^p\Big)^{1/p},$$

2.2. The Solution of the SDE

Assumption **C** and $N_{\sigma,1} = 0$ imply that b is locally Lipschitz and σ is globally Lipschitz. Moreover, from assumptions **C**, **M**, and **P** we obtain that there exists $C > 0$ such that for any $x, y \in \mathbb{R}^d$,

$$y^\top \nabla_x b(x) y \leq L_b |y|^2, \tag{5}$$
$$|\sigma(x)|^2 \leq C(1 + |x|^2), \tag{6}$$
$$|\nabla_x \sigma(x)|^2 \leq C, \tag{7}$$
$$|b(x)|^2 \leq C(1 + |x|^{2N}), \tag{8}$$
$$|\nabla_x b(x)|^2 \leq C(1 + |x|^{2N_{b,1}}). \tag{9}$$

From assumption **C**, (6), and Theorems 3.6 in [15], we know that there exists a unique global solution $X(t,0,x)$ of the SDE (1). From Theorems 9.1 and 9.5 in [15], we know that $(X(t,0,x))_{t\geq 0}$ is a time-homogeneous $\{\mathcal{F}_t\}_{t\geq 0}$-adapted Markov process and we have

$$E\left(\int_0^T |X(t,0,x)|^2 dt\right) < \infty. \tag{10}$$

Moreover, from Theorem 4.1 in [15] we know that for any $p \geq 2$ there exists a constant $\alpha_p > 0$ such that we have

$$E[|X(t,0,x)|^p] \leq 2^{\frac{p-2}{2}}(1+|x|^p)e^{p\alpha_p t}, \quad t \in [0,T]. \tag{11}$$

From Theorem 2.2 in [9], we know that the map $t \to X(t)(\omega)$ is $\mathbb{P}$ almost surely continuous, and for any $p \geq 2$ we have $X \in S^p := S^p([0,T],\mathbb{R}^d)$ and there exists $C > 0$ depending on p, b, and σ such that

$$E\left[\sup_{t\in[0,T]} |X(t,0,x)|^p\right] < C(|x|^p + 1). \tag{12}$$

From inequalities Equations (6)–(9), (12), and assumption **C**, for any multi-index $\alpha = (\alpha_1,\dots,\alpha_d)$ and any $p \geq 2$ we obtain

$$E\left[\sup_{t\in[0,T]} |\partial_\alpha b(X(t))|^p\right] < \infty,\ E\left[\sup_{t\in[0,T]} |\partial_\alpha \sigma(X(t))|^p\right] < \infty. \tag{13}$$

From Corollary 3.5 and Theorem 3.21 in [9], we know that X is Malliavin-differentiable and $\mathcal{D}X \in S^p\big([0,T],L^2([0,T])\big)$ for any $p \geq 2$:

$$E\left[\left(\sup_{t\in[0,T]}\int_0^T |\mathcal{D}_s X(t)|^2 ds\right)^{p/2}\right] < \infty \tag{14}$$

Thus, from (12) and (14) we have $X \in \mathbb{D}^{1,p}(S^p)$ for any $p \geq 2$, so $X \in \mathbb{D}^{1,\infty}(S^p) = \cap_{p\geq 2}\mathbb{D}^{1,p}(S^p)$. Thus, similarly with the case in Theorem 2.2.1 in [16] of globally Lipschitz coefficients, for any $t \in [0,T]$ we have $X(t) \in \mathbb{D}^{1,\infty}$.

For any fixed $s \in [0,T]$ and $i = 1,\dots,m$, from (6), (5), (10), and Theorem 2.5 in [9], we have for any $p \geq 2$,

$$E\left[\sup_{t\in[0,T]} |\mathcal{D}_s^i X(t)|^p\right] \leq CE\Big[|\sigma^i(X(s,0,x))|^p\Big] \leq C(1+E[|X(s))|^p])$$

This and (11) imply that for any $t \in [0,T]$ and any $p \geq 2$,

$$\|X(t,0,x)\|_{1,p}^p = E[|X(t,0,x)|^p] + E\left[\left|\int_0^T |\mathcal{D}_s X(t,0,x)|^2 ds\right|^{p/2}\right] \leq C_{1,p}(T)(1+|x|^p), \tag{15}$$

where $C_{1,p}(T) > 0$ depends on p, T, b, and σ.

In [10], we extend this result and show that under assumptions **C**, **M**, and **P**, $X^i(t)$ belongs to $\mathbb{D}^\infty$ for all $t \in [0,T]$, and $i = 1,\dots,d$. Moreover, for any $t \in [0,T]$, $p \geq 2$, $k = 1,2,\dots$, there exist $C_{k,p}(T), \beta_{k,p} > 0$ depending on p, k, T, b, and σ such that

$$\|X(t,0,x)\|_{k,p}^p \leq C_{k,p}(T)(1+|x|^{\beta_{k,p}}). \tag{16}$$

3. The Main Result

From Theorem 4.9 in [9] we know that under assumptions **C**, **M**, and **P** the matrix-valued SDE

$$J(t) = I_d + \int_0^t \nabla_x b(X(s,0,x))J(s)ds + \int_0^t \nabla_x \sigma(X(s,0,x))J(s)dW(s), \tag{17}$$

$t \in [0,T]$, has a unique solution $J \in S^p\Big([0,T], \mathbb{R}^{d\times d}\Big)$, $p \geq 2$, and for any $t \in [0,T]$ the map $x \to X(t,0,x)$ is differentiable $\mathbb{P}-$ a.s. and as $\epsilon \to 0$,

$$\frac{X(t,0,x+\epsilon h)(\omega) - X(t,0,x)(\omega)}{\epsilon} \to hJ(t)(\omega)\ \mathbb{P}\ a.s.. \tag{18}$$

We consider the matrix-valued SDE

$$K(t) = I_d - \int_0^t K(s)[\nabla_x b(X(s,0,x)) - <\nabla_x\sigma, \nabla_x\sigma> (X(s,0,x))]ds$$
$$- \int_0^t K(s)\nabla_x\sigma(X(s,0,x)dW(s), \quad t \in [0,T]. \tag{19}$$

From Theorem 2.5 and Propositions 4.13 and 4.14 in [9], we know that under assumptions **C**, **M**, and **P** we have $K(t)J(t) = I_d$ for all $t \in [0,T]$ $\mathbb{P}-$a.s.. Consequently, the Jacobian matrix $J(t)$ is $\mathbb{P}$-a.s. invertible for any choice of $t \in [0,T]$, and $J(t)^{-1} = K(t)$ $\mathbb{P}-$a.s.. In [9], it was noted that since $-y^\top\nabla_x b(X(s,0,x))y$ is not bounded from above by a constant $\mathbb{P}-$a.s. for any choice of y with $|y| = 1$, an explicit solution of Equation (19) can be written path-wise, but it might not have finite moments.

Let $J_s(t) = J(t)J(s)^{-1}$, $t > s$. Under assumptions **C**, **M**, and **P** from Proposition 5.1 in [9], we know that we have

$$J_s(t) = I_d + \int_s^t \nabla_x b(X(r,0,x))J_s(r)dr + \int_s^t \nabla_x\sigma(X(r,0,x))J_s(r)dW(r), \tag{20}$$

and the Malliavin derivative of X can be expressed for $t > s$ as $\mathcal{D}_s X(t,0,x) = J_s(t)\sigma(X(s,0,x))$. The Malliavin matrix $Q(t)$ is defined by

$$Q(t,x) := \int_0^t \mathcal{D}_s X(t,0,x)\mathcal{D}_s X(t,0,x)^\top ds = J(t)C(t,x)J(t)^\top \tag{21}$$
$$C(t,x) := \int_0^t J(s)^{-1}\sigma(X(s,0,x))\sigma(X(s,0,x))^\top(J(s)^{-1})^\top ds \tag{22}$$

The Lie bracket of the $C^1(\mathbb{R}^d, \mathbb{R}^d)$ vector fields $V = \sum_{i=1}^d V^i \frac{\partial}{\partial x_i}$, $U = \sum_{i=1}^d U^i \frac{\partial}{\partial x_i}$ is defined as $[V,U](x) = \partial U(x)V(x) - \partial V(x)U(x)$, where $\partial U = (\partial_i U^j)_{i,j=1,\ldots d}$, $\partial V = (\partial_i V^j)_{i,j=1,\ldots d}$ are the Jacobian matrices of U and V, respectively. Let us denote $\sigma^0 = b - \frac{1}{2}\sum_{i=1}^m \sum_{j=1}^d \sigma_j^i \partial_j \sigma^i$ and let $\sigma^0, \ldots, \sigma^m$ be the corresponding vector fields:

$$\sigma^0(x) = \sum_{i=1}^d \sigma_i^0(x)\frac{\partial}{\partial x_i}, \quad \sigma^j(x) = \sum_{i=1}^d \sigma_i^j(x)\frac{\partial}{\partial x_i}, \quad j = 1, \ldots, m.$$

We construct by recurrence the sets $\Sigma_0 = \{\sigma^j, j = 1, \ldots, m\}$, $\Sigma_k = \{[\sigma^j, V], j = 0, \ldots, m,$ $V \in \Sigma_{k-1}\}$, $k \geq 1$, and $\Sigma_\infty = \cup_{k=1}^\infty \Sigma_k$. We denote by $\Sigma_k(x)$ the subset of $\mathbb{R}^m$ obtained by freezing the variable $x \in \mathbb{R}^d$ in the vector fields of Σ_k. For $x \in \mathbb{R}^d$ we consider Hörmander's hypothesis:

H(x): The vector space $Span\{\Sigma_\infty(x)\} = \mathbb{R}^d$.

If we have the ellipticity condition at $x \in \mathbb{R}^d$, i.e., for $A(x) = \sigma(x)\sigma(x)^\top$ there exists C>0 such that $y^\top A(y)y \geq C|y|^2$ for any $y \in \mathbb{R}^d$, then Hörmander's hypothesis **H(x)** holds. The interesting applications appear when $A(x)$ is degenerate at x.

Example 1. *It is easy to check that assumptions* **C**, **M**, *and* **P** *hold for the coefficients of the following stochastic version with multiplicative noise of the Ginzburg–Landau equation [2] used in the theory of superconductivity to describe a phase transition:*

$$dX(t) = \left(\mu + \left(\eta + \frac{1}{2}\bar{\sigma}^2\right)X(t) - \lambda X(t)^3\right)dt + \bar{\sigma}X(t)dW(t), \quad X(0) = x_0 \in \mathbb{R}, \tag{23}$$

where $t \in [0,T]$, $\eta \geq 0$, $\mu, \lambda, \bar{\sigma} > 0$. *We have* $A(x) = \bar{\sigma}x^2$, *so* $A(0) = 0$. *However, a simple calculation shows that Hörmander's hypothesis* **H(x)** *holds for any* $x \in \mathbb{R}$.

As in the Appendix in [5], let $\mathcal{A} = \{\emptyset\} \cup \bigcup_{i=1}^{\infty}(\{0,\ldots,m\})^i$. Given $\alpha = (\alpha_1,\ldots,\alpha_i) \in \mathcal{A}\backslash\{\emptyset\}$, we define $\alpha_* = \alpha_i$ and

$$\alpha' = \begin{cases} \emptyset, & \text{if } i = 1 \\ (\alpha_1,\ldots,\alpha_{i-1}), & \text{if } i \geq 2. \end{cases}$$

Given $\alpha \in \mathcal{A}$, set

$$|\alpha| = \begin{cases} 0 & \text{if } \alpha = \emptyset \\ i & \text{if } \alpha \in (\{0,\ldots,m\})^i, \end{cases} \qquad \|\alpha\| = \begin{cases} 0 & \text{if } \alpha = \emptyset \\ |\alpha| + \text{card}\{j : \alpha_j = 0\} & \text{if } |\alpha| \geq 1. \end{cases}$$

We define $T_{(\alpha)}$ and $I^{(\alpha)}(t)$ inductively on $|\alpha|$ by

$$T_{(\alpha)}(V) = \begin{cases} V & \text{if } \alpha = \emptyset \\ [\sigma^{\alpha_*}, T_{(\alpha')}(V)] & \text{if } \alpha \neq \emptyset, \end{cases}, \quad V \in C^\infty(\mathbb{R}^d,\mathbb{R}^d)$$

$$I^{(\alpha)}(t) = \begin{cases} 1 & \text{if } \alpha = \emptyset, \\ \int_0^t I^{(\alpha')}(s) \circ dW^{\alpha^*}(s) & \text{if } |\alpha| \geq 1, \end{cases}$$

where we consider $W^0(t) = t$, $t \in [0,T]$.

Given $L \geq 1$, we define for any $x, \eta \in \mathbb{R}^d$

$$\mathcal{V}_L(x,\eta) = \sum_{k=1}^{m} \sum_{\|\alpha\| \leq L-1} < T_{(\alpha)}(\sigma^k)(x), \eta >^2, \quad \mathcal{V}_L(x) = \inf_{|\eta|=1} \mathcal{V}_L(x,\eta) \wedge 1.$$

Let

$$U_L := \{x \in \mathbb{R}^d, \mathcal{V}_L(x) > 0\}, \quad U = \bigcup_{L=1}^{\infty} U_L$$

Notice that for $L \in \mathbb{N}^* = \{1,2,\ldots\}$, the hypothesis

$\mathrm{H_L}(x)$: $Span\{\phi(x), \phi \in \cup_{i=1}^{L}\Sigma_i\} = \mathbb{R}^d$

is equivalent with $x \in U_L$. As in [1], we consider the following assumption:
UH: For some integer $L_0 > 0$, we have $C_{L_0} := \inf_{x \in \mathbb{R}^d} V_{L_0}(x) > 0$.
Notice that assumption **UH** implies $U = \mathbb{R}^d$ and Hörmander's hypothesis **H(x)** is true for any $x \in \mathbb{R}^d$.

Suppose that **H(x)**, **C**, **M**, and **P** hold. Based on assumption **H(x)**, (13), Formulas (21) and (22) for the Malliavin matrix $Q(t,x)$, and proceeding as in the proof of Theorem 2.3.2 in [16], we can show that the Malliavin matrix $Q(t,x)$ is invertible almost surely. Thus, since from (15) we also know that $X(t,0,x) \in \mathbb{D}^{1,p}$ for any $p \geq 2$, this implies that the law of $X(t,0,x)$ is absolutely continuous with respect to the Lebesgue measure (Theorem 2.2.1 in [16]). Here, we replace assumption **H(x)** with assumption **UH** and we obtain an exponential bound for the density of the law of $X(t,0,x)$ with respect to the Lebesgue measure.

Theorem 1. *Let X be the solution of SDE (1) and suppose that the assumptions* **C**, **M**, **P**, *and* **UH** *hold. Then, for any $t \in (0,T]$ and any $x \in \mathbb{R}^d$ the law of the random vector $X(t,0,x)$ is absolutely continuous with respect to the Lebesgue measure, and for the density $y \to p_t(x,y)$ the following inequalities hold:*

$$p_t(x,y) \leq \frac{K_0(T)(1+|x|^{Q_0})}{t^{q_0}} \exp\left(-C_0 \frac{(|x-y| \wedge 1)^2}{t(1+|x|)^{2N}}\right), \tag{24}$$

$$\left|\partial_y^\alpha p_t(x,y)\right| \leq \frac{K_\alpha(T)(1+|x|^{Q_\alpha})}{t^{q_\alpha}} \exp\left(-C_\alpha \frac{(|x-y| \wedge 1)^2}{t(1+|x|)^{2N}}\right), \tag{25}$$

for any $x,y \in \mathbb{R}^d$, $t \in (0,T]$, $t \leq 1 \wedge (|y-x| \wedge 1)/(4M(x))$, where N is as in (8) and $M(x) = \sup_{z \in B(x,1)} \{\|\sigma(z)\| \vee |b(z)|\}$. Here, the non-decreasing functions K_0 and K_α and the positive real numbers C_0, C_α, Q_0, and Q_α depend on $L \in \mathbb{N}^*$ such that $x \in U_L$, and on the coefficients b and σ.

The proof is included in Section 5.

Example 2. *For the coefficients of the SDE (23), a simple calculation shows that assumption* **UH** *holds with $L_0 = 3$. Thus, we can apply Theorem 1 and obtain the exponential bounds (24) and (25).*

Example 3. *The following SDE includes a family of nonlinear mean-reverting models for interest rates [4]:*

$$dX(t) = \left(\mu - \lambda X(t)^3\right)dt + \bar{\sigma} X(t) dW(t), \quad X(0) = x_0 > 0, \tag{26}$$

where $t \in [0,T]$, $\mu, \lambda, \bar{\sigma} > 0$. We can easily check that the assumptions **C**, **M**, **P**, *and* **UH** *are met, so we can apply Theorem 1 and obtain the exponential bounds (24) and (25).*

4. Results About the Malliavin Matrix

Notice that we can write Equations (1), (17), (19), and (20) in Stratonovich form as follows:

$$X(t) = x + \int_0^t \sigma^0(X(u))du + \sum_{i=1}^m \int_0^t \sigma^i(X(u)) \circ dW^i(u), \tag{27}$$

$$J(t) = I_d + \int_0^t \nabla_x \sigma^0(X(s))J(s)ds + \sum_{i=1}^m \int_0^t \nabla_x \sigma^i(X(s))J(s) \circ dW^i(s), \tag{28}$$

$$J^{-1}(t) = I_d - \int_0^t J^{-1}(s)\nabla_x \sigma^0(X(s))ds - \sum_{i=1}^m \int_0^t J^{-1}(s)\nabla_x \sigma^i(X(s)) \circ dW^i(s), \tag{29}$$

$$J_s(t) = I_d + \int_s^t \nabla_x \sigma^0(X(r))J_s(r)dr + \sum_{i=1}^m \int_s^t \nabla_x \sigma^i(X(r))J_s(r) \circ dW^i(r). \tag{30}$$

Given $V \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$, we use Ito's formula and Equations (27) and (29) (Equation 2.10 in [5], Equation 2.63, p. 130 in [16]) to obtain

$$J^{-1}(t)V(X(t)) = V(x) + \int_0^t J^{-1}(s)[\sigma^0, V](X(s))ds + \sum_{i=1}^m \int_0^t J^{-1}(s)[\sigma^i, V](X(s)) \circ dW^i(s) \quad (31)$$

Theorem 2. *Suppose that assumptions* **C**, **M**, *and* **P** *hold. For any $L \geq 1$ and $0 < \epsilon \leq 1$ there exist $C_{L,\epsilon}, \lambda(L,\epsilon), \mu_{L,\epsilon} > 0$ such that for all $x \in \mathbb{R}^d$ and $V \in C_p^\infty(\mathbb{R}^d, \mathbb{R}^d)$ we have*

$$J^{-1}(t)V(X(t)) = \sum_{\|\alpha\| \leq L-1} T_{(\alpha)}(V)(x) I^{(\alpha)}(t) + R_L(t,x,V) \quad (32)$$

for $K \geq K_0(L,x,V) > 1$, with

$$\sup_{0<t<1} \mathbb{P}\left(\frac{1}{t^L} \int_0^{t/K} |R_L(s,x,V)|^2 ds \geq \frac{1}{K^{L+1-\epsilon}} \right) \leq C_{L,\epsilon} \exp\left(\frac{-\lambda(L,\epsilon) K^{\mu_{L,\epsilon}}}{(1+M(x))^2} \right) \quad (33)$$

with

$$M(x) = \max\{\|\sigma^i\|_{C_b^2(B(x,1),\mathbb{R}^d)}, i = 0, \ldots, m\} \vee \max\{\|T^{(\alpha)}(V)\|_{C_b^0(B(x,1),\mathbb{R}^d)}, |\alpha| \leq L+1\}$$

The proof is included in Appendix A.

Theorem 3. *Suppose that assumptions* **C**, **M**, *and* **P** *hold. For any $L \geq 1$ there exists $C(L)$, $\tilde{C}(L) > 0$, $\lambda(L)$, $\tilde{\lambda}(L) > 0$, and μ_L, $\tilde{\mu}_L \in (0,1]$, all of them independent of $\sigma^0, \ldots, \sigma^m$, such that for all $t \in (0,1]$ and all $K \geq 1$ we have*

$$P\left(\frac{\tilde{\lambda}\left(\frac{t}{K^{1/(L+1)}}, x \right)}{t^L} \leq \frac{1}{K} \right) \leq \tilde{C}(L) \exp\left(-\frac{\tilde{\lambda}_L (\mathcal{V}_L(x)^{L+2} K)^{\tilde{\mu}_L}}{(1+M(x))^2} \right) \quad (34)$$

$$P\left(\frac{\lambda\left(\frac{t}{K^{1/(L+1)}}, x \right)}{t^L} \leq \frac{1}{K} \right) \leq C(L) \exp\left(-\frac{\lambda_L (\mathcal{V}_L(x)^{L+2} K)^{\mu_L}}{(1+M(x))^2} \right), \quad (35)$$

where

$$\tilde{\lambda}(s,x) = \inf_{|\eta|=1} < \eta, C(s,x)\eta >, \quad \lambda(s,x) = \inf_{|\eta|=1} < \eta, Q(s,x)\eta > \quad (36)$$

$$M(x) = \max\{\|T^{(\alpha)}(\sigma^k)\|_{C_b^2(B(x,1),\mathbb{R}^d)}, k = 0, \ldots, m, |\alpha| \leq L+1\} \quad (37)$$

The proof is given in Appendix B.

Let us denote

$$\Delta(t,x) := \det(Q(t,x)), \quad \tilde{\Delta}(t,x) := \det(C(t,x))$$

Equations (21), (22), and (36) imply that $C(t,x)$ and $Q(t,x)$ are positive semi-definite and both $\tilde{\lambda}(t,x)$ and $\lambda(t,x)$ are non-decreasing with respect to t.

Theorem 4. *Suppose that assumptions* **C**, **M**, *and* **P** *hold, and let $L \in \mathbb{N}^*$. For any $p \geq 1$ there exists $K(L,p)$, $\mu(L) > 0$, also depending on the coefficients b and σ, such that we have for any $t \in (0,1]$ and any $x \in U_L$*

$$E\left[\left|\frac{1}{\Delta(t,x)}\right|^p\right] \le K(L,p)\frac{(1+|x|^2)^{p\mu(L)}}{(\mathcal{V}_L(x)^{1+2/L}t)^{pdL}} \tag{38}$$

Proof. We know from (22) that $Q(t,x)$ is positive semi-definite, so that the smallest eigenvalue of $Q(t,x)$ is equal with $\lambda(t,x)$, and we have $(\lambda(t,x))^d \le \det(Q(t,x))$. These imply that for any $p \ge 1$,

$$E\left[\left|\frac{1}{(\Delta(t,x))^p}\right|\right] \le E\left[\frac{1}{\lambda(t,x)^{pd}}\right]. \tag{39}$$

Next, since $t \to \lambda(t,x)$ is non-decreasing, from (35) we obtain for any $t \in (0,1]$ and $K \ge 1$,

$$\mathbb{P}\left(\frac{\lambda(t,x)}{t^L} \le \frac{1}{K}\right) \le \mathbb{P}\left(\frac{\lambda(t/K^{1/(L+1)},x)}{t^L} \le \frac{1}{K}\right) \le C(L)\exp\left(-\frac{\lambda_L(\mathcal{V}_L(x)^{L+2}K)^{\mu_L}}{(1+M(x))^2}\right),$$

where $C(L), \lambda_L > 0$, $\mu_L \in (0,1]$, and $M(x)$ are as in Theorem 3. Using this, we obtain

$$\begin{aligned}
E\left[\frac{1}{\lambda(t,x)^{pd}}\right] &= \int_0^\infty pdy^{pd-1}\mathbb{P}\left(\lambda(t,x)^{-1} > y\right)dy = \int_0^\infty pdy^{pd-1}\mathbb{P}\left(\frac{\lambda(t,x)}{t^L} < \frac{1}{yt^L}\right)dy \\
&= \int_0^{1/A} pdy^{pd-1}\mathbb{P}\left(\frac{\lambda(t,x)}{t^L} < \frac{1}{yt^L}\right)dy + \int_{1/A}^\infty pdy^{pd-1}\mathbb{P}\left(\frac{\lambda(t,x)}{t^L} < \frac{1}{yt^L}\right)dy \\
&\le pd\left(\frac{1}{A}\right)^{pd} + \int_{1/A}^\infty pdy^{pd-1}C(L)\exp\left(-\frac{\lambda_L(Ay)^{\mu_L}}{(1+M(x))^2}\right)dy \\
&= pd\left(\frac{1}{A}\right)^{pd} + pd\frac{C(L)}{\mu_L}\left(\frac{1}{A}\right)^{pd}\int_1^\infty z^{(pd)/\mu_L-1}\exp\left(-\frac{\lambda_L z}{(1+M(x))^2}\right)dz \\
&\le pd\left(\frac{1}{A}\right)^{pd} + pd\frac{C(L)}{\mu_L}\left(\frac{1}{A}\right)^{pd}\int_1^\infty z^{k-1}\exp\left(-\frac{\lambda_L z}{(1+M(x))^2}\right)dz,
\end{aligned}$$

where

$$A := \mathcal{V}_L(x)^{L+2}t^L \in (0,1], \quad k = \left\lfloor\frac{pd}{\mu_L}\right\rfloor + 1 \ge 1.$$

By repeatedly applying integration by parts, we obtain

$$E\left[\frac{1}{\lambda(t,x)^{pd}}\right] \le C_1(L,p)\left(\frac{1}{A}\right)^{pd}(1+M(x))^{2pd/\mu_L+1}$$

From assumption **C** and (6)–(9) we obtain

$$M(x)^2 \le C_1(1+|x|^{2N}).$$

This yields

$$E\left[\frac{1}{\lambda(t,x)^{pd}}\right] \le K(L,p)\frac{(1+|x|^2)^{2Npd/\mu_L+N}}{(\mathcal{V}_L(x)^{1+2/L}t)^{pdL}} \le K(L,p)\frac{(1+|x|^2)^{Npd(2/\mu_L+1)}}{(\mathcal{V}_L(x)^{1+2/L}t)^{pdL}}$$

Thus, substituting $\mu_L \in (0,1]$ into (39) we obtain (38). □

5. Proof of Theorem 1

Proof. From Theorem 4 we obtain that $1/\Delta(t,x) \in \bigcap_{p\ge1} L^p(\Omega,\mathbb{R})$, and since $X(t,0,x) \in \mathbb{D}^\infty$, $X(t,0,x)$ is non-degenerate for any $t \in (0, T\wedge 1]$, $x \in \mathbb{R}^d$ (see Definition 2.1.1 in

[16]). Thus, we have the integration by parts formulas (Proposition 2.1.4 in [16]): for any $\phi \in C_p^\infty(R^d)$, $G \in \mathbb{D}^\infty$ and any index α, there exists $H_\alpha(X(t,0,x),G) \in \mathbb{D}^\infty$ such that

$$E[\partial_\alpha \phi(X(t,0,x))G] = E[\phi((X(t,0,x))H_\alpha(X(t,0,x),G)].$$

Moreover, for any $1 < p < q_1 < \infty$ there exist constants $C_{p,q} > 0$, $\beta, \gamma > 1$ and $k_1, k_2 \in \mathbb{N}^*$ such that

$$\|H_\alpha(X(t,0,x),G)\|_p \le C_{p,q_1} \left\| \frac{1}{\Delta(t,x)} \right\|_\beta^{k_1} \|\mathcal{D}X(t,0,x)\|_{|\alpha|,\gamma}^{k_2} \|G\|_{|\alpha|,q_1}. \tag{40}$$

Using this, it can be shown (Proposition 2.1.5 in [16]) that the density p_t belongs to the Schwarz space $\mathcal{S}(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{R} : f \in C^\infty(\mathbb{R}^d), \sup_{x\in\mathbb{R}^d} |x|^k |\partial_\alpha f(x)| < \infty$ for any $k \ge 1$, and any index $\alpha\}$. Moreover, for any $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$ such that $y_i > 0 \vee x_i$, $i = 1, \ldots, d$, we have

$$p_t(x,y) = E[\mathbb{1}_{I_y}(X(t,0,x))H_{(1,\ldots,1)}(X(t,0,x),1)], \quad I_y = \prod_{i=1}^d [y_i, \infty).$$

Using the Cauchy–Schwarz inequality, (40), (38) in Theorem 4, and (16) we obtain that for any $x \in \mathbb{R}^d$, $t \in (0, T \wedge 1]$,

$$\begin{aligned} p_t(x,y) &\le E[\mathbb{1}_{I_y}(X(t,0,x))]^{1/2} E[|H_{(1,\ldots,d)}(X(t,0,x),1)|^2]^{1/2} \\ &\le (\mathbb{P}(X(t,0,x) \in I_y))^{1/2} C \left\| \frac{1}{\Delta(t,x)} \right\|_\beta^{k_1} \|\mathcal{D}X(t,0,x)\|_{d,\gamma}^{k_2} \\ &\le C K_L(t) \frac{(1+|x|^Q)}{t^{dLk_1}} (\mathbb{P}(X(t,0,x) \in I_y))^{1/2}, \end{aligned} \tag{41}$$

where $L \in \mathbb{N}^*$ is as in Theorem 4, $K_L(\cdot)$ is non-decreasing, $C > 0$, $\beta, \gamma > 1$, and $k_1, k_2 \in \mathbb{N}^*$.

Let

$$\tau = \inf_{s\in[0,T]} \{|X(s,0,x) - x| \ge 1/2\} \wedge T, \quad \xi = \inf_{s\in[0,T]} \{|X(s,0,x) - x| \ge 1\} \wedge T$$

If $|y - x| < 1$, we have

$$\begin{aligned} &\mathbb{P}(X(t,0,x) \in I_y)) = \mathbb{P}\left(\bigcap_{i=1}^d \{X_i(t,x) \ge y_i\} \right) = \mathbb{P}\left(\bigcap_{i=1}^d \{X_i(t,x) - x_i \ge y_i - x_i\} \right) \\ &\le \mathbb{P}(|X(t,0,x) - x| \ge |y-x|) \\ &\le \mathbb{P}\left(\sup_{0\le s\le t\wedge\xi} \left| \int_0^s b(X(u,0,x))du + \sum_{i=1}^m \int_0^s \sigma^i(X(u,0,x))dW^i(u) \right| \ge |y-x| \right) \\ &\le \mathbb{P}\left(\sup_{0\le s\le t\wedge\xi} \left(\int_0^s |b(X(u,0,x))|du + \sum_{i=1}^m \int_0^s |\sigma^i(X(u,0,x))|dW^i(u) \right) \ge |y-x| \right) \end{aligned}$$

For $s \le t \wedge \xi$ we have $|X(u,0,x) - x| \le 1$ for any $u \in [0,s]$, so $X(u,0,x) \in B(x,1)$ and $|b(X(u,0,x))| \le M(x)$, $|\sigma(X(u,0,x))| \le M(x)$ for any $u \in [0,s]$. This implies

$$\int_0^s |b(X(u,0,x))|du \le tM(x) \le |y-x|/2,$$

for any $t \le |y-x|/(2M(x))$. We also have

$$\int_0^s |\sigma^i(X(u,0,x))|^2 du \le M(x)^2 t, \quad i = 1,\dots,m.$$

Hence, for any $t \le |y-x|/(2M(x))$ we obtain

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\le s\le t\wedge\xi}\left(\int_0^s |b(X(u,0,x))|du + \sum_{i=1}^m \int_0^s |\sigma^i(X(u,0,x))|dW^i(u)\right) \ge |y-x|\right)\\ &\le 2d\exp\left(-\frac{|y-x|^2}{8dtM(x)^2}\right)\end{aligned}$$

Here, we have applied Lemma 8.5 from Chapter V, Section 8 [17] for each component of X.

Similarly, if $|y-x|>1$

$$\begin{aligned}&\mathbb{P}(X(t,0,x)\in I_y)) \le \mathbb{P}(|X(t,0,x)-x| \ge |y-x|)\\ &\le \mathbb{P}(\tau \le t) = \mathbb{P}\left(\sup_{0\le s\le t\wedge\xi} |X(s,0,x)-x| \ge 1/2\right)\end{aligned}$$

$$\begin{aligned}&= \mathbb{P}\left(\sup_{0\le s\le t\wedge\xi}\left|\int_0^s b(X(u,0,x))du + \sum_{i=1}^m \int_0^s \sigma^i(X(u,0,x))dW^i(u)\right| \ge 1/2\right)\\ &\le \mathbb{P}\left(\sup_{0\le s\le t\wedge\xi}\left(\int_0^s |b(X(u,0,x))|du + \sum_{i=1}^m \int_0^s |\sigma^i(X(u,0,x))|dW^i(u)\right) \ge 1/2\right)\end{aligned}$$

For $s \le t\wedge\xi$ we have $|X(u,0,x)-x| \le 1$ for any $u\in[0,s]$, so $X(u,0,x)\in B(x,1)$ and $|b(X(u,0,x))| \le M(x)$, $|\sigma(X(u,0,x))| \le M(x)$ for any $u\in[0,s]$. This implies

$$\int_0^s |b(X(u,0,x))|du \le tM(x) \le 1/4,$$

for any $t \le 1/(4M(x))$. We also have

$$\int_0^s |\sigma^i(X(u,0,x))|^2 du \le M(x)^2 t, \quad i = 1,\dots,m.$$

Hence, for any $t \le 1/(4M(x))$ we obtain

$$\begin{aligned}&\mathbb{P}\left(\sup_{0\le s\le t\wedge\xi}\left(\int_0^s |b(X(u,0,x))|du + \sum_{i=1}^m \int_0^s |\sigma^i(X(u,0,x))|dW^i(u)\right) \ge 1/2\right)\\ &\le 2d\exp\left(-\frac{1}{16dtM(x)^2}\right)\end{aligned}$$

Here, we have applied Lemma 8.5 in Chapter V, Section 8 [17] for each component of X.

Thus, we obtain for $t \le (|y-x|\wedge 1)/(4M(x))$

$$\mathbb{P}(X(t,0,x)\in I_y)) \le 2d\exp\left(-\frac{(|y-x|^2\wedge 1)}{16dtM(x)^2}\right) \tag{42}$$

Notice that from (6) and (8) we obtain

$$\begin{aligned}&M(x)^2 \le C\sup_{z\in B(x,1)}(1+|z|)^{2N} \le C\sup_{z\in B(x,1)}(1+|z-x|+|x|)^{2N}\\ &\le C(2+|x|)^{2N} \le 2^{2N}C(1+|x|)^{2N}\end{aligned}$$

Substituting this into (42), we obtain, for $t \leq (|y-x| \wedge 1)/(4M(x))$,

$$\mathbb{P}(X(t,0,x) \in I_y)) \leq C_1 \exp\left(-C_2 \frac{(|y-x|^2 \wedge 1)}{t(1+|x|)^{2N}}\right) \tag{43}$$

This inequality and (41) imply (24) for $t \leq 1 \wedge (|y-x| \wedge 1)/(4M(x))$.

Next, we prove (25). From Proposition 2.1.5 [16] we know that for any index α, and for any $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$ such that $y_i > 0 \vee x_i$, $i = 1, \ldots, d$, we have

$$\partial_y^\alpha p_t(x,y) = (-1)^{|\alpha|} E\left[\mathbb{1}_{I_y}(X(t,0,x)) H_\alpha\left(X(t,0,x), H_{(1,\ldots,1)}(X(t,0,x),1)\right)\right],$$

where $I_y = \prod_{i=1}^d [y_i, \infty)$. Using this formula, the proof is similar to the proof of (24). □

6. Conclusions and Future Work

We have proved exponential bounds for the density of the law of the solution of the autonomous SDE (1). This result is based on the Malliavin differentiability of any order for the solution of (1) with coefficients satisfying the assumptions **C**, **M**, and **P**.

Here, we obtain exponential bounds for the density $p_t(x,y)$ and for its partial derivatives with respect to y. As future work, we plan to also find bounds for the partial derivatives with respect to x and t. This would fully extend the results in [5] to SDEs with non-globally Lipschitz coefficients. Moreover, as we have mentioned before, the exponential bounds for the density can be used to obtain an expansion of the error for numerical schemes. Symplectic methods for general stochastic Hamiltonian systems are fully implicit [18,19], and their convergence can be proved under non-globally Lipschitz assumptions. This paper was motivated by our interest in obtaining an expansion of the error for these symplectic schemes. In the stochastic case, there is no theoretical proof of the better long term accuracy of the symplectic schemes compared with non-symplectic ones, and the study of the error could be an important step in solving this problem.

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Appendix A. Proof of Theorem 2

Proof. The proof is similar with the proof of Theorem 2.12 in [5]. By repeated application of (31), we see that we have

$$\begin{aligned}
J^{-1}(t)V(X(t)) &= V(x) + \sum_{i=0}^{m} \int_0^t J^{-1}(s)[\sigma^i, V](X(s)) \circ dW^i(s) = V(x) \\
&+ \sum_{i=0}^{m} [\sigma^i, V](x) \int_0^t 1 \circ dW^i(s) + \sum_{i,j=0}^{m} \int_0^t \int_0^s J^{-1}(s)[\sigma^j, [\sigma^i, V]](X(s)) \circ dW^j(u) \circ dW^i(s) \\
&= V(x) + \sum_{i=0}^{m} [\sigma^i, V](x) \int_0^t 1 \circ dW^i(s) + \sum_{i,j=0}^{m} [\sigma^j, [\sigma^i, V]](x) \int_0^t \int_0^s 1 \circ dW^j(u) \circ dW^i(s) \\
&+ \sum_{i,j,k=0}^{m} \int_0^t \int_0^s \int_0^u J^{-1}(s)[\sigma^k, [\sigma^j, [\sigma^i, V]]](X(s)) \circ dW^k(v) \circ dW^j(u) \circ dW^i(s) \\
&= \sum_{|\alpha| \leq L-1} T_{(\alpha)}(V)(x) I^{(\alpha)}(t) + \sum_{|\alpha| = L} S^{(\alpha)}(t, Z_{(\alpha)})
\end{aligned} \tag{A1}$$

where as in the proof of Theorem 2.12 [5], we set (see also Equation 2.63, p. 130 [16])

$$S^{(\alpha)}(t, Z_{(\alpha)}) = \begin{cases} Z_{(\alpha)}(t) & \text{if } \alpha = \emptyset, \\ \int_0^t S^{(\alpha')}(s, Z_{(\alpha)}) \circ dW^{\alpha^*}(s) & \text{if } |\alpha| \geq 1, \end{cases}$$

$$Z_{(\alpha)}(t) = J^{-1}(t)T_{(\alpha)}(V)(X(t)) = T_{(\alpha)}(V)(x) + \sum_{i=0}^{m} \int_0^t J^{-1}(s)[\sigma^i, T_{(\alpha)}(V)](X(s)) \circ dW^i(s)$$

$$= T_{(\alpha)}(V)(x) + \sum_{i=1}^{m} \int_1^t J^{-1}(s)[\sigma^i, T_{(\alpha)}(V)](X(s))dW^i(s)$$

$$+ \int_0^t J^{-1}(s)\left([\sigma^0, T_{(\alpha)}(V)] + \frac{1}{2}\sum_{i=1}^{m}[\sigma^i, [\sigma^i, T_{(\alpha)}(V)]]\right)(X(s))ds$$

Notice that $S^{(\alpha)}(t, 1) = I^{(\alpha)}(t)$.

From (A1), we have expansion (32) with

$$R_L(t, x, V) = \sum_{|\alpha|=L} S^{(\alpha)}(t, Z_{(\alpha)}) + \sum_{\substack{\|\alpha\| \geq L \\ |\alpha| \leq L-1}} T_{(\alpha)}(V)(x)I^{(\alpha)}(t)$$

For $f \in C([0,1])$, we have for any $t \in (0,1]$ and $K \geq 1$,

$$\int_0^{t/K} |f(s)|^2 ds = \int_0^{t/K} \left(\frac{|f(s)|}{s^{L/2-\epsilon/4}}\right)^2 s^{L-\epsilon/2} ds \leq \sup_{0<s\leq 1/K} \left(\frac{|f(s)|}{s^{L/2-\epsilon/4}}\right)^2 \int_0^{t/K} s^{L-\epsilon/2} ds$$

$$= \frac{2}{2L-\epsilon+2} \sup_{0<s\leq 1/K} \frac{|f(s)|^2}{s^{L-\epsilon/2}} \left(\frac{t}{K}\right)^{L-\epsilon/2+1} \leq \frac{2}{2L-\epsilon} \sup_{0<s\leq 1/K} \frac{|f(s)|^2}{s^{L-\epsilon/2}} \left(\frac{t}{K}\right)^{L-\epsilon/2+1}$$

Hence, for any $0 < t \leq 1$, $K \geq 1$ we have

$$\mathbb{P}\left(\frac{1}{t^L}\int_0^{t/K} |R_L(s, x, V)|^2 ds \geq \frac{1}{K^{L+1-\epsilon}}\right) \leq \mathbb{P}\left(\frac{2}{2L-\epsilon} \sup_{0<s\leq 1/K} \frac{|R_L(s, x, V)|^2}{s^{L-\epsilon/2}} t^{1-\epsilon/2} \geq K^{\epsilon/2}\right)$$

$$\leq \mathbb{P}\left(\frac{2}{2L-\epsilon} \sup_{0<s\leq 1/K} \frac{|R_L(s, x, V)|^2}{s^{L-\epsilon/2}} \geq K^{\epsilon/2}\right) \leq \mathbb{P}\left(\frac{2}{2L-\epsilon}\left(\sup_{0<s\leq 1/K} \frac{|R_L(s, x, V)|}{s^{L/2-\epsilon/4}}\right)^2 \geq K^{\epsilon/2}\right)$$

$$= \mathbb{P}\left(\sup_{0<s\leq 1/K} \frac{|R_L(s, x, V)|}{s^{L/2-\epsilon/4}} \geq \left(\frac{2L-\epsilon}{2}\right)^{1/2} K^{\epsilon/4}\right)$$

$$\leq \sum_{|\alpha|=L} \mathbb{P}\left(\sup_{0<s\leq 1/K} \frac{|S^{(\alpha)}(s, Z_{(\alpha)})|}{s^{L/2-\epsilon/4}} \geq K_1\right) + \sum_{\substack{\|\alpha\| \geq L \\ |\alpha| \leq L-1}} \mathbb{P}\left(\sup_{0<s\leq 1} \frac{|T_{(\alpha)}(V)(x)I^{(\alpha)}(s)|}{s^{L/2-\epsilon/4}} \geq K_1\right),$$

where $K_1 = \left(\frac{2L-\epsilon}{2}\right)^{1/2} \frac{K^{\epsilon/4}}{N}$, with $N = \text{card}\{\alpha \in \mathcal{A}, |\alpha| = L\} + \text{card}\{\alpha \in \mathcal{A}, |\alpha| \leq L-1, \|\alpha\| \geq L\}$.

To handle the terms of the first sum, let us define

$$Y_i^{(\alpha)}(s) = J^{-1}(s)[\sigma^i, T_{(\alpha)}(V)](X(s)), \quad i = 1, \ldots, m$$

$$Y_0^{(\alpha)}(s) = J^{-1}(s)\left([\sigma^0, T_{(\alpha)}(V)] + \frac{1}{2}\sum_{i=1}^{m}[\sigma^i, [\sigma^i, T_{(\alpha)}(V)]]\right)$$

We obtain, for any $\alpha \in \mathcal{A}$, $|\alpha| = L$,

$$\mathbb{P}\left(\sup_{0<s\le 1/K}\frac{|S^{(\alpha)}(s,Z_{(\alpha)})|}{s^{L/2-\epsilon/4}}\ge K_1\right)\le \mathbb{P}\left(\sup_{0<s\le 1/K}\frac{|S^{(\alpha)}(s,Z_{(\alpha)})|}{s^{\|\alpha\|/2-\epsilon/4}}\ge K_1\right)$$
$$\le \mathbb{P}\left(\sup_{0<s\le 1/K}\frac{|S^{(\alpha)}(s,Z_{(\alpha)})|}{s^{\|\alpha\|/2-\epsilon/4}}\ge K_1, \sup_{0<s\le 1/K}|Z_{(\alpha)}(s)|\le K_2, \left(\sum_{i=1}^m\int_0^{1/K}|Y_i^{(\alpha)}(s)|^2ds\right)^{1/2}\le K_2\right)$$
$$+\mathbb{P}\left(\sup_{0<s\le 1/K}|Z_{(\alpha)}(s)|\ge K_2\right)+\mathbb{P}\left(\left(\sum_{i=1}^m\int_0^{1/K}|Y_i^{(\alpha)}(s)|^2ds\right)^{1/2}\ge K_2\right)$$
$$=: P_1(\alpha)+P_2(\alpha)+P_3(\alpha),$$

where $K_2 = K_1^{2-\|\alpha\|} \ge K_1^{2-2L}$ for a K large enough because $L \le \|\alpha\| \le 2|\alpha| = 2L$. From Theorem A.5 [5] we know that there exist $C(L,\epsilon) > 0$, $\lambda(L,\epsilon) > 0$ such that

$$P_1(\alpha)\le C(L,\epsilon)\exp(-\lambda_1(L,\epsilon)K_2)\le C(L,\epsilon)\exp\left(-\lambda_1(L,\epsilon)K_1^{2-2L}\right).$$

Thus, there exist $C_1(L,\epsilon) > 0$, $\lambda_1(L,\epsilon) > 0$, $\mu_1(L,\epsilon) > 0$ such that

$$\sum_{|\alpha|=L}P_1(\alpha)\le C_1(L,\epsilon)\exp\left(-\lambda_1(L,\epsilon)K^{\mu_1(L,\epsilon)}\right).$$

For $K > 1$, let

$$\tau = \inf\{s\ge 0: \|J^{-1}(s)-I_d\|\vee|X(s)-x|\ge 1/2\}\wedge\ T$$
$$\xi = \inf\{s\ge 0: \|J^{-1}(s)-I_d\|\vee|X(s)-x|\ge 1\}\wedge T$$

We denote $F(s) = (X(s), J^{-1}(s))^{\top}$. We have

$$\mathbb{P}\left(\sup_{0<s\le 1/K}|Z_{(\alpha)}(s)-Z_{(\alpha)}(0)|\ge K_2\right)$$
$$\le \mathbb{P}\left(\sup_{0<s\le 1/K}|Z_{(\alpha)}(s)-Z_{(\alpha)}(0)|\ge K_2, \tau>1/K\right)+\mathbb{P}(\tau\le 1/K)$$
$$\le \mathbb{P}\left(\sup_{0<s\le 1\wedge\tau}|Z_{(\alpha)}(s)-Z_{(\alpha)}(0)|\ge K_2\right)+\mathbb{P}\left(\sup_{0<s\le 1/K\wedge\xi}|F(s)-F(0)|\ge 1/2\right)$$

For a K large enough, we apply Lemma 8.5 in Chapter V, Section 8 [17] for each component of $Z_{(\alpha)}$ and F and we obtain

$$P_2(\alpha)\le C_1\exp\left(-\lambda_1K^2/(1+M(x))^2\right)+C_2\exp\left(-\lambda_2K/(1+M(x))^2\right)$$

Notice that we have

$$P_3(\alpha)\le\sum_{i=1}^m\mathbb{P}\left(\int_0^{1/K}|Y_i^{(\alpha)}(s)|^2ds\ge\frac{K_2^2}{m}\right)\le\sum_{i=1}^m\mathbb{P}\left(\sup_{0<s\le 1/K}|Y_i^{(\alpha)}(s)|^2\ge\frac{K_2^2}{m}\right)$$
$$\le\sum_{i=1}^m\mathbb{P}\left(\left(\sup_{0<s\le 1/K}|Y_i^{(\alpha)}(s)|\right)^2\ge\frac{K_2^2}{m}\right)=\sum_{i=1}^m\mathbb{P}\left(\sup_{0<s\le 1/K}|Y_i^{(\alpha)}(s)|\ge\frac{K_2}{\sqrt{m}}\right)$$

For $Y_i^{(\alpha)}$, $i = 1,\ldots,m$, we have an SDE similar to the one for $Z_{(\alpha)}$ but with $[\sigma^i, T_{(\alpha)}(V)]$ replacing $T_{(\alpha)}(V)$, so we can treat $P_2(\alpha)$ and the terms of $P_3(\alpha)$ similarly.

Finally, since $S^{(\alpha)}(t,1) = I^{(\alpha)}(t)$, we obtain for any $\alpha \in \mathcal{A}, |\alpha| \le L-1, \|\alpha\| \ge L$

$$\mathbb{P}\left(\sup_{0<s\le 1} \frac{|T_{(\alpha)}(V)(x)I^{(\alpha)}(s)|}{s^{L/2-\epsilon/4}} \ge K_1\right) \le \mathbb{P}\left(\sup_{0<s\le 1} \frac{|I^{(\alpha)}(s)|}{s^{L/2-\epsilon/4}} \ge \frac{K_1}{M(x)^2}\right)$$
$$\le \mathbb{P}\left(\sup_{0<s\le 1} \frac{|S^{(\alpha)}(s,1)|}{s^{\|\alpha\|/2-\epsilon/4}} \ge \frac{K_1}{2M(x)^2}\right)$$

If K is large enough, using Theorem A.5 [5] we know that there exist $C(L,\epsilon) > 0$, $\lambda(L,\epsilon) > 0$ and $\mu_1(L,\epsilon) > 0$ such that

$$\sum_{\substack{\|\alpha\|\ge L \\ |\alpha|\le L-1}} \mathbb{P}\left(\sup_{0<s\le 1} \frac{|T_{(\alpha)}(V)(x)I^{(\alpha)}(s)|}{s^{L/2-\epsilon/4}} \ge K_1\right) \le C(L,\epsilon)\exp\left(-\frac{\lambda(L,\epsilon)K^{\mu_1(L,\epsilon)}}{(1+M(x))^2}\right)$$

□

Appendix B. Proof of Theorem 3

Proof. This proof is similar to the proof of Theorem 2.17 [5]. From (22) and (36), notice that $C(t,x)$ is positive semi-definite and both $C(t,x)$ and $\tilde{\lambda}(t,x)$ are non-decreasing with respect to t, so it is enough to prove (34) for $K \ge 1$ large enough. Since for any $a,b \ge 0$ we have $(a+b)^2 \ge a^2/2 - b^2$, from (22) and (32) we have for any $t \in (0,1]$, $K > 1$ and $\eta \in \mathbb{R}^d$, $|\eta| = 1$,

$$< \eta, C(t/K,x)\eta >= \sum_{k=1}^{m} \int_0^{t/K} < J(s)^{-1}\sigma^k(X(s)), \eta >^2 \; ds$$
$$= \sum_{k=1}^{m} \int_0^{t/K} < \sum_{\|\alpha\|\le L-1} T_{(\alpha)}(\sigma^k)(x)I^{(\alpha)}(s) + R_L(s,x,\sigma^k), \eta >^2 ds$$
$$= \sum_{k=1}^{m} \int_0^{t/K} \left(< \sum_{\|\alpha\|\le L-1} T_{(\alpha)}(\sigma^k)(x)I^{(\alpha)}(s), \eta > + < R_L(s,x,\sigma^k), \eta >\right)^2 ds$$
$$\ge \frac{1}{2}\sum_{k=1}^{m} \int_0^{t/K} < \sum_{\|\alpha\|\le L-1} T_{(\alpha)}(\sigma^k)(x)I^{(\alpha)}(s), \eta >^2 ds - \sum_{k=1}^{m} \int_0^{t/K} < R_L(s,x,\sigma^k), \eta >^2 ds$$
$$\ge \frac{1}{2}\sum_{k=1}^{m} \int_0^{t/K} \left(< \sum_{\|\alpha\|\le L-1} T_{(\alpha)}(\sigma^k)(x), \eta > I^{(\alpha)}(s)\right)^2 ds - \sum_{k=1}^{m} \int_0^{t/K} |R_L(s,x,\sigma^k)|^2 ds$$

This implies

$$\inf_{|\eta|=1} \sum_{k=1}^{m} \int_0^{t/K} \left(< \sum_{\|\alpha\|\le L-1} T_{(\alpha)}(\sigma^k)(x), \eta > I^{(\alpha)}(s)\right)^2 ds$$
$$\le 2\inf_{|\eta|=1} < \eta, C(t/K,x)\eta > +2\sum_{k=1}^{m} \int_0^{t/K} |R_L(s,x,\sigma^k)|^2 ds$$
$$= 2\tilde{\lambda}(t/K,x) + 2\sum_{k=1}^{m} \int_0^{t/K} |R_L(s,x,\sigma^k)|^2 ds \tag{A2}$$

Let

$$M_k(x,\eta) := \sum_{\|\alpha\|\le L-1} < T_{(\alpha)}(\sigma^k)(x), \eta >^2$$

Since for any $\eta \in \mathbb{R}^d$, $|\eta| = 1$,

$$\mathcal{V}_L(x,\eta) = \sum_{k=1}^{m} M_k(x,\eta) \geq \mathcal{V}_L(x), \qquad \sum_{\|\alpha\| \leq L-1} \frac{< T_{(\alpha)}(\sigma^k)(x), \eta >^2}{M_k(x,\eta)} = 1$$

we obtain

$$\begin{aligned}
&\inf_{|\eta|=1} \sum_{k=1}^{m} \int_0^{t/K} \left(< \sum_{\|\alpha\| \leq L-1} T_{(\alpha)}(\sigma^k)(x), \eta > I^{(\alpha)}(s) \right)^2 ds \\
&= \inf_{|\eta|=1} \sum_{k=1, M_k(x,\eta)>0}^{m} M_k(x,\eta) \int_0^{t/K} \left(\sum_{\|\alpha\| \leq L-1} \frac{< T_{(\alpha)}(\sigma^k)(x), \eta >}{\sqrt{M_k(x,\eta)}} I^{(\alpha)}(s) \right)^2 ds \\
&\geq \inf_{|\eta|=1} \sum_{k=1}^{m} M_k(x,\eta) \inf \left\{ \int_0^{t/K} \left(\sum_{\|\alpha\| \leq L-1} b_\alpha I^{(\alpha)}(s) \right)^2 ds : \sum_{\|\alpha\| \leq L-1} b_\alpha^2 = 1 \right\} \\
&= \inf \left\{ \int_0^{t/K} \left(\sum_{\|\alpha\| \leq L-1} b_\alpha I^{(\alpha)}(s) \right)^2 ds : \sum_{\|\alpha\| \leq L-1} b_\alpha^2 = 1 \right\} \inf_{|\eta|=1} \mathcal{V}_L(x,\eta) \\
&\geq \inf \left\{ \int_0^{t/K} \left(\sum_{\|\alpha\| \leq L-1} b_\alpha I^{(\alpha)}(s) \right)^2 ds : \sum_{\|\alpha\| \leq L-1} b_\alpha^2 = 1 \right\} \mathcal{V}_L(x)
\end{aligned}$$

Here, if $M_k(x,\eta) = 0$, then $< T_{(\alpha)}(\sigma^k)(x), \eta >= 0$ for any $\|\alpha\| \leq L-1$, so the term corresponding to such a k is 0.

Thus, for any $L \geq 1$, (A2) yields for any $\epsilon \in (0,1)$, $t \in (0,1]$, and $K \geq 1$,

$$\begin{aligned}
&\mathbb{P}\left(\frac{\tilde{\lambda}(t/K,x)}{t^L} \leq \frac{1}{K^{L+1-\epsilon}} \right) \leq \mathbb{P}\Bigg(\frac{\mathcal{V}_L(x)}{2t^L} \inf \left\{ \int_0^{t/K} \left(\sum_{\|\alpha\| \leq L-1} b_\alpha I^{(\alpha)}(s) \right)^2 ds : \sum_{\|\alpha\| \leq L-1} b_\alpha^2 = 1 \right\} \\
&- \frac{1}{t^L} \sum_{k=1}^{m} \int_0^{t/K} |R_L(s,x,\sigma^k)|^2 ds \leq \frac{1}{K^{L+1-\epsilon}} \Bigg) \leq \mathbb{P}\left(\frac{1}{t^L} \sum_{k=1}^{m} \int_0^{t/K} |R_L(s,x,\sigma^k)|^2 ds \geq \frac{1}{K^{L+1-\epsilon}} \right) \\
&+ \mathbb{P}\left(\frac{\mathcal{V}_L(x)}{2t^L} \inf \left\{ \int_0^{t/K} \left(\sum_{\|\alpha\| \leq L-1} b_\alpha I^{(\alpha)}(s) \right)^2 ds : \sum_{\|\alpha\| \leq L-1} b_\alpha^2 = 1 \right\} \leq \frac{2}{K^{L+1-\epsilon}} \right) \\
&\leq \sum_{k=1}^{m} \mathbb{P}\left(\frac{1}{t^L} \int_0^{t/K} |R_L(s,x,\sigma^k)|^2 ds \geq \frac{1}{mK^{L+1-\epsilon}} \right) \\
&+ \mathbb{P}\left(\left(\frac{K}{t} \right)^L \inf \left\{ \int_0^{t/K} \left(\sum_{\|\alpha\| \leq L-1} b_\alpha I^{(\alpha)}(s) \right)^2 ds : \sum_{\|\alpha\| \leq L-1} b_\alpha^2 = 1 \right\} \leq \frac{4}{\mathcal{V}_L(x) K^{1-\epsilon}} \right)
\end{aligned}$$

By Theorem A.6 [5], there exist $C_1(L,\epsilon), \mu_1(L,\epsilon) > 0$ such that for all $t \in (0,1]$,

$$\begin{aligned}
&\mathbb{P}\left(\left(\frac{K}{t} \right)^L \inf \left\{ \int_0^{t/K} \left(\sum_{\|\alpha\| \leq L-1} b_\alpha I^{(\alpha)}(s) \right)^2 ds : \sum_{\|\alpha\| \leq L-1} b_\alpha^2 = 1 \right\} \leq \frac{4}{\mathcal{V}_L(x) K^{1-\epsilon}} \right) \\
&\leq C_1(L,\epsilon) \exp\left(-\left(\frac{\mathcal{V}_L(x) K^{1-\epsilon}}{4} \right)^{\mu_1(L,\epsilon)} \right)
\end{aligned}$$

By Theorem 2, there exist $C_2(L,\epsilon), \lambda_2(L,\epsilon), \mu_2(L,\epsilon) > 0$ such that for all $t \in (0,1]$ and $K \geq 1$ large enough,

$$\sum_{k=1}^{m} \mathbb{P}\left(\frac{1}{t^L}\sum_{k=1}^{m}\int_0^{t/K} |R_L(s,x,\sigma^k)|^2 ds \geq \frac{1}{mK^{L+1-\epsilon}}\right)$$
$$\leq mC_2(L,\epsilon)\exp\left(-\frac{\lambda_2(L,\epsilon)\left(m^{1/(L+1-\epsilon)}K\right)^{\mu_2(L,\epsilon)}}{(1+M(x))^2}\right)$$

Replacing K with $K^{1/(L+1)}$ and then taking $\epsilon = 1/(L+2)$ and using $\mathcal{V}_L(x)) \leq 1$, we obtain that there exist $\tilde{C}(L) > 0$, $\tilde{\lambda}(L) > 0$, and $\tilde{\mu}_L > 0$, all of them independent of $\sigma^0, \ldots, \sigma^m$, such that for all $t \in (0,1]$ and any $K \geq 1$ large enough,

$$\mathbb{P}\left(\frac{\tilde{\lambda}(t/K^{1/(L+1)},x)}{t^L} \leq \frac{1}{K}\right) \leq \mathbb{P}\left(\frac{\tilde{\lambda}(t/K^{1/(L+1)},x)}{t^L} \leq \frac{1}{K^{(L+1-\epsilon)/(L+1)}}\right)$$
$$\leq C_1(L,\epsilon)\exp\left(-\left(\frac{\mathcal{V}_L(x)K^{(1-\epsilon)/(L+1)}}{4}\right)^{\mu_1(L,\epsilon)}\right)$$
$$+ mC_2(L,\epsilon)\exp\left(-\frac{\lambda_2(L,\epsilon)\left(m^{1/(L+1-\epsilon)}K^{1/(L+1)}\right)^{\mu_2(L,\epsilon)}}{(1+M(x))^2}\right)$$

$$\leq C_1(L)\exp\left(-\left(\frac{\mathcal{V}_L(x)K^{1/(L+2)}}{4}\right)^{\mu_1(L)}\right)$$
$$+ mC_2(L)\exp\left(-\frac{\lambda_2(L)\left(m^{(L+2)/(L^2+3L+1)}K^{1/(L+1)}\right)^{\mu_2(L)}}{(1+M(x))^2}\right)$$
$$\leq C_1(L)\exp\left(-\frac{1}{4^{\mu_1(L)}}\frac{((\mathcal{V}_L(x))^{L+2}K)^{\mu_1(L)/(L+2)}}{(1+M(x))^2}\right)$$
$$+ mC_2(L)\exp\left(-\frac{\lambda_3(L)((\mathcal{V}_L(x))^{L+2}K)^{\mu_2(L)/(L+1)}}{(1+M(x))^2}\right)$$
$$\leq \tilde{C}(L)\exp\left(-\frac{\tilde{\lambda}_L\left(\mathcal{V}_L(x)^{L+2}K\right)^{\tilde{\mu}_L}}{(1+M(x))^2}\right)$$

For $K \geq 1$ large enough we can consider $\tilde{\mu}_L \in (0,1]$.

Next, notice that for any $\eta \in \mathbb{R}^d$, $|\eta| = 1$,

$$< \eta, Q(s,x)\eta > = < J(s)^\top \eta, C(s,x)J(s)^\top \eta >$$
$$= |J(s)^\top\eta|^2 < \frac{J(s)^\top\eta}{|J(s)^\top\eta|}, C(s,x)\frac{J(s)^\top\eta}{|J(s)^\top\eta|} \geq |J(s)^\top\eta|^2\tilde{\lambda}(s,x)$$

Let

$$\tau = \inf\{s \geq 0 : \|J(s) - I_d\| \geq 1/2\} \wedge T, \quad \tau_1 = \inf\{s \geq 0 : \|J(s) - I_d\| \geq 1\} \wedge T$$
$$\xi = \inf\{s \geq 0 : |X(s) - x| \geq 1/2\} \wedge T, \quad \xi_1 = \inf\{s \geq 0 : |X(s) - x| \geq 1\} \wedge T.$$

Notice that for $s \leq \tau$ we have

$$|J(s)^\top \eta|^2 = |(J(s) - I_d)^\top \eta + \eta|^2 \geq \frac{1}{2}|\eta|^2 - |(J(s) - I_d)^\top \eta|^2$$
$$\geq \frac{1}{2} - \|J(s) - I_d\|^2 |\eta|^2 \geq \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

Hence, for $s \leq \tau$ and for any $\eta \in \mathbb{R}^d$, $|\eta| = 1$,

$$< \eta, Q(s,x)\eta > \geq \frac{1}{4}\tilde{\lambda}(s,x)$$

Thus, there exist $C(L) > 0$, $\lambda(L) > 0$, and $\mu_L > 0$, all of them independent of $\sigma^0, \ldots, \sigma^m$, such that for all $t \in (0,1)$ and all $K \geq 1$ large enough we have

$$\mathbb{P}\left(\frac{\lambda\left(\frac{t}{K^{1/(L+1)}}, x\right)}{t^L} \leq \frac{1}{K}\right) \leq \mathbb{P}\left(\frac{\tilde{\lambda}\left(\frac{t}{K^{1/(L+1)}}, x\right)}{t^L} \leq \frac{4}{K}, \tau > \frac{1}{K^{1/(L+1)}}\right)$$
$$+ \mathbb{P}\left(\tau \leq \frac{1}{K^{1/(L+1)}}, \xi > \frac{1}{K^{1/(L+1)}}\right) + \mathbb{P}\left(\xi \leq \frac{1}{K^{1/(L+1)}}\right)$$
$$\leq \mathbb{P}\left(\frac{\tilde{\lambda}\left(\frac{t}{K^{1/(L+1)}}, x\right)}{t^L} \leq \frac{4}{K}\right) + \mathbb{P}\left(\sup_{0<s\leq\tau_1\wedge\frac{1}{K^{1/(L+1)}}} \|J(s) - I_d\| \geq 1/2, \xi > \frac{1}{K^{1/(L+1)}}\right)$$

$$+ \mathbb{P}\left(\sup_{0<s\leq\xi_1\wedge\frac{1}{K^{1/(L+1)}}} |X(s) - x| \geq 1/2\right)$$
$$\leq C_0(L) \exp\left(-\frac{\lambda_0(L)\left(\mathcal{V}_L(x)^{L+2}K\right)^{\mu_0(L)}}{(1+M(x))^2}\right) + C_1(L) \exp\left(-\frac{\lambda_1(L)K^{1/(L+1)}}{(1+M(x))^2}\right)$$
$$+ C_2(L) \exp\left(-\frac{\lambda_2(L)K^{1/(L+1)}}{(1+M(x))^2}\right)$$
$$\leq C_0(L) \exp\left(-\frac{\lambda_0(L)\left(\mathcal{V}_L(x)^{L+2}K\right)^{\mu_0(L)}}{(1+M(x))^2}\right) + C_1(L) \exp\left(-\frac{\lambda_1(L)\left(\mathcal{V}_L(x)^{L+2}K\right)^{1/(L+1)}}{(1+M(x))^2}\right)$$
$$+ C_2(L) \exp\left(-\frac{\lambda_2(L)\left(\mathcal{V}_L(x)^{L+2}K\right)^{1/(L+1)}}{(1+M(x))^2}\right)$$
$$\leq C(L) \exp\left(-\frac{\lambda_L\left(\mathcal{V}_L(x)^{L+2}K\right)^{\mu_L}}{(1+M(x))^2}\right)$$

Here, we have used $\mathcal{V}_L(x) \leq 1$, and for the first probability in the third inequality we have used (34) (we have $4/K$ instead of $1/K$ but we can adjust the constants from the beginning of the proof of (34)). For the second and third probabilities in the third inequality for a $K > 1$ large enough, we have applied Lemma 8.5 from Chapter V, Section 8 [17] for each component of X and J. $\square$

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