PURE POINT DIFFRACTION AND MEAN, BESICOVITCH AND WEYL ALMOST PERIODICITY

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Abstract. We show that a translation bounded measure has pure point diffraction if and only if it is mean almost periodic. We then go on and show that a translation bounded measure solves what we call the phase problem if and only if it is Besicovitch almost periodic. Finally, we show that a translation bounded measure solves the phase problem independent of the underlying van Hove sequence if and only if it is Weyl almost periodic. These results solve fundamental issues in the theory of pure point diffraction and answer questions of Lagarias.

Contents

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Introduction

This article deals with the harmonic analysis behind pure point diffraction. This topic has received substantial attention since the discovery of quasicrystals some thirty five years ago. Indeed, the pivotal article [11] by Bombieri / Taylor written right after the discovery of quasicrystals has in its title the question 'Which distributions of matter diffract?'. Of course, in order to answer this question one needs to be more specific: How is the distribution of matter modeled? What is meant by 'diffract'? In this section, we will discuss this and present our results. For further details and precise definition of certain concepts appearing along our discussion, we refer the reader to Section 1.

As has become the custom in the last two decades, distribution of matter is modeled by a measure in Euclidean space. This measure should satisfy a uniform boundedness condition, known as translation boundedness. This setting covers both Delone sets and bounded densities. In fact, as far as our investigation here is concerned, there is no reason to restrict to the Euclidean space. Instead we will from now on consider translation bounded measures on a locally compact, σ-compact Abelian group G. The dual group of G, i.e. the set of all continuous group homomorphisms from G into the circle is denoted by $\widehat{G}$.

According to the mathematical setup developed by Hof [22] (dealing with the Euclidean case) and extended by Schlottmann [51] (considering the group case), diffraction then comes about after one fixes a van Hove sequence $\mathcal{A} = (A_n)$ of subsets of the group G. Such a sequence is characterized by having the boundary of its members become arbitrarily small compared to the volume for large n. The Eberlein convolution of the measure μ with the complex conjugate of its reflection along the sequence $\mathcal{A}$ is then known as autocorrelation of μ and denoted by $\gamma_{\mathcal{A}}$ (provided it exists). This Eberlein convolution is positive definite and, hence, possesses a Fourier transform. This Fourier transform is known as the diffraction measure of μ along $\mathcal{A}$ and is denoted by $\widehat{\gamma_{\mathcal{A}}}$. It is this diffraction measure that models the outcome of diffraction experiments. In our context the natural first question is the following:

Question 1 (Characterization of pure point diffraction). Let a van Hove sequence $\mathcal{A}$ be given, and let μ be a measure with autocorrelation $\gamma_{\mathcal{A}}$. Under which conditions is the diffraction measure $\widehat{\gamma_{\mathcal{A}}}$ a pure point measure?

This is a long standing problem and clearly a most relevant question in our context. However, it only deals with a partial aspect of the situation as the diffraction measure only gives information on the diffraction amplitudes. It does not contain any information on the phases. The real issue of diffraction concerns the phases. Accordingly, the topic of phases is a central focus of Lagarias' article [27] on the problem of diffraction.

The article [27] has been fairly influential. In particular, various works in recent years have been devoted to answer questions from this article. This includes the work of Lev / Olevski [29] on Poisson summation type formulae and generalizations of Cordoba's theorem, the work of Favorov [18] on the failure of certain such generalizations in dimension bigger than one, and the work of Kellendonk / Sadun [25] and Kellendonk / Lenz [24] on the existence of sets with pure point diffraction without finite local complexity or Meyer property.

Following Lagarias,[1] we state the phase problem in the following way: Consider a measure μ with pure point diffraction supported on the set $E \subset \widehat{G}$. How can one associate phase information $a_\chi \in \mathbb{C}$, $\chi \in E$, such that both the Fourier transform of μ formally equals $\sum_{\chi\in E} a_\chi \delta_\chi$ and the consistent phase property

$$\widehat{\gamma} = \sum_{\chi\in E} |a_\chi|^2 \, \delta_\chi \tag{CPP}$$

holds? As already discussed in [27] when dealing with the phase problem, one first has to deal with what is meant by the Fourier transform of μ being formally equal to $\sum_{\chi\in E} a_\chi \delta_\chi$. Here, we take the point of view that this means that for each $\chi \in \widehat{G}$ the Fourier–Bohr coefficient

$$\lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} \overline{\chi(t)} \, \mathrm{d}\mu(t)$$

of μ exists along the sequence $\mathcal{A}$. Then, the phase problem can be stated in a precise mathematical form as our second question:

Question 2 (Phase Problem). Let a van Hove sequence $\mathcal{A}$ be given. When does a measure μ satisfy the following three properties?

- (P1) The autocorrelation $\gamma_{\mathcal{A}}$ of μ exists along $\mathcal{A}$, and the corresponding diffraction measure $\widehat{\gamma_{\mathcal{A}}}$ is pure point.
- (P2) For each $\chi \in \widehat{G}$ the Fourier–Bohr coefficient of μ exists along $\mathcal{A}$.
- (P3) The consistent phase property holds.

So far, everything is developed with respect to a fixed van Hove sequence. However, it is natural to aim for independence of the van Hove sequence. This leads us to the third question:

Question 3 (Uniform Phase Problem). When does a measure μ solve the phase problem for every van Hove sequence $\mathcal{A}$ with Fourier–Bohr coefficients and diffraction independent of the actual van Hove sequence?

[1] We use notation slightly different from the notation in [27]. Note also that the setting of [27] is restricted to Delone sets in Euclidean space, and [27] assumes that the autocorrelation exists for any van Hove sequence, whereas here we just assume existence along one fixed van Hove sequence.

In our article we provide complete answers to all three questions in terms of almost periodicity properties of μ. Our main results can be stated as follows:

Result 1 (Theorem 2.13). Let μ be translation bounded with autocorrelation $\gamma_{\mathcal{A}}$ along $\mathcal{A}$. Then, $\widehat{\gamma_{\mathcal{A}}}$ is pure point if and only if μ is mean almost periodic.

Result 2 (Theorem 3.36). Let a van Hove sequence $\mathcal{A}$ be given. Then, the translation bounded measure μ solves the phase problem (along $\mathcal{A}$) if and only if μ is Besicovitch almost periodic.

Result 3 (Theorem 4.16). The translation bounded measure μ solves the phase problem independent of the van Hove sequence if and only if μ is Weyl almost periodic.

Result 1 solves a long standing open problem with some partial results obtained earlier. For Delone sets in $\mathbb{R}^d$, a characterization of pure point diffraction has been given by Gouéré in [20]. As we will discuss below, his characterization is just an alternative description of mean almost periodicity for Delone sets. Thus, his result is a special case of our result (see Theorem 2.15). For Meyer sets a sufficient condition for pure point diffraction is given by Baake / Moody in [7]. Here, again, we can show that their condition actually is a description of means almost periodicity in the context of Meyer sets. So, we not only recover their result but in fact show that their condition is not only sufficient but also necessary (see Theorem 2.18).

Result 2 and Result 3 are completely unprecedented. They settle fundamental issues as witnessed by the mentioned article of Lagarias [27]. Indeed, it has already been discussed how that article focuses on the phase problem solved by Result 2 and Result 3. Moreover, the discussion in that article suggests to tackle the problem via suitable notions of almost periodicity. To be more specific, we need some more notation. A Patterson set in the sense of [27] is a Delone set in Euclidean space such that its autocorrelation exists for any van Hove sequence, is independent of the van Hove sequence, and has as its Fourier transform a pure point measure. Let now $\mathcal{B}$ be a suitable vector space of almost periodic functions in Euclidean space satisfying three natural additional assumptions viz a Parseval type condition, a Riesz–Fischer property and translation invariance. Then, Lagarias calls a Delone set Λ in Euclidean space a $\mathcal{B}$-quasicrystal or a $\mathcal{B}$-Besicovitch almost periodic set if

$$\sum_{x\in\Lambda} \varphi(\cdot - x) \in \mathcal{B}$$

holds for each infinitely many differentiable function φ with compact support. In the introduction to his article, Lagarias writes (p 64): '*…it remains to determine a good class $\mathcal{B}$ that gives a reasonable theory.*' and further on '*It is natural to hope that a suitable class of $\mathcal{B}$-Besicovitch almost periodic sets will all be Patterson sets and have the consistent phase property given in (3.9), but this is an open problem.*' Now, our results can clearly be understood to answer these issues. In fact, our result specifically can be seen as answers to Problems 4.6, 4.7 and 4.8 mentioned in the problem session of [27]. This deserves some further discussion: Problem 4.6 asks for a class $\mathcal{B}$ of almost periodic functions on Euclidean space such that their $\mathcal{B}$-quasicrystals satisfy:

(a) each $\mathcal{B}$-quasicrystal satisfies (CPP).
(b) every Patterson set coming from a cut and project scheme is a $\mathcal{B}$-quasicrystal.
(c) every selfreplicating Delone set which is a Patterson set is a $\mathcal{B}$-quasicrystal.

Our results show that the choice $\mathcal{B}$ as the 2-almost periodic Besicovitch functions provides a solution to Problem 4.6: This choice of $\mathcal{B}$ entails that every $\mathcal{B}$-quasicrystal is a Besicovitch almost periodic measure and, by Result 2, each such measure satisfies (CPP). Moreover, by Result 2 again, each Patterson set satisfying (CPP) belongs to $\mathcal{B}$. Hence, (b) and (c) are satisfied[2].

Problem 4.7 asks whether every $\mathcal{B}$-quasicrystal is a Patterson set. Now, this is not the case for the choice of $\mathcal{B}$ as 2-almost periodic Besicovitch functions. The reason is that in Result 2 we do not obtain existence of the autocorrelation along any an Hove sequence (but just along one fixed van Hove sequence). So, our Result 2 solves only a weakened version of Problem 4.7. On the other hand, our Result 3 implies existence of the autocorrelation along any van Hove sequence. So, Problem 4.7 is solved if one takes as $\mathcal{B}$ the Weyl almost periodic functions.

Strictly speaking, however, the Weyl almost periodic functions do not qualify as a $\mathcal{B}$-class as they do not satisfy the Riesz–Fischer property. On the other hand, we can show in Section 5 that any $\mathcal{B}$-class satisfying Riesz–Fischer property and Parseval must actually agree with the 2-Besicovitch almost periodic functions under some mild additional assumptions. Now, with the choice of $\mathcal{B}$ as 2-Besicovitch almost periodic functions one always ends up with some quasicrystals for which the autocorrelation does not exist for all van Hove sequences. Thus, it seems that one can not expect a full solution to Problem 4.7 when insisting on Riesz–Fischer property and Parseval equality.

Problem 4.8 deals with a translation bounded measure μ of $\mathcal{B}$ whose Fourier transform is formally given by $\sum a_\xi \delta_\xi$. It asks whether a_ξ must be the Fourier–Bohr coefficient (if this Fourier–Bohr coefficient exists). Now, this is (trivially) true in our context if one chooses for $\mathcal{B}$ the Besicovitch 2-space as we have just defined the formal Fourier expansion via the Fourier–Bohr coefficients.

Result 2 and Result 3 are not only of conceptual interest but also of direct consequence. Result 2 sheds a new and different light on model sets of maximal density. Such model sets have received attention in recent years [5, 26]. They have the particular feature that - unlike most other basic models for aperiodic order - here the actual choice of the van Hove sequence matters. As we show below, they can rather directly be seen to be Besicovitch almost periodic. Given this, Result 2 allows one to recover most of the fundamental results obtained for such models in the mentioned works (Theorem 3.40). Result 3 gives a new perspective on a class of almost periodic measures recently introduced by Meyer [39]. Meyer showed that regular model sets (in Euclidean space) belong to this class but did not give any results on their diffraction. Here we show that this class of Meyer is contained in the class of Weyl almost periodic measures (Corollary 4.26). Then, Result 3 provides a rather complete picture of the diffraction of sets in this class of Meyer.

[2] The article [27] does not completely specify what is meant by Patterson set coming from a cut and project scheme. We understand this to mean regular model sets.

A few words on methods are in order. As discussed above diffraction theory starts with a translation bounded measure. The Eberlein convolution (along a given van Hove sequence) is then used to form its autocorrelation. The Fourier transform of the autocorrelation is the diffraction measure. A key insight in the present article is that this theory can naturally be placed within the context of group representations. Specifically, the autocorrelation gives rise to an (pre-)Hilbert space structure on a certain space of functions, on which the group acts continuously by isometries. The diffraction measure then appears as a kind of 'universal' spectral measures of this group representation. In this way, tools from representation theory become available in the study of diffraction. A convenient way of formalizing this part of our approach is given by the concept of $\mathcal{A}$-representation introduced below (for a van Hove sequence $\mathcal{A} = (A_n)$). Such a representation is a linear G-invariant map $N : C_{\mathsf{c}}(G) \longrightarrow L^1_{loc}(G)$ with the additional property that the means

$$\lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} N(\varphi)(s)\, \overline{N(\psi)(s)}\, \mathrm{d}s =: \langle N(\varphi), N(\psi)\rangle$$

exist for all $\varphi, \psi \in C_{\mathsf{c}}(G)$. Under a mild additional assumption any such representation comes with a measure σ on $\widehat{G}$ such that $t \mapsto \langle N(\varphi), T_t N(\varphi)\rangle$ is just the Fourier transform of the finite measure $|\widehat{\varphi}|^2\sigma$ for any $\varphi \in C_{\mathsf{c}}(G)$. The application of this general approach to diffraction of measures is achieved by considering for a measure μ on G the map $N = N_\mu$ defined on $C_{\mathsf{c}}(G)$ by

$$N_\mu(\varphi) := \mu * \varphi\, .$$

In this situation, the measure σ can then be seen to be just the 'usual' diffraction measure considered in the literature.

A second key insight of the present article is that almost periodicity properties of the functions in the range of such an $\mathcal{A}$-representation N store all pieces of information relevant to us to deal with pure point diffraction and its strengthened variants. Our main results are then obtained by combining the framework of $\mathcal{A}$-representations with a thorough study of the relevant sets of almost periodic functions (together with the translation action on them). To provide such a study can be seen as a core of the article.

We single out three types of almost periodicity. These are mean almost periodicity, Besicovitch almost periodicity and Weyl almost periodicity. All these concepts are natural generalizations of Bohr almost periodicity. They arise by replacing the supremum norm by a seminorm arising from averaging along a van Hove sequence in respective characterizations of Bohr almost periodic functions.

While very natural, the concept of mean almost periodicity seems not to have been investigated before. On the other hand, Besicovitch and Weyl almost periodic functions have been considered in the literature, mostly in connection with differential equations i.e. in the one dimensional Euclidean situation. Still, parts of the theory have also been considered for more general groups. Here, we thoroughly develop the theory in the context of σ-compact, locally compact Abelian groups and point out earlier results along the way.

Our approach relies on group representations and Eberlein convolution. Accordingly, the treatment of the group action on Besicovitch spaces by translation and discussion of the Eberlein convolution of Besicovitch almost periodic functions are central to us. As these can not be found in the literature, we include full discussion. As for Weyl almost periodicity, a key element for us is to characterize this within the Besicovitch class by uniformity with respect to the van Hove sequences. This seems to be new.

Two additional advantages of our concept of $\mathcal{A}$-representation may be worth mentioning. First of all, it makes the underlying mathematics very transparent. In particular, it is clear that the domain of N is rather irrelevant. The crucial ingredient is the range of N being contained in certain classes of almost periodic functions. In fact, the domain $C_{\mathsf{c}}(G)$ of N could be replaced by any other subalgebra of continuous functions which is closed under convolution and whose image under Fourier transform is dense in a suitable L^2-space. In particular, if G is the Euclidean space, we could develop a completely analogous theory based on (tempered) distributions by considering $\mathcal{A}$-representations N mapping smooth functions with compact support (or elements of the Schwartz space) into the set of functions on G. Then, any (tempered) distribution ϱ would give a map $N = N_{\varrho}$ defined by $N_{\varrho}(\varphi) := \varrho * \varphi$.

Secondly, we feel that this concept seems to be appropriate in terms of modeling. After all, there is no intrinsic reason to prefer measures over distributions. The only thing relevant is that - irrespective of how the distribution of matter in question is modeled - one should be able to pair it with functions. This is exactly what is achieved by our concept of $\mathcal{A}$-representation. In the context of dynamical systems related ideas were developed in [31].

A standard tool in the investigation of aperiodic order is the use of dynamical systems. Here, the basic idea is to gather together all 'distributions of matter' with the 'same' local features. The arising set will be invariant under translation and compact and, hence, can be considered as a dynamical system. This approach makes powerful methods from dynamical systems available to the investigation of aperiodic order and has led to numerous results. In particular, it has been instrumental in proving pure point diffraction for many concrete models, see e.g. the recent monograph of Baake / Grimm [3] for discussion. On the conceptual level, this approach has enabled Lee / Moody / Solomyak to characterize pure point diffraction (of suitable Delone sets) via pure point spectrum of the associated dynamical system [28]. Their result has been generalized in various directions in [20, 6, 34, 31] in the last two decades.

Given this, it is natural to ask for applications of our approach to the dynamical system setting. Two such applications are discussed in this article. Both are set within the class of translation bounded measure dynamical systems (TMDS). Such dynamical systems are by now a standard setting for the description of aperiodic order via dynamical systems.

Our first application concerns characterization of pure point spectrum for TMDS and calculation of the eigenfunctions via mean almost periodicity and Besicovitch almost periodicity of the measures in question (Theorem 6.8 and Corollary 6.11).

The other application gives an characterization of Weyl almost periodicity of a measure via their associated dynamical system. It shows that the measure is Weyl almost periodic if and only if the associated dynamical system is uniquely ergodic with pure point spectrum and continuous eigenfunctions (Theorem 6.15). Within the context of aperiodic order this result can be seen as a relevant step in understanding why typical models for aperiodic order yield uniquely ergodic dynamical systems with pure point spectrum and continuous eigenfunctions. Within the context of dynamical system the result may also be of interest. Recent results by

Downarowicz / Glasner [14] (for actions of $\mathbb{Z}$) and Gröger/ Fuhrmann / Lenz [21] (for actions of more general groups) characterize mean equicontinuity of a dynamical system via exactly unique ergodicity, pure point spectrum and continuous eigenfunctions. In this context, our result says that mean equicontinuity of a transitive TMDS means Weyl almost periodicity of the involved measures.

In a companion article we study mean, Besicovitch, and Weyl almost periodicity for general dynamical systems [33].

This article is organized as follows: In Section 1, we present the setting and discuss the necessary concepts and tools for our considerations. In particular, we define the autocorrelation and the Fourier–Bohr coefficients of a measure. We then discuss the fundamental seminorms and the associated Besicovitch and Weyl type spaces and introduce the framework of $\mathcal{A}$-representations.

Section 2 is devoted to mean almost periodicity. We first introduce and discuss this notion for functions and measures and then turn to our first main result, Theorem 2.13. Finally, we discuss applications and show how our result contains the mentioned earlier results of [20] and [7].

Section 3 deals with Besicovitch almost periodicity. This section is the core of our article. We first present a thorough study of Besicovitch almost periodic functions. In particular, we show that the p-Besicovitch almost periodic functions form a Banach space for every $p \geq 1$. For $p = 2$ this space is even a Hilbert space with a natural orthonormal basis given by the characters of the group. Expansion with respect to this orthonormal basis gives a convincing Fourier type theory and is the basis for our solution to Problem 2. This solution is presented in Theorem 3.36. As an application, we give in Theorem 3.40 and its proof a new approach to results of [5, 26] dealing with weak model sets.

Our study of Weyl almost periodicity is given in Section 4. One key insight is that Weyl almost periodicity can be understood as simultaneous Besicovitch almost periodicity for all van Hove sequences. Given this, Result 3 is a rather direct consequence of Result 2. Details are given in Theorem 4.16 and its proof. As an application, we discuss a (slight generalization of a) concept of almost periodicity recently introduced by Meyer. As shown by Meyer, this type of almost periodicity is present in regular model sets in Euclidean space. Indeed, finding a concept of almost periodicity present in such models was exactly the motivation for Meyer. Here, we show in Corollary 4.24 that this form of almost periodicity entails Weyl almost periodicity. Given Result 3, this complements the results of Meyer by providing the missing diffraction theory for this form of almost periodicity.

To a certain extent, Besicovitch almost periodic functions and Weyl almost periodic functions are unavoidable when one deals with pure point diffraction. In this sense, there is a uniqueness to our solution of the phase problem. Details are discussed in Section 5. Our discussion of dynamical systems is given in Section 6.

The article is concluded by appendices, dealing with cut and project schemes, semimeasures and a counterexample respectively.

1. Key players and fundamental facts

In this section, we introduce the main concepts in our investigations. These are the autocorrelation and its Fourier transform, the diffraction measure, as well as the associated Fourier–Bohr coefficients and certain seminorms arising from averaging. All these quantities live on σ-compact, locally compact Abelian groups, and we start this section with a discussion of basic concepts related to such groups. At the end of this section, we introduce the abstract framework of diffraction theory captured by our notion of $\mathcal{A}$-representation.

1.1. **Basic setting.** For the entire paper G denotes a locally compact (Hausdorff), σ-compact Abelian group. The associated Haar measure is denoted by θ_G. For the Haar measure of a set $A \subseteq G$ we often write $|A|$ instead of $\theta_G(A)$. Integration of an integrable function $f : G \longrightarrow \mathbb{C}$ with respect to θ_G is often written as $\int_G f(s)\, \mathrm{d}s$. For $p \geq 1$ we denote by $L^p_{loc}(G)$ the space of all measurable $f : G \longrightarrow \mathbb{C}$ with $\int_K |f(t)|^p\, \mathrm{d}t < \infty$ for all compact $K \subseteq G$.

We use the familiar symbols $C(G)$, $C_{\mathrm{c}}(G)$, $C_{\mathrm{u}}(G)$ and $C_0(G)$ for the spaces of continuous, compactly supported continuous, bounded uniformly continuous, and continuous functions vanishing at infinity, which map from G to $\mathbb{C}$. For any function g on G and element $t \in G$, the functions $\widetilde{g}$, $\tau_t g$ and $g^\dagger$ are defined by

$$\widetilde{g}(s) := \overline{g(-s)}, \quad (\tau_t g)(s) := g(s-t) \quad \text{ and } \quad g^\dagger(s) := g(-s)\,.$$

The dual group $\widehat{G}$ of G is the set of all continuous group homomorphisms from G to $\{z \in \mathbb{C} : |z| = 1\}$. It becomes a topological space in a natural way, see e.g. [10]. The **Fourier transform** of $g \in C_{\mathrm{c}}(G)$ is the function

$$\widehat{g} : \widehat{G} \longrightarrow \mathbb{C}\,, \qquad \chi \mapsto \int_G \overline{\chi(t)}\, g(t)\, \mathrm{d}t\,.$$

For basic properties of the Fourier transform we refer the reader to [49].

A subset Λ of G is **relatively dense** if there exists a compact set $K \subseteq G$ with

$$G = \bigcup_{t\in\Lambda} (t+K).$$

A subset Λ of G is **uniformly discrete** if there exists an open neighborhood U of the identity with $(x+U) \cap (y+U) = \emptyset$ for all $x, y \in \Lambda$ with $x \neq y$. A subset Λ of G is a **Delone set** if it is both relatively dense and uniform discrete.

A **Radon measure** μ on G is a linear functional on $C_{\mathrm{c}}(G)$ such that, for every compact subset $K \subseteq G$, there is a constant $a_K > 0$ with

$$|\mu(\varphi)| \leq a_K\, \|\varphi\|_\infty$$

for all $\varphi \in C_{\mathrm{c}}(G)$ with $\mathrm{supp}(\varphi) \subseteq K$. Here, $\|\varphi\|_\infty$ denotes the supremum norm of φ. Subsequently, we will simply call μ a **measure**. For general background on measures we recommend [50] or [48, Appendix].

For a measure μ on G and $t \in G$, we define $\widetilde{\mu}$, $\tau_t\mu$ and $\mu^\dagger$ by

$$\widetilde{\mu}(g) := \overline{\mu(\widetilde{g})}\,, \quad (\tau_t\mu)(g) := \mu(T_{-t} g) \quad \text{ and } \quad \mu^\dagger(g) := \mu(g^\dagger)\,.$$

Any measure μ gives rise to a positive measure $|\mu|$ with $|\mu(\varphi)| \leq |\mu|(|\varphi|)$ for all $\varphi \in C_{\mathrm{c}}(G)$ called the **total variation of** μ (see [45, Thm. 6.5.6] or [48, Appendix] for the definition of

$|\mu|$). A measure μ on G is **finite** if $|\mu|(G) < \infty$ holds. A measure μ on G is called **translation bounded** if

$$\|\mu\|_V := \sup_{t\in G} |\mu|(t+V) < \infty$$

holds for one (and then each) open relatively compact subset $V \subseteq G$. This is equivalent to $\mu * \varphi \in C_{\mathsf{u}}(G)$ for all $\varphi \in C_{\mathsf{c}}(G)$ [1, 43]. The space of all translation bounded measures on G is denoted by $\mathcal{M}^\infty(G)$.

For $\varphi \in C_{\mathsf{u}}(G), \psi \in C_{\mathsf{c}}(G)$ or $\varphi \in C_{\mathsf{c}}(G), \psi \in C_{\mathsf{u}}(G)$, the **convolution** $\varphi * \psi \in C_{\mathsf{u}}(G)$ is defined via

$$(\varphi * \psi)(t) = \int_G \varphi(t-s)\, \psi(s)\, \mathrm{d}s$$

for $t \in G$. If both φ and ψ belong to $C_{\mathsf{c}}(G)$ so does $\varphi * \psi$. The convolution of a finite measure μ and a translation bounded measure ν is the measure $\mu * \nu$ defined by

$$(\mu * \nu)(\varphi) = \int_G \int_G \varphi(s+t)\, \mathrm{d}\mu(s)\, \mathrm{d}\nu(t)\,.$$

For $f \in C_{\mathsf{u}}(G)$ and $\varepsilon > 0$ there exists a neighborhood U of 0 in G with $\|f - f*\varphi\|_\infty \leq \varepsilon$ for all $\varphi \in C_{\mathsf{c}}(G)$ with support contained in U and $\int_G \varphi(t)\, \mathrm{d}t = 1$. This allows one to find an **approximate unit** i.e. a net (φ_α) in $C_{\mathsf{c}}(G)$ with $\varphi_\alpha * f \to f$ with respect to $\|\cdot\|_\infty$ for all $f \in C_{\mathsf{u}}(G)$ and support of every φ_α contained in one fixed open relatively compact neighborhood of $0 \in G$.

Finally, μ is called **positive definite** if $\mu(\varphi * \widetilde{\varphi}) \geq 0$ holds for all $\varphi \in C_{\mathsf{c}}(G)$. Any positive definite measure γ admits a (unique) measure $\widehat{\gamma}$ on $\widehat{G}$ with [10, 43]

$$\int_{\widehat{G}} |\check{\varphi}|^2 \, \mathrm{d}\widehat{\gamma} = \int_G (\varphi * \widetilde{\varphi})\, \mathrm{d}\gamma$$

for all $\varphi \in C_{\mathrm{c}}(G)$. The measure $\widehat{\gamma}$ is called the **Fourier transform** of γ.

A function $f \in C_{\mathsf{u}}(G)$ is called **Bohr almost periodic** if the closure of $\{\tau_t f \;:\; t \in G\}$ is compact in $(C_{\mathsf{u}}(G), \|\cdot\|_\infty)$. Equivalently, f is Bohr almost periodic if and only if for all $\varepsilon > 0$ the set $\{t \in G : \|f - \tau_t f\|_\infty < \varepsilon\}$ is relatively dense. The set of all Bohr almost periodic functions is denoted by $SAP(G)$ (for 'strongly almost periodic functions'). It is a subalgebra of the set of continuous bounded function on G and is closed with respect to the supremum norm.

Finite linear combinations of elements of $\widehat{G}$ are called **trigonometric polynomials**. The trigonometric polynomials are a dense subalgebra of $SAP(G)$. A measure $\mu \in \mathcal{M}^\infty(G)$ is called **strongly almost periodic** if, for all $\varphi \in C_{\mathsf{c}}(G)$, the function $\mu * \varphi$ is Bohr almost periodic. The set of all strongly almost periodic measures is denoted by $\mathcal{SAP}(G)$. Almost periodic measures are particularly relevant to spectral theory as a positive definite measure γ on G is strongly almost periodic if and only if $\widehat{\gamma}$ is pure point [17, 43].

Averaging will play a main role in our considerations. The corresponding basics will be discussed next. We will need the concept of van Hove sequences. While the notion of a Følner sequence suffices when dealing with functions, the van Hove property is essential for calculating the mean of a measure (see [43, Lem. 4.10.6, Lem. 4.10.7]).

Definition 1.1. A sequence (A_n) of relatively compact open subsets of G is called a **van Hove** sequence if, for each compact set $K \subseteq G$, we have

$$\lim_{n\to\infty} \frac{|\partial^K A_n|}{|A_n|} = 0\,,$$

where the K**-boundary** $\partial^K A$ of an open set A is defined as

$$\partial^K A := \big(\overline{A+K} \setminus A\big) \cup \big(((G\backslash A) - K) \cap \overline{A}\,\big)\,.$$

A locally compact Abelian group admits a van Hove sequence if and only if it admits a Følner sequence, if and only if G is σ-compact [53] (compare [51]). It is for this reason that assume σ-compactness throughout this article.

Whenever f belongs to $L^1_{loc}(G)$ and $\mathcal{A}$ is a van Hove sequence we define the **mean of** f **along** $\mathcal{A}$ by

$$M_{\mathcal{A}}(f) := \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} f(t)\,\mathrm{d}t$$

if the limit exists. In this case we also say that **the mean** $M_{\mathcal{A}}(f)$ **exists**.

For bounded measurable functions existence of the means along arbitrary van Hove sequences is characterized as follows.

Proposition 1.2 (Amenability of functions)**.** *Let $f : G \longrightarrow \mathbb{C}$ be bounded and measurable. Let $\mathcal{A}$ be a van Hove sequence. Then, the following statements are equivalent:*

(i) *The limit $\lim\limits_{n\to\infty} \frac{1}{|A_n|} \int_{s+A_n} f(t)\, dt$ exists uniformly in $s \in G$.*

(ii) *For each van Hove sequence $\mathcal{B}$ the limit $\lim\limits_{n\to\infty} \frac{1}{|B_n|} \int_{s+B_n} f(t)\, dt$ exists uniformly in $s \in G$ and is independent of the van Hove sequence.*

(iii) *The limit $\lim\limits_{n\to\infty} \frac{1}{|B_n|} \int_{B_n} f(t)\, dt$ exists for every van Hove sequence $\mathcal{B}$.*

Proof. (i) $\Longrightarrow$ (ii) This follows from Proposition D.1 (see [15, Thm. 3.1] or [43, Prop. 4.5.6] for $f \in C_{\mathsf{u}}(G)$).

(ii) $\Longrightarrow$(iii) is obvious.

(iii) $\Longrightarrow$ (i) Assume by contradiction that

$$\lim_{n\to\infty} \frac{1}{|A_n|} \int_{s+A_n} f(t)\,\mathrm{d}t$$

does not exist uniformly in s. Note that by (d) the limit

$$M_{\mathcal{A}}(f) = \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} f(t)\,\mathrm{d}t$$

exists. Then, there exists some $\varepsilon > 0$, an increasing sequence $n_1 < n_2 < \ldots < n_k < \ldots$ and some $s_k \in G$ such that, for all k, we have

$$\left| \frac{1}{|A_{n_k}|} \int_{s_k + A_{n_k}} f(t)\,\mathrm{d}t - M_{\mathcal{A}}(f) \right| > \varepsilon\,. \tag{1}$$

Next, define $B_k = s_k + A_{n_k}$. Then, (B_k) is a van Hove sequence, and hence, by (iii)

$$M_{\mathcal{B}}(f) = \lim_{k\to\infty} \frac{1}{|B_k|} \int_{B_k} f(t)\,\mathrm{d}t$$

exists and $|M_{\mathcal{B}}(f) - M_{\mathcal{A}}(f)| \geq \varepsilon$ holds by (1). On the other hand, we can consider the van Hove sequence $\mathcal{C}$ with $C_{2n} = B_n$ and $C_{2n+1} = A_n$ for all natural numbers n. Then, by (iii) again the limit of f along this sequence exists and this shows $M_{\mathcal{A}}(f) = M_{\mathcal{B}}(f)$ giving a contradiction. □

A bounded measurable function $f : G \longrightarrow \mathbb{C}$ satisfying one of the equivalent conditions of the preceding proposition is called **amenable**.

The **Eberlein convolution** $f \circledast_{\mathcal{A}} g$ of measurable functions $f, g : G \longrightarrow \mathbb{C}$ is defined as the function

$$f \circledast_{\mathcal{A}} g : G \longrightarrow \mathbb{C}\,, \qquad t \mapsto M_{\mathcal{A}}(fg(t-\cdot)) = \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} f(s)\, g(t-s)\,\mathrm{d}s,$$

if the integrals in question and the limit exist for all $t \in G$, [15, 43]. Similarly, the **Eberlein convolution** $\mu \circledast_{\mathcal{A}} \nu$ of measures μ and ν on G is the defined as the vague limit

$$\mu \circledast_{\mathcal{A}} \nu = \lim_{n\to\infty} \frac{1}{|A_n|} (\mu|_{A_n} * \nu|_{-A_n})$$

if this limit exists. Here, $\mu|_{A_n}$ denotes the restriction of the measure μ to the set A_n. Note that the convolution in the definition makes sense as $\mu|_{A_n}$ and $\nu|_{-A_n}$ are finite measures by compactness of the A_n. If the van Hove sequence $\mathcal{A}$ is clear from the context, we drop it in the notation.

Existence of the Eberlein convolution of translation bounded measures can be characterized as follows.

Proposition 1.3. *Let μ and ν be translation bounded measures on G. Let $\mathcal{A}$ be a van Hove sequence. Then, the following assertions are equivalent:*

(i) *The Eberlein convolution $\mu \circledast_{\mathcal{A}} \widetilde{\nu}$ exists.*
(ii) *For all $\varphi, \psi \in C_{\mathsf{c}}(G)$ the mean $M_{\mathcal{A}}(\mu * \varphi \cdot \overline{\nu * \psi})$ exists.*

If (i) and (ii) hold, then

$$M_{\mathcal{A}}(\mu * \varphi \cdot \overline{\nu * \psi}) = \big((\mu \circledast_{\mathcal{A}} \widetilde{\nu}) * \varphi * \widetilde{\psi}\big)(0)\,.$$

holds for all $\varphi, \psi \in C_{\mathsf{c}}(G)$.

Proof. This is essentially contained in (the proof of) [35, Lem. 7.1]. Specifically, this lemma states that (i) implies (ii) and that the last statement holds and the proof shows that the reverse implication holds as well. For the convenience of the reader we include some details.

For $n \in \mathbb{N}$, we define the measure m_n on G by $m_n := \frac{1}{|A_n|} \mu|_{A_n} * \widetilde{\nu}|_{-A_n}$ and the map $M_n : C_{\mathsf{c}}(G) \longrightarrow \mathbb{C}$ by $M_n(\varphi) = \frac{1}{|A_n|} | \int_{A_n} \varphi(t)\,\mathrm{d}t$.

Now, (i) is the statement that the sequence $(m_n(\varphi))$ converges for all $\varphi \in C_{\mathsf{c}}(G)$. Of course, this is equivalent to convergence of the sequence $((m_n * \varphi)(0))$ for all $\varphi \in C_{\mathsf{c}}(G)$. In fact, (i) is actually equivalent to convergence of the sequence $((m_n * \varphi * \widetilde{\psi})(0))$ for all $\varphi, \psi \in C_{\mathsf{c}}(G)$.

To see this consider $\varphi \in C_{\mathsf{c}}(G)$ arbitrary. Let $K \in G$ be compact with φ vanishing outside of K. Choose an open, relatively compact neighborhood U of $0 \in G$. Then, for any $\varepsilon > 0$ we can find a $\psi \in C_{\mathsf{c}}(G)$ supported in U with $\|\varphi - \varphi * \widetilde{\psi}\|_\infty < \varepsilon$. This gives

$$|m_n(\varphi) - m_n(\varphi * \widetilde{\psi})| \leq |m_n|(K + \overline{U})\, \|\varphi - \varphi * \widetilde{\psi}\|_\infty \leq \varepsilon\, |m_n|(K + \overline{U})\,.$$

Due to the translation boundedness of μ and ν, the sequence $\big(|m_n|(K + \overline{U})\big)$ can be seen to be bounded (compare Lemma 1.1 in [51]) and the desired statement follows.

Now, the proof of [35, Lem. 7.1] contains the line

$$\lim_{n\to\infty} \Big|(m_n * \varphi * \widetilde{\psi})(0) - M_n((\mu * \varphi) \cdot \overline{\nu * \psi}))\Big| = 0\,.$$

This shows that (ii) is equivalent to the convergence of $\big((m_n * \varphi * \widetilde{\psi})(0)\big)$ for all $\varphi, \psi \in C_{\mathsf{c}}(G)$, which is equivalent to (i) by our considerations above. This finishes the proof. $\square$

As a consequence of the preceding proposition, we can easily see that the Eberlein convolution of functions in $C_{\mathsf{u}}(G)$ agrees with the Eberlein convolution of the corresponding measures.

Proposition 1.4. *Let $\mathcal{A}$ be a van Hove sequence. Let $f, g \in C_{\mathsf{u}}(G)$ be given. Then, the following assertions are equivalent:*

(i) *The Eberlein convolution $f \circledast_{\mathcal{A}} g$ exists.*
(ii) *The Eberlein convolution $(f\theta_G) \circledast_{\mathcal{A}} (g\theta_G)$ exists.*

If (i) and (ii) holds we have $(f\theta_G) \circledast_{\mathcal{A}} (g\theta_G) = (f \circledast_{\mathcal{A}} g)\, \theta_G$. Moreover, in this case $f \circledast_{\mathcal{A}} g$ belongs to $C_{\mathsf{u}}(G)$.

Proof. As $(f\theta_G) * \varphi = f * \varphi$ and similarly $(g\theta_G) * \psi = g * \psi$ for all $\varphi, \psi \in C_{\mathsf{c}}(G)$, the previous proposition easily gives that (ii) is equivalent to existence of the means $M_{\mathcal{A}}(f * \varphi \cdot g * \psi(-\cdot))$ for all $\varphi, \psi \in C_{\mathsf{c}}(G)$. As f, g are uniformly continuous, this can easily be seen to be equivalent to existence of $f \circledast_{\mathcal{A}} g$. Moreover, uniform continuity of f, g easily gives that $t \mapsto M_{\mathcal{A}}(f \cdot g(t - \cdot))$ is uniformly continuous (if it exists at all). $\square$

Definition 1.5 (Upper mean and uniform upper mean)**.** Given a van Hove sequence $\mathcal{A} = (A_n)$, we can define the **upper mean** $\overline{M}_{\mathcal{A}}$ and the **uniform upper mean** $\overline{uM}_{\mathcal{A}}$ on $L^1_{loc}(G)$ via

$$\overline{M}_{\mathcal{A}}(f) := \limsup_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} f(t)\, \mathrm{d}t\,,$$
$$\overline{uM}_{\mathcal{A}}(f) := \limsup_{n\to\infty} \sup_{x\in G} \frac{1}{|A_n|} \int_{x + A_n} f(t)\, \mathrm{d}t\,.$$

Finally, we introduce one more class of functions that we will meet. A function $f \in C_{\mathsf{u}}(G)$ is called **weakly almost periodic** if the closure $\{\tau_t f\, :\, t \in G\}$ is compact in the weak topology of the Banach space $(C_{\mathsf{u}}(G), \|\cdot\|_\infty)$. Any weakly almost periodic f admits a (unique) decomposition $f = g + h$ with g being Bohr almost periodic and h being amenable with $M(|h|) = 0$, see e.g. [43]. As any strongly almost periodic function is amenable, we infer in particular that any weakly almost periodic function is amenable. A measure μ is called

weakly almost periodic if $\mu * \varphi$ is a weakly almost periodic function for all $\varphi \in C_{\mathsf{c}}(G)$. Weakly almost periodic measures were recently investigated in [35].

1.2. **Diffraction theory for measures: autocorrelation and Fourier–Bohr coefficients.** In this section, we introduce the autocorrelation and study some of its properties as well as its Fourier transform. These are the main objects of interest to us in the article.

Definition 1.6 (Autocorrelation)**.** Let μ be a measure on G and let $\mathcal{A} = (A_n)$ be a van Hove sequence. If the Eberlein convolution $\mu \circledast_{\mathcal{A}} \widetilde{\mu}$ exists, it is called the **autocorrelation of** μ **along** $\mathcal{A}$ and denoted by $\gamma_{\mathcal{A}}$ or just γ (if $\mathcal{A}$ is clear from the context).

Remark 1.7. For translation bounded measures in second countable groups G the existence of the limit in the definition is rather a matter of convention. Indeed, the limit will always exist for a suitable subsequence of (A_n).

By the standard argument of 'mixing' van Hove sequences (compare proof of Proposition 1.2) we obtain the following.

Proposition 1.8. *Let μ be a translation bounded measure on G. If the autocorrelation of μ exists along each van Hove sequence, then it is independent of the van Hove sequence.*

In the situation of the proposition, we say that μ has a **unique autocorrelation**.

It is easy to see that for any finite measure ν, the measure $\nu * \widetilde{\nu}$ is positive definite, and that vague limits of positive definite measures are positive definite [43]. Moreover, the Fourier transform of any positive definite measure exists. Thus, we immediately obtain the following corollary.

Corollary 1.9. *Let γ be the autocorrelation of the measure μ with respect to the van Hove sequence $\mathcal{A}$. Then, γ is positive definite. In particular, its Fourier transform $\widehat{\gamma}$ exists.*

In the situation of the corollary, we refer to $\widehat{\gamma}$ as the **diffraction** or **diffraction measure** of μ (with respect to $\mathcal{A}$).

For a given van Hove sequence (A_n), the **Fourier–Bohr** coefficient of the measure μ on G at $\chi \in \widehat{G}$ is defined as

$$a_{\chi}^{\mathcal{A}}(\mu) = \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} \overline{\chi(t)}\, \mathrm{d}\mu(t)\,,$$

if the limit exists. We then say that the **Fourier–Bohr coefficient of** μ **exists (along** $\mathcal{A}$**)**. If (A_n) can be replaced by $A_n + s_n$ with an arbitrary sequence (s_n) in G, we say that the **Fourier–Bohr coefficient of** μ **exists uniformly on** G **(along** $\mathcal{A}$**)**. Similarly, for a locally integrable function f on G, we define the **Fourier–Bohr coefficient** of f as that of $f\theta_G$ and write $a_{\chi}^{\mathcal{A}}(f)$ for it.

We complete the section by discussing the connection between the Fourier–Bohr coefficients of a measure $\mu \in \mathcal{M}^{\infty}(G)$ and the Fourier–Bohr coefficients of $\mu * \varphi$ for $\varphi \in C_{\mathsf{c}}(G)$.

Lemma 1.10. *Let $\mathcal{A}$ be a van Hove sequence, $\mu \in \mathcal{M}^{\infty}(G), \varphi \in C_{\mathsf{c}}(G)$ and $\chi \in \widehat{G}$. Set $K := supp(\varphi)$. Then, for all $s \in G$, we have*

$$\left| \left(\int_{s+A_n} (\varphi * \mu)(t)\, \overline{\chi(t)}\, dt \right) - (\widehat{\varphi}(\{\chi\})) \int_{s+A_n} \overline{\chi(t)}\, d\mu(t) \right| \leq \big\| |\varphi| * |\mu| \big\|_{\infty} |\partial^{K}(A_n)|\,.$$

Proof. By a standard application of Fubini, we have

$$D := \left| \left(\int_{s+A_n} (\varphi * \mu)(z)\, \overline{\chi(z)}\, \mathrm{d}z \right) - (\widehat{\varphi}(\{\chi\})) \int_{s+A_n} \overline{\chi(t)}\, \mathrm{d}\mu(t) \right|$$
$$= \left| \int_G \int_G \left((1_{x+A_n}(r) - 1_{x+A_n}(t))\, \varphi(r-t) \right) \overline{\chi(r)}\, \mathrm{d}r\, \mathrm{d}\mu(t) \right| .$$

A simple computation shows that $(1_{s+A_n}(z) - 1_{s+A_n}(t))\, \varphi(r-t) = 0$ unless we have $r \in \partial^K(s+A_n) = s + \partial^K(A_n)$. Therefore, we obtain

$$D \leq \int_G \int_{s+\partial^K(A_n)} |\varphi(r-t)|\, \mathrm{d}r\, \mathrm{d}|\mu|(t) \leq \big\| |\varphi| * |\mu| \big\|_\infty |\partial^K(A_n)| .$$

This finishes the proof. □

We note an immediate consequence of the lemma.

Corollary 1.11. *Let $\mathcal{A}$ be a van Hove sequence, $\mu \in \mathcal{M}^\infty(G)$, $\chi \in \widehat{G}$ and $s \in G$ be given.*

(a) *If $\widehat{\varphi}(\chi) \neq 0$ for $\varphi \in C_{\mathsf{c}}(G)$ and the Fourier–Bohr coefficient $a_\chi^{\mathcal{A}}(\mu*\varphi)$ exists (uniformly on G), then the Fourier–Bohr coefficient $a_\chi^{\mathcal{A}}(\mu)$ exists (uniformly on G).*

(b) *If the Fourier–Bohr coefficient $a_\chi^{\mathcal{A}}(\mu)$ exists (uniformly on G), then for all $\varphi \in C_{\mathsf{c}}(G)$, the Fourier–Bohr coefficient $a_\chi^{\mathcal{A}}(\mu*\varphi)$ exists (uniformly on G) and satisfies the identity $a_\chi^{\mathcal{A}}(\mu*\varphi) = \widehat{\varphi}(\chi) a_\chi^{\mathcal{A}}(\mu)$.*

By combining Corollary 1.11 with Proposition 1.2 we also get the next corollary.

Corollary 1.12. *Let $\mathcal{A}$ be a van Hove sequence, $\mu \in \mathcal{M}^\infty(G)$ and $\chi \in \widehat{G}$ be given. Then, the Fourier–Bohr coefficient $a_\chi^{\mathcal{A}}(\mu)$ exist uniformly in $s \in G$ if and only if the Fourier–Bohr coefficients $a_\chi^{\mathcal{B}}(\mu)$ exist with respect to any van Hove sequence $\mathcal{B}$.*

1.3. **Besicovitch and Weyl seminorms and associated spaces.** One basic tool in our considerations will be two seminorms and the corresponding spaces. These are introduced in this section.

Let $1 \leq p < \infty$, and let $\mathcal{A}$ be a van Hove sequence. For $f \in L^p_{loc}(G)$ define

$$\|f\|_{b,p,\mathcal{A}} := \left(\overline{M}_{\mathcal{A}}(|f|^p)\right)^{\frac{1}{p}} ,$$
$$\|f\|_{w,p,\mathcal{A}} := \left(\overline{uM}_{\mathcal{A}}(|f|^p)\right)^{\frac{1}{p}} .$$

Moreover, we set

$$BL^p_{\mathcal{A}}(G) := \{ f \in L^p_{loc}(G) : \|f\|_{b,p,\mathcal{A}} < \infty \} , WL^p_{\mathcal{A}}(G) \quad := \{ f \in L^p_{loc}(G) : \|f\|_{w,p,\mathcal{A}} < \infty \} .$$

Lemma 1.13 (Basic properties of the seminorms)**.** *Let $1 \leq p < \infty$, and let $\mathcal{A}$ be a van Hove sequence. Then,*

(a) *The maps $\|\cdot\|_{b,p,\mathcal{A}}$ and $\|\cdot\|_{w,p,\mathcal{A}}$ define seminorms on $BL^p_{\mathcal{A}}(G)$ and $WL^p_{\mathcal{A}}(G)$, respectively.*

(b) *For all $f \in WL^p_{\mathcal{A}}(G)$, we have $\|f\|_{b,p,\mathcal{A}} \leq \|f\|_{w,p,\mathcal{A}}$.*

(c) *If $f \in BL^p_{\mathcal{A}}(G) \cap L^\infty(G)$, then, for all $t \in G$, we have*

$$\|f\|_{b,p,\mathcal{A}} = \|\tau_t f\|_{b,p,\mathcal{A}} .$$

(d) *For all $f \in WL^p_{\mathcal{A}}(G)$ and all $t \in G$, we have*
$$\|f\|_{w,p,\mathcal{A}} = \|\tau_t f\|_{w,p,\mathcal{A}} \,.$$

Proof. (a) The only thing which is not obvious is the triangle inequality. This follows immediately from the triangle inequality for $L^p(G)$. Indeed, for each $f, g \in L^p_{loc}(G)$ and all $m \in \mathbb{N}$, we have
$$\frac{1}{|A_m|^{\frac{1}{p}}} \left(\int_{A_m} |f(t)+g(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}} \leq \frac{1}{|A_m|^{\frac{1}{p}}} \left(\int_{A_m} |f(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}} + \frac{1}{|A_m|^{\frac{1}{p}}} \left(\int_{A_m} |g(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}} .$$
Taking the limsup gives the desired inequality. The proof for the uniform limit is identical.

(b) is obvious from the definition.

(c) We have
$$\begin{aligned}
&\left| \frac{1}{|A_m|} \int_{A_m} |f(z)|^p \, \mathrm{d}z - \frac{1}{|A_m|} \int_{A_m} |\tau_t f(s)|^p \, \mathrm{d}s \right| \\
&\qquad = \left| \frac{1}{|A_m|} \int_{A_m} |f(s)|^p \, \mathrm{d}t - \frac{1}{|A_m|} \int_{A_m} |f(s-t)|^p \, \mathrm{d}t \right| \\
&\qquad = \left| \frac{1}{|A_m|} \int_{A_m} |f(z)|^p \, \mathrm{d}z - \frac{1}{|A_m|} \int_{t+A_m} |f(z)|^p \, \mathrm{d}z \right| \\
&\qquad = \left| \frac{1}{|A_m|} \int_{A_m \,\Delta\, (t+A_m)} |f(z)|^p \, \mathrm{d}z \right| \leq \frac{|A_m \,\Delta\, (t+A_m)|}{|A_m|} \|f\|_\infty^p \,.
\end{aligned}$$
Therefore, by the Følner condition, we get that
$$\lim_{m\to\infty} \left(\frac{1}{|A_m|} \int_{A_m} |f(z)|^p \, \mathrm{d}z - \frac{1}{|A_m|} \int_{A_m} |\tau_t f(s)|^p \, \mathrm{d}s \right) = 0$$
and hence
$$\limsup_{m\to\infty} \frac{1}{|A_m|} \int_{A_m} |f(z)|^p \, \mathrm{d}z = \limsup_{m\to\infty} \frac{1}{|A_m|} \int_{A_m} |\tau_t f(s)|^p \, \mathrm{d}s \,.$$

(d) Follows immediately from the definition. Indeed,
$$\begin{aligned}
\|\tau_t f\|^p_{w,p,\mathcal{A}} &= \limsup_{m\to\infty} \sup_{x\in G} \frac{1}{|A_m|} \int_{x+A_m} |\tau_t f(s)|^p \, \mathrm{d}s \\
&= \limsup_{m\to\infty} \sup_{x\in G} \frac{1}{|A_m|} \int_{t+x+A_m} |f(s)|^p \, \mathrm{d}s = \|f\|^p_{w,p,\mathcal{A}} \,.
\end{aligned}$$
This finishes the proof. □

Remark 1.14. For general (not necessarily bounded) functions f it may well be that f belongs to $BL^p_{\mathcal{A}}(G)$ and $\tau_t f$ does not belong to $BL^p_{\mathcal{A}}(G)$ for some $t \in G$. Consider for example the case $G = \mathbb{R}$. Let two strictly increasing sequences (a_n) and (b_n) converging to ∞ be given and define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = a_n$ for $b_n \leq x \leq b_n + 1$ and $-b_n - 1 \leq x \leq -b_n$ and $f(x) = 0$ else. Let $A_n = [-b_n, b_n]$. Then, by suitably adjusting (b_n) and (a_n) we can have $\|f\|_{b,p,\mathcal{A}} = 0$ and $\|\tau_t f\|_{b,p,\mathcal{A}} = \infty$ for all $t \neq 0$.

We refer to $\|\cdot\|_{b,p,\mathcal{A}}$ as the **Besicovitch** p-seminorm and to $\|\cdot\|_{w,p,\mathcal{A}}$ as the **Weyl** p-seminorm.

Remark 1.15. For $G = \mathbb{R}$ and special choices of $\mathcal{A}$, the spaces $BL^p_{\mathcal{A}}(G)$ were investigated by Marcinkiewicz [37] under the name 'Besicovitch space'. Later these spaces were then called Marcinkiewicz spaces, see e.g. [13].

We next give some standard inequalities involving the mean and square mean. Such estimates were used in [15, 17] for weakly almost periodic functions.

Lemma 1.16. *Let $1 \le p < \infty$ be arbitrary and q the conjugate exponent of p (i.e $1/p + 1/q = 1$). Let $\mathcal{A}$ be a van Hove sequence.*

(a) *For each $f \in L^p_{loc}(G)$ and $g \in L^q_{loc}(G)$ we have*
$$\|fg\|_{b,1,\mathcal{A}} \le \|f\|_{b,p,\mathcal{A}} \|g\|_{b,q,\mathcal{A}} \,.$$
In particular, for each $f \in L^1_{loc}(G)$ we have $\|f\|_{b,1,\mathcal{A}} \le \|f\|_{b,p,\mathcal{A}}$.

(b) *For each $f \in L^1_{loc}(G) \cap L^\infty(G)$ and each van Hove sequence $\mathcal{A}$, we have*
$$\|f\|^p_{b,p,\mathcal{A}} \le \|f\|^{p-1}_\infty \, \|f\|_{b,1,\mathcal{A}} \,.$$

Verbatim the same statements hold with $\|\cdot\|_{b,p,\mathcal{A}}$ replaced by $\|\cdot\|_{w,p,\mathcal{A}}$.

Proof. (a) By the Hölder's inequality, we have
$$\frac{1}{|A_n|} \int_{A_n} |f(x)g(x)| \, \mathrm{d}x \le \left(\frac{1}{|A_n|} \int_{A_n} |f(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \left(\frac{1}{|A_n|} \int_{A_n} |g(x)|^q \, \mathrm{d}x \right)^{\frac{1}{q}} .$$
Taking $\limsup_{n\to\infty}$ on both sides yields the first statement. The last statement follows by considering $g = 1$.

(b) The inequality
$$\frac{1}{|A_n|} \int_{A_n} |f(x)|^p \, \mathrm{d}x \le \|f\|^{p-1}_\infty \left(\frac{1}{|A_n|} \int_{A_n} |f(x)| \, \mathrm{d}x \right) ,$$
is obvious, and again, taking $\limsup_{n\to\infty}$ on both sides finishes the proof.

The preceding proofs carry over with $\|\cdot\|_{b,p,\mathcal{A}}$ replaced by $\|\cdot\|_{w,p,\mathcal{A}}$ and this yields the last statement of the lemma. ☐

We note the following consequence of Lemma 1.16.

Lemma 1.17 (Inclusion of spaces)**.** *Let $1 \le p < q < \infty$ be arbitrary, and let $\mathcal{A}$ be a van Hove sequence. Then, for all $f \in L^q_{loc}(G)$, we have*
$$\|f\|_{b,p,\mathcal{A}} \le \|f\|_{b,q,\mathcal{A}} \qquad \textit{and} \qquad \|f\|_{w,p,\mathcal{A}} \le \|f\|_{w,q,\mathcal{A}} \,.$$

Proof. We only discuss the Besicovitch norm. The argument for the Weyl norm is identical. We have
$$\|f\|^p_{b,p,\mathcal{A}} = \||f|^p\|_{b,1,\mathcal{A}} \le \||f|^p\|_{b,\frac{q}{p},\mathcal{A}} = \|f\|^p_{b,q,\mathcal{A}} \,.$$
The claim follows. ☐

We now come to the crucial completeness result. The result is certainly known. As we could not find it in the form stated here we include a proof (see [43] for related reasoning).

Theorem 1.18 (Completeness of $BL^p_{\mathcal{A}}(G)$)**.** *For each $p \geq 1$ and every van Hove sequence $\mathcal{A}$, the space $(BL^p_{\mathcal{A}}(G), \|\cdot\|_{b,p,\mathcal{A}})$ is complete.*

Proof. We distinguish two cases.

Case 1: G is compact. It suffices to show that $BL^p_{\mathcal{A}}(G)$ is the space $L^p(G)$ of measurable $f : G \longrightarrow \mathbb{C}$ with $\int_G |f|^p \,\mathrm{d}t < \infty$ and $\|f\|_{b,p,\mathcal{A}} = \big(\int_G |f|^p \,\mathrm{d}t\big)^{1/p}$ holds.

Via a standard renormalization, we can assume without loss of generality that $|G| = 1$. Setting $K = G$ in the definition of the van Hove sequence, we see that $G\backslash A_m \subseteq \partial^K(A_m)$, and hence, by the definition of the van Hove sequence, we get

$$\limsup_{m\to\infty} |G\backslash A_m| \leq \limsup_{m\to\infty} \frac{|G\backslash A_m|}{|A_m|} \leq \limsup_{m\to\infty} \frac{\partial^K(A_m)}{|A_m|} = 0\,.$$

From here, it follows immediately that, for all $f \in L^p(G)$, we have

$$\|f\|_{b,p,\mathcal{A}} = \lim_{m\to\infty} \left(\frac{1}{|A_m|}\int_{A_m} |f(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}} = \|f\|_p\,.$$

It is easy to deduce from here that $(BL^p_{loc}(G), \|\cdot\|_{p,b,\mathcal{A}}) = (L^p(G), \|\cdot\|_p)$ and the desired claim follows.

Case 2: G is not compact. Hence, $|G| = \infty$ holds. By Proposition 2.2 of [13] it suffices to find for each $f \in BL^p_{\mathcal{A}}(G)$ with $\|f\|_{b,p,\mathcal{A}} > 0$ an $f^* \in BL^p_{\mathcal{A}}(G)$ satisfying the following two properties:

$$\|f - f^*\|_{b,p,\mathcal{A}} = 0 \qquad \text{and} \qquad \sup_{n\in\mathbb{N}} \frac{1}{|A_n|}\left(\int_{A_n} |f^*(t)|^p \,\mathrm{d}t\right)^{1/p} \leq 2\,\|f\|_{b,p,\mathcal{A}}\,.$$

To do so, choose a natural number N large enough such that

$$\left(\frac{1}{|A_n|}\int_{A_n} |f(t)|^p \,\mathrm{d}t\right)^{1/p} \leq 2\|f\|_{b,p,\mathcal{A}} \qquad \text{for all } n \geq N\,.$$

Now consider f^* with $f^* = 0$ on the relatively compact $A_1 \cup A_2 \cdots \cup A_N$ and $f^* = f$ else. By construction, f^* has the second desired property. Thus, it remains to show $\|f - f^*\|_{b,p,\mathcal{A}} = 0$. This in turn follows immediately once we show $|A_n| \to \infty, n \to \infty$. Let $n > 0$. Since G is not compact, $|G| = \infty$. Therefore, we can find a compact set $0 \in K$ such that $|K| > 2n$. Since $0 \in K$, we immediately get that $A_m \subseteq A_m + K \subseteq A_m \cup \partial^K(A_m)$ for all m. The van Hove property then gives $\lim_{m\to\infty} \frac{|A_m+K|}{|A_m|} = 1$. Therefore, there exists some M such that, for all $m > M$, we have $\frac{|A_m+K|}{|A_m|} < 2$. Since A_m is non-empty, there exists some $t_m \in A_m$. Then, for each $m > M$ we have

$$|A_m| > \frac{1}{2}\,|A_m + K| > \frac{1}{2}\,|t_m + K| = \frac{|K|}{2} > n\,.$$

This finishes the proof. □

Finally, we turn to establishing a (continuous) translation action. As shown in Remark 1.14 the Besicovitch semi-norm is far from being invariant under translations and it may even be that translates of a function with finite Besicovitch semi-norm do not belong to Besicovitch space. Thus, we can not hope to find a translation action on the whole space. To remedy this, we will restrict our attention to a subspace. We define for $1 \leq p < \infty$

$$BC^p_{\mathcal{A}}(G) := \text{Closure of } C_{\mathsf{u}}(G) \text{ in } BL^p_{\mathcal{A}}(G) \text{ with respect to } \|\cdot\|_{b,p,\mathcal{A}}.$$

Elements in $BC^p_{\mathcal{A}}(G)$ can naturally be approximated by their cut-off functions. To make this precise, we define for $L \in (0,\infty)$ the cut-off c_L at L by

$$c_L : \mathbb{C} \longrightarrow \mathbb{C}, \qquad c_L(z) = \begin{cases} z, & |z| \leq L\,, \\ L\frac{z}{|z|}, & \text{otherwise}\,. \end{cases} \tag{2}$$

Then, $|c_L(z) - c_L(w)| \leq |z-w|$ for all $z, w \in \mathbb{C}$ and this implies

$$\|c_L(f) - c_L(g)\|_{b,p,\mathcal{A}} \leq \|f-g\|_{b,p,\mathcal{A}}$$

for all $f, g \in BC^p_{\mathcal{A}}(G)$, where $c_L(f)(t) := c_L(f(t))$.

Proposition 1.19. *For $f \in BC^p_{\mathcal{A}}(G)$, we have $c_n(f) \to f$ in $(BC^p_{\mathcal{A}}(G), \|\cdot\|_{b,p,\mathcal{A}})$ as $n \to \infty$.*

Proof. Let $\varepsilon > 0$ be arbitrary. Then, we can choose $g \in C_{\mathsf{u}}(G)$ with $\|f-g\|_{b,p,\mathcal{A}} < \varepsilon/2$. Then, for all $n \in \mathbb{N}$ with $n \geq \|g\|_\infty$ we have $c_n(g) = g$ and, hence, find

$$\|f - c_n(f)\|_{b,p,\mathcal{A}} \leq \|f-g\|_{b,p,\mathcal{A}} + \|c_n(g) - c_n(f)\|_{b,p,\mathcal{A}} < \varepsilon,$$

where we used $\|c_n(f) - c_n(g)\|_{b,p,\mathcal{A}} \leq \|f-g\|_{b,p,\mathcal{A}}$. □

On $BC^p_{\mathcal{A}}(G)$ we introduce the equivalence relation $\equiv$ with $f \equiv g$ whenever $\|f-g\|_{b,p,\mathcal{A}} = 0$. Then, $\|\cdot\|_{b,p,\mathcal{A}}$ descends to a norm on the quotient $BC^p_{\mathcal{A}}(G)/\equiv$ making it into a complete space. Then, it is not hard to establish the following crucial feature of this space: For each $t \in G$ the map $\tau_t : C_{\mathsf{u}}(G) \longrightarrow C_{\mathsf{u}}(G)$ can be extended uniquely to a continuous map T_t on $BC^p_{\mathcal{A}}(G)/\equiv$. The map T_t is isometric for each $t \in G$, and for each $[f] \in BC^p_{\mathcal{A}}(G)/\equiv$ the map

$$G \longrightarrow BC^p_{\mathcal{A}}(G)/\equiv, \qquad t \mapsto T_t[f]\,,$$

is continuous.

For $\varphi \in C_{\mathsf{c}}(G)$ we define the operator $T(\varphi)$ of convolution with φ on $BC^p_{\mathcal{A}}(G)/\equiv$ by setting

$$T(\varphi)[f] := \int_G \varphi(s)\, T_s[f]\,\mathrm{d}s$$

for $[f] \in BC^p_{\mathcal{A}}(G)$. Here, the integral is defined via Riemannian sums (which is possible as $G \longrightarrow BC^p_{\mathcal{A}}(G)$, $s \mapsto \varphi(s)T_s[f]$, is continuous with compact support). As each T_t, $t \in G$, is an isometry, the inequality $\|T(\varphi)\| \leq \|\varphi\|_1$ holds for all $\varphi \in C_{\mathsf{c}}(G)$. It is not hard to see that $T(\varphi)$ agrees on $C_{\mathsf{u}}(G)$ with convolution by φ, i.e.

$$T(\varphi)[f] = [f * \varphi]$$

holds for all $\varphi \in C_{\mathsf{c}}(G)$ and $f \in C_{\mathsf{u}}(G)$. Indeed, the function $G \times G \longrightarrow \mathbb{C}$, $(t,s) \mapsto \varphi(s)f(t-s)$ is bounded and uniformly continuous and this shows that the approximation of $\int_G \varphi(s)\, T_s[f]\,\mathrm{d}s$ by Riemannian sums is close to $f * \varphi$ in uniform norm and hence also in $BC^p_{\mathcal{A}}(G)$.

In particular, if (φ_α) is an approximate identity, we find that

$$T(\varphi_\alpha)[f] \to [f],$$

first for all $f \in C_{\mathsf{u}}(G)$ and, then, by uniform boundedness of the $T(\varphi_\alpha)$, for all $f \in BC^p_{\mathcal{A}}(G)$.

Moreover, we easily see by a direct computation that $\tau_t f_n$ converges to $\tau_t f$ for all $t \in G$ and $\varphi * f_n$ converges to $\varphi * f$ with respect to $\|\cdot\|_{b,p,\mathcal{A}}$ whenever f is bounded and measurable, f_n belongs to $C_{\mathsf{u}}(G)$ and (f_n) converges to f with respect to $\|\cdot\|_{b,p,\mathcal{A}}$. So, we obtain

$$T_t[f] = [\tau_t f] \quad \text{ as well as } \quad T(\varphi)[f] = [\varphi * f]$$

for all bounded $f \in BC^p_{\mathcal{A}}(G)$.

Proposition 1.20 (Compatibility with translations)**.** *Let $\mathcal{A}$ be a van Hove sequence. Let $f \in BC^p_{\mathcal{A}}(G)$ be given. If $\tau_t f \in BC^p_{\mathcal{A}}(G)$ for some $t \in G$, then $T_t[f] = [\tau_t f]$.*

Proof. By Proposition 1.19, we have $c_n(f) \to f$ as well as $c_n(\tau_t f) \to \tau_t f$. Moreover, as $(c_n(f))$ is bounded by construction, we have $T_t[c_n(f)] = [\tau_t c_n(f)] = [c_n(\tau_t f)]$. Putting this together, we arrive at the desired conclusion. ☐

In subsequent parts of the article, we will consider the situation that we are given a subspace $\mathcal{S}'$ of $C_{\mathsf{u}}(G)$, which is invariant under translation and closed in $\|\cdot\|_\infty$. Hence, this subspace is also invariant under taking convolutions with elements from $C_{\mathsf{c}}(G)$. We will be interested in the closure $\mathcal{S}$ of this subspace in $BC^p_{\mathcal{A}}(G)$ equipped with $\|\cdot\|_{b,p,\mathcal{A}}$. Clearly, translation action and convolution then descend from $BC^p_{\mathcal{A}}(G)$ to $\mathcal{S}$ and the above considerations holds for $\mathcal{S}$ as well. Specifically, we find the following:

Proposition 1.21 (Translation action and convolution)**.** *Let $\mathcal{S}'$ be a subspace of $C_{\mathsf{u}}(G)$, which is invariant under translation and closed in $\|\cdot\|_\infty$. Let $\mathcal{S}$ be its closure in $BL^p_{\mathcal{A}}(G)$.*

- (a) *For each $t \in G$, there exists a (unique) continuous map $T_t : \mathcal{S}/\equiv \longrightarrow \mathcal{S}/\equiv$ extending the translation τ_t on $\mathcal{S}'$. Each T_t is an isometry.*
- (b) *The map $G \longrightarrow \mathcal{S}/\equiv$, $t \mapsto T_t[f]$, is continuous for each $f \in \mathcal{S}$.*
- (c) *$T_t \circ T_s = T_{t+s}$ and $T_0 = Id$ hold for all $t, s \in G$.*
- (d) *For each $\varphi \in C_{\mathsf{c}}(G)$, there exists a unique continuous map $T(\varphi) : \mathcal{S}/\equiv \longrightarrow \mathcal{S}/\equiv$ with $T(\varphi)[f] = [f * \varphi]$ for all $f \in \mathcal{S}'$.*
- (e) *If (φ_α) is an approximate identity, then $T(\varphi_\alpha)[f] \to [f]$ for all $f \in \mathcal{S}$.*
- (f) *For all $f \in \mathcal{S} \cap L^\infty(G)$, we have $\tau_t f \in \mathcal{S}$ for all $t \in G$ and $T_t[f] = [\tau_t f]$.*
- (g) *For all $f \in \mathcal{S} \cap L^\infty(G)$, we have $f * \varphi \in \mathcal{S}$ for all $\varphi \in C_{\mathsf{c}}(G)$ and $T(\varphi)[f] = [f * \varphi]$.*

One can even extend validity of $T(\varphi)[f] = [f * \varphi]$ to all $f \in BC^1_{\mathcal{A}}(G)$ with $f\theta \in \mathcal{M}^\infty(G)$. This is discussed next. We need some preparation.

Lemma 1.22. *Let $\mathcal{A}$ be a van Hove sequence and $p \geq 1$ be given. Let $f \in BL^p_{\mathcal{A}}(G)$ and $\varphi \in C_{\mathsf{c}}(G)$. If $f\theta_G \in \mathcal{M}^\infty(G)$, then*

$$\|f * \varphi\|_{b,p,\mathcal{A}} \leq 2\, \|f\|_{b,p,\mathcal{A}}\, \|\varphi\|_1 \, .$$

Proof. For each n, we have by Young's convolution inequality

$$\int_G \left| \int_G 1_{A_n}(t-s)\, f(t-s)\, \varphi(s)\, \mathrm{d}s \right|^p \mathrm{d}t \leq \|\varphi\|_1^p \int_{A_n} |f(t)|^p \, \mathrm{d}t \, .$$

We also have

$$\left|\int_G 1_{A_n}(t)\, f(t-s)\,\varphi(s)\,\mathrm{d}s\right| \leq \left|\int_G 1_{A_n}(t-s)\, f(t-s)\,\varphi(s)\,\mathrm{d}s\right| + 1_{\partial^K(A_n)}(t)\,\|f*\varphi\|_\infty\,.$$

Next, using the standard inequality $(a+b)^p \leq 2^p(a^p+b^p)$, we get

$$\begin{aligned}\left|\int_G 1_{A_n}(t)\, f(t-s)\,\varphi(s)\,\mathrm{d}s\right|^p \leq 2^p &\left|\int_G 1_{A_n}(t-s)\, f(t-s)\,\varphi(s)\,\mathrm{d}s\right|^p \\ &+ 2^p\left(1_{\partial^K(A_n)}(t)\|f*\varphi\|_\infty\right)^p\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_G 1_{A_n}(t)\left|\int_G f(t-s)\,\varphi(s)\,\mathrm{d}s\right|^p \mathrm{d}t \leq 2^p &\int_G\left|\int_G 1_{A_n}(t-s)\, f(t-s)\,\varphi(s)\,\mathrm{d}s\right|^p \mathrm{d}t \\ &+ \int_G 1_{\partial^K(A_n)}(t)\,\|f*\varphi\|_\infty^p\,\mathrm{d}t\,.\end{aligned}$$

Using the van Hove property and boundedness of $f*\varphi$ (which follows from $f\theta_G \in \mathcal{M}^\infty(G)$, we get the claim. □

We also note the following.

Proposition 1.23 (f vs $f\theta_G$)**.** *Let $\mathcal{S}'$ be a subspace of $C_{\mathsf{u}}(G)$ which is invariant under translation and closed in $\|\cdot\|_\infty$. Let $\mathcal{S}$ be its closure in $BL^p_{\mathcal{A}}(G)$ with respect to $\|\cdot\|_{b,p,\mathcal{A}}$. Then, the following holds:*

- (a) *$f*\varphi \in \mathcal{S}'$ for all $f \in \mathcal{S}'$ and $\varphi \in C_{\mathsf{c}}(G)$.*
- (b) *Let $f \in L^p_{loc}(G)$ such that $f\theta_G$ is a translation bounded measure, and assume that (f_n) is a sequence in $\mathcal{S}'$ with $f_n \to f$ with respect to $\|\cdot\|_{b,p,\mathcal{A}}$. Then, $f*\varphi$ belongs to $\mathcal{S}$ and $(f_n*\varphi)$ converges to $f*\varphi$ with respect to $\|\cdot\|_{b,p,\mathcal{A}}$.*
- (c) *For $f \in BL^p_{\mathcal{A}}(G)$ with $f\theta_G \in \mathcal{M}^\infty(G)$ the following assertions are equivalent:*
  - (i) *f belongs to $BC^p_{\mathcal{A}}(G)$ and $f*\varphi \in \mathcal{S}$ for all $\varphi \in C_{\mathsf{c}}(G)$.*
  - (ii) *f belongs to $\mathcal{S}$.*

  In particular, $f \in C_{\mathsf{u}}(G)$ belongs to $\mathcal{S}$ if and only if $f*\varphi$ belongs to $\mathcal{S}$ for all $\varphi \in C_{\mathsf{c}}(G)$.

Proof. (a) This follows easily as $\mathcal{S}'$ is closed under translations and with respect to $\|\cdot\|_\infty$.

(b) As $\mathcal{S}$ is closed, it suffices to show that $(f_n*\varphi)$ converges to $f*\varphi$ with respect to $\|\cdot\|_{b,p,\mathcal{A}}$. By Lemma 1.22, we have

$$\|f*\varphi - f_n*\varphi\|_{b,p,\mathcal{A}} \leq 2\,\|f-f_n\|_{b,p,\mathcal{A}}\,\|\varphi\|_1$$

and the desired statement follows.

(c) The implication (ii)$\Longrightarrow$ (i) follows from (b). It remains to show (i)$\Longrightarrow$(ii): As, by (i), the function $f*\varphi$ belongs to $\mathcal{S}$ for all $\varphi \in C_{\mathsf{c}}(G)$ and $\mathcal{S}$ is closed, it suffices to show that $\|f-f*\varphi\|_{b,p,\mathcal{A}}$ becomes arbitrarily small. Now, for $\varphi \in C_{\mathsf{c}}(G)$ with $\|\varphi\|_1 \leq 1$ and $g \in C_{\mathsf{u}}(G)$, we find

$$\|f-f*\varphi\|_{b,p,\mathcal{A}} \quad\leq\quad \|f-g\|_{b,p,\mathcal{A}} + \|g-g*\varphi\|_{b,p,\mathcal{A}} + \|g*\varphi - f*\varphi\|_{b,p,\mathcal{A}}$$

$$\leq \quad \|f-g\|_{b,p,\mathcal{A}} + \|g-g*\varphi\|_\infty + 2\,\|g-f\|_{b,p,\mathcal{A}},$$

where we used Lemma 1.22 and $\|\cdot\|_{b,p,\mathcal{A}} \leq \|\cdot\|_\infty$. Now, the right hand side can be made arbitrarily small, by first choosing g sufficiently close to f (which is possible due to the assumption $f \in BC^p_{\mathcal{A}}(G)$) and then choosing $\varphi \in C_{\mathsf{c}}(G)$ with $\|g-g*\varphi\|_\infty$ sufficiently small (which is possible due to $g \in C_{\mathsf{u}}(G)$). This proves (ii). □

On the $WL^p_{\mathcal{A}}(G)$ the situation is different. For each $t \in G$, the map τ_t gives an isometric map from $WL^p_{\mathcal{A}}(G)$ into itself. In general this map will not be continuous in $t \in G$. However, it can easily be seen to give a continuous map on

$$WC^p_{\mathcal{A}}(G) := \text{Closure of } C_{\mathsf{u}}(G) \text{ in } WL^p_{\mathcal{A}}(G).$$

Clearly, each element of $WC^p_{\mathcal{A}}(G)$ belongs to $BC^p_{\mathcal{A}}(G)$ as well.

All spaces of almost periodic functions considered in the remainder of the article will be subspaces of $BC^p_{\mathcal{A}}(G)$. Hence, they can and will be equipped with a continuous translation.

1.4. **Diffraction theory for $\mathcal{A}$-representations.** In this section, we develop an abstract version of diffraction theory. It covers the diffraction theory developed above for translation bounded measures.

Let $\mathcal{H}$ be a vector space with semi-inner product $\langle\cdot,\cdot\rangle$ and associated seminorm $\|\cdot\|$. Let G act continuously on $\mathcal{H}$ via isometries T_t, $t \in G$. Then, a short computation shows that $g : G \longrightarrow \mathbb{C}$, $t \mapsto \langle f, T_t f\rangle$ satisfies

$$\sum_{j,k=1}^{n} c_i\,\overline{c_j} g(t_i - t_j) = \Big\| \sum_{j=1}^{n} c_j T_{t_j} f \Big\|^2 \geq 0$$

for $n \in \mathbb{N}$ and arbitrary $c_1, \ldots, c_n \in \mathbb{C}$ and $t_1, \ldots, t_n \in G$. Hence, this function is positive definite for each $f \in \mathcal{H}$, and is clearly continuous. Thus, by a result of Bochner, there exists for each $f \in \mathcal{H}$ a unique positive finite measure σ_f on $\widehat{G}$ with

$$\langle f, T_t f\rangle = \int_{\widehat{G}} \chi(t)\,\mathrm{d}\sigma_f(\chi)$$

for all $t \in G$. This measure is called the **spectral measure** of f.

Whenever we have a continuous action of G by isometries T_t on $\mathcal{H}$, we can define the operator

$$T(\varphi) : \mathcal{H} \longrightarrow \mathcal{H}, \qquad f \mapsto \int_G \varphi(s) T_s f\,\mathrm{d}s,$$

for $\varphi \in C_{\mathsf{c}}(G)$. Then, $T(\varphi)$ will be a bounded operator with $\|T(\varphi)\| \leq \|\varphi\|_1$. The spectral measure is compatible with taking convolutions in the following sense.

Proposition 1.24. *Let $\mathcal{H}$ be a vector space with semi-inner product $\langle\cdot,\cdot\rangle$. Let G act continuously on $\mathcal{H}$ via isometries T_t, $t \in G$. Then,*

$$\sigma_{T(\varphi)f} = |\widehat{\varphi}|^2 \sigma_f.$$

Proof. A direct computation shows that both $\sigma_{T(\varphi)f}$ and $|\widehat{\varphi}|^2\sigma_f$ have the same (inverse) Fourier transform:

$$\begin{aligned}\langle T(\varphi)f, T_tT(\varphi)f\rangle &= \int_G\int_G \varphi(s)\,\overline{\varphi(r)}\,\langle f, T_{t-s+r}f\rangle\,\mathrm{d}s\,\mathrm{d}r\\ &= \int_G\int_G \varphi(s)\,\overline{\varphi(r)}\int_{\widehat{G}}\chi(t-s+r)\,\mathrm{d}\sigma_f(\chi)\,\mathrm{d}s\,\mathrm{d}r\\ &= \int_{\widehat{G}}\chi(t)\,|\widehat{\varphi}(\chi)|^2\,\mathrm{d}\sigma_f(\chi)\,.\end{aligned}$$

By uniqueness of Fourier transform the desired claim follows. □

We will be particularly interested in the situation that all spectral measures are pure point measures. To put this in context consider a continuous representation T on a Hilbert space. An $f\in\mathcal{H}$ with $f\neq 0$ is called an **eigenfunction** of T to the **eigenvalue** $\chi\in\widehat{G}$ if

$$T_tf = \chi(t)f$$

for all $t\in G$. Then, T is said to have **pure point spectrum** if there exists an orthonormal basis of eigenfunctions. As is well known (and not hard to see), pure point spectrum of T is equivalent to all measures σ_f being pure point measures. This suggests to look for criteria ensuring that a spectral measure is a pure point measure. The following characterization is well-known, see e.g. [34] for a recent discussion.

Proposition 1.25. *Let $\mathcal{H}$ be a vector space with semi-inner product $\langle\cdot,\cdot\rangle$ and associated seminorm $\|\cdot\|$. Let G act continuously on $\mathcal{H}$ via isometries T_t, $t\in G$. Then, for $f\in\mathcal{H}$ the following assertions are equivalent:*

(i) *The spectral measure σ_f is pure point.*
(ii) *The function $G\longrightarrow\mathbb{C}$, $t\mapsto\langle f, T_tf\rangle$, is Bohr almost periodic.*
(iii) *The function $G\longrightarrow\mathcal{H}$, $t\mapsto T_tf$, is almost periodic in the sense that for each $\varepsilon>0$ the set $\{t\in G : \|f-T_tf\|<\varepsilon\}$ is relatively dense.*

Proof. The equivalence between (i) and (ii) is a classical result of Wiener.

(ii)$\Longrightarrow$(iii): Let $\varepsilon>0$ be given. By (ii) the set of $t\in G$ with $|\langle f,f\rangle-\langle f,T_tf\rangle|<\varepsilon/2$ is relatively dense. Now, a direct computation using that T_t is an isometry gives for each such $t\in G$

$$\|f-T_tf\|^2 = 2(\langle f,f\rangle - \Re\langle f,T_tf\rangle)\leq 2|\langle f,f\rangle-\langle f,T_tf\rangle|<\varepsilon$$

and (iii) follows.

(iii)$\Longrightarrow$(ii): Set $F(t)=\langle f,T_tf\rangle$. Let $\varepsilon>0$. Then, for each $t\in G$ with $\|f-T_tf\|<\varepsilon/(\|f\|+1)$, we have

$$|F(t+s)-F(s)| = |\langle f,T_{t+s}f\rangle-\langle f,T_sf\rangle|\leq\|T_{t+s}f-T_sf\|\,\|f\|=\|T_tf-f\|\,\|f\|<\varepsilon$$

and (ii) follows. □

We will now consider special representations. An $\mathcal{N}$-representation of G is a quadruple $(N, \mathcal{H}, \langle\cdot,\cdot\rangle, T)$ consisting of a vector space $\mathcal{H}$ with semi-inner product $\langle\cdot,\cdot\rangle$ together with a continuous action T_t, $t \in G$, of G on $\mathcal{H}$ by isometries and a linear G-equivariant map

$$N : C_{\mathsf{c}}(G) \longrightarrow \mathcal{H} .$$

When we are given an $\mathcal{N}$-representation, we denote the spectral measure of $N(\varphi)$ by σ_φ (instead of $\sigma_{N(\varphi)}$). We will often just refer to $N : C_{\mathsf{c}}(G) \longrightarrow \mathcal{H}$ or even just N as an $\mathcal{N}$-representation.

For us, the situation where $\mathcal{H}$ is a Hilbert space and $N(C_{\mathsf{c}}(G))$ is dense in $\mathcal{H}$ will be particularly relevant. We then speak about an $\mathcal{N}$-representation **on a Hilbert space with dense range**. Note that this is not so much an assumption but rather a matter of convenience. Indeed, whenever $N : C_{\mathsf{c}}(G) \longrightarrow \mathcal{H}$ is an $\mathcal{N}$-representation, we can always factor out elements with vanishing seminorm and then take the completion of $N(C_{\mathsf{c}}(G))$.

We call the $\mathcal{N}$-representation N **intertwining** if

$$T(\varphi)N(\psi) = T(\psi)N(\varphi)$$

holds for all $\varphi, \psi \in C_{\mathsf{c}}(G)$.

Proposition 1.26 (Intertwining follows from continuity)**.** *If N is continuous (with respect to inductive limit topology) then N is intertwining.*

Proof. A short computation using the continuity of N gives

$$T(\varphi)N(\psi) = \int_G \varphi(s)\, T_s N(\psi)\,\mathrm{d}s = \int_G N(\varphi(s)\tau_s\psi)\,\mathrm{d}s = N(\varphi * \psi) .$$

From this we directly see that N is intertwining as $\varphi * \psi = \psi * \varphi$. $\square$

Definition 1.27. Define $K_2(G)$ to be the subspace of $C_{\mathsf{c}}(G)$ spanned by $\{\varphi * \psi : \varphi, \psi \in C_{\mathsf{c}}\}$. A linear map $\vartheta : K_2(G) \to \mathbb{C}$ is called **semi-measure**. A semi-measure ϑ is **Fourier transformable** if there exists a measure $\widehat{\vartheta}$ on $\widehat{G}$ such that, for all $\varphi \in C_{\mathsf{c}}(G)$, we have $|\check{\varphi}|^2 \in L^1(|\widehat{\vartheta}|)$ and

$$\vartheta(\varphi * \widetilde{\varphi}) = \widehat{\vartheta}(|\check{\varphi}|^2) .$$

In this case we call the measure $\widehat{\vartheta}$ the **Fourier transform** of ϑ.

Note that, given a semi-measure ϑ, for all $\psi \in K_2(G)$, we can define the convolution

$$(\vartheta * \psi)(t) := \vartheta(\psi(t - \cdot)) .$$

We say that N possesses the **semi-autocorrelation** η if η is a Fourier transformable semi-measure with

$$\langle N(\varphi), N(\psi)\rangle = (\eta * \varphi * \widetilde{\psi})(0) \tag{3}$$

for all $\varphi, \psi \in C_{\mathsf{c}}(G)$. We say that N possesses an **autocorrelation** if η is a measure. Finally, we say that N possesses the **diffraction measure** σ if σ is a positive measure on $\widehat{G}$ with $|\widehat{\varphi}|^2\sigma = \sigma_\varphi$ for all $\varphi \in C_{\mathsf{c}}(G)$.

Note here that, whenever a semi-measure satisfying (3) exists, it is Fourier transformable by Remark C.7.

Lemma 1.28. *Let $N : C_{\mathsf{c}}(G) \longrightarrow \mathcal{H}$ be an $\mathcal{N}$-representation. Then, the following assertions are equivalent:*

(i) *N is intertwining.*
(ii *N possesses a semi-autocorrelation η.*
(iii) *N possesses a diffraction measure σ.*

If one of these equivalent conditions holds then $\widehat{\eta} = \sigma$.

Proof. (i)$\Longrightarrow$(iii): By Lemma C.3 in Appendix C, it suffices to show that

$$|\widehat{\varphi}|^2\, \sigma_\psi = |\widehat{\psi}|^2\, \sigma_\varphi$$

for all $\varphi, \psi \in C_{\mathsf{c}}(G)$. This, however, is immediate from (i) and Proposition 1.24.

(iii)$\Longrightarrow$(i): From (iii and polarisation, we find

$$\langle N(\varphi), T_t N(\psi)\rangle = \int_{\widehat{G}} \chi(t)\, \widehat{\varphi}(\chi)\, \overline{\widehat{\psi}(\chi)}\, \mathrm{d}\sigma(\chi)$$

for all $t \in G$ and $\varphi, \psi \in C_{\mathsf{c}}(G)$. Given this, a direct computation similar to the one in the proof of Proposition 1.24 shows

$$\langle T(\varphi)N(\psi), T(\varrho)N(\xi)\rangle = \int_{\widehat{G}} \widehat{\varphi}(\chi)\, \widehat{\psi}(\chi)\, \overline{\widehat{\varrho}(\chi)\, \widehat{\xi}(\chi)}\, \mathrm{d}\sigma(\chi)$$

for all $\varphi, \psi, \varrho, \xi \in C_{\mathsf{c}}(G)$. This then easily gives

$$\|T(\varphi)N(\psi) - T(\psi)N(\varphi)\|^2 = 0.$$

(ii)$\Longrightarrow$ (iii): Let σ be the Fourier transform of η. From the defining properties of σ, η and the spectral measure we find for all $t \in G$

$$\int_{\widehat{G}} \chi(t)\, |\widehat{\varphi}|^2\, \mathrm{d}\sigma(\chi) = (\eta * \varphi * \widetilde{\tau_t \varphi})(0) = \langle N(\varphi), T_t N(\varphi)\rangle = \int_{\widehat{G}} \chi(t)\, \mathrm{d}\sigma_\varphi(\chi)\,.$$

As this holds for all $t \in G$ we conclude (iii).

(iii)$\Longrightarrow$(ii): By (iii) the measures $|\widehat{\varphi}|^2\sigma$ agree with σ_φ and, hence, are finite for all $\varphi \in C_{\mathsf{c}}(G)$. Hence, σ is weakly admissible in the sense of Appendix C. Then, Proposition C.4 gives existence of a semi-measure η whose Fourier transform is σ. Then, (ii) follows as η satisfies

$$(\eta * \varphi * \widetilde{\psi})(0) = \eta((\varphi * \widetilde{\psi})^\dagger) = \int_{\widehat{G}} \widehat{\varphi}(\chi)\, \overline{\widehat{\psi}(\chi)}\, \mathrm{d}\sigma(\chi) = \langle N(\varphi), N(\psi)\rangle$$

for all $\varphi, \psi \in \mathbb{C}(G)$. Here, the last equality follows from (iii) and polarisation.

The last statement has been shown along the proofs of (iii)$\Longrightarrow$(ii) and (ii)$\Longrightarrow$(iii). $\square$

In the situation of the preceding lemma $L^2(\widehat{G}, \sigma)$ admits a natural continuous action of G by multiplication, i.e. via

$$(t \cdot f)(\chi) = \chi(t) f(\chi).$$

We will always think about $L^2(\widehat{G}, \sigma)$ as equipped with this action.

Theorem 1.29. *Let $N : C_{\mathsf{c}}(G) \longrightarrow \mathcal{H}$ be an intertwining $\mathcal{N}$-representation on a Hilbert space with dense range. Then, there exists a unique unitary map*

$$U : L^2(\widehat{G}, \sigma) \longrightarrow \mathcal{H}$$

with $\widehat{\varphi} \mapsto N(\varphi)$ for all $\varphi \in C_{\mathsf{c}}(G)$. This map is a G-map.

Proof. We have

$$\|\widehat{\varphi}\|^2_{L^2(\widehat{G},\sigma)} = \int |\widehat{\varphi}|^2 \,\mathrm{d}\sigma = \int \mathrm{d}\sigma_{\varphi} = \|N(\varphi)\|^2 .$$

This shows that the map is well-defined and isometric on the subspace $L := \{\widehat{\varphi} : \varphi \in C_{\mathsf{c}}(G)\} \subset L^2(\widehat{G}, \sigma)$. Hence, it can be extended to an isometric map on the closure of L. This closure is $L^2(\widehat{G}, \sigma)$. The map has dense range as N has dense range. As it is an isometry, it must then be unitary. Finally, note that the map is a G-map on L as N is a G-map. □

As a unitary map completely preserve spectral features, we immediately obtain the next result.

Corollary 1.30. *Let N be intertwining with dense image and assume furthermore that $\mathcal{H}$ is a Hilbert space. Then, T on $\mathcal{H}$ has pure point spectrum if and only if σ is a pure point measure. In this case, each eigenvalue has multiplicity one and the functions $c_\chi := U(1_\chi)$ for $\chi \in \widehat{G}$ with $\sigma(\{\chi\}) \neq 0$ form a canonical orthogonal system with dense span in $\mathcal{H}$ satisfying $\langle c_\chi, c_\chi \rangle = \sigma(\{\chi\})$.*

Consider the situation of the corollary. Now, assume that we are given additionally an orthonormal basis of eigenfunctions e_χ of $\mathcal{H}$. Then we can define the **Fourier coefficient** A_χ **of** N **(with respect to** (e_χ)**)** as the unique factor with

$$A_\chi e_\chi = c_\chi .$$

Then, the following will be true

$$|A_\chi|^2 = \|c_\chi\|^2 = \sigma(\{\chi\})$$

and

$$\langle N(\varphi), e_\chi \rangle = \langle \widehat{\varphi}, \frac{1_\chi}{A_\chi} \rangle \; = \widehat{\varphi}(\chi)\, A_\chi ,$$

where we used $\sigma(\{\chi\}) = A_\chi \cdot \overline{A_\chi}$ in the last step.

Now, let $\mathcal{A}$ be a van Hove sequence. Consider $BC^2_{\mathcal{A}}(G)$ (which is the closure of $C_{\mathsf{u}}(G)$ with respect to $\|\cdot\|_{b,2,\mathcal{A}}$) and note that it allows for an action of G by translations T_t, $t \in G$ (see Section 1).

A linear map

$$N : C_{\mathsf{c}}(G) \longrightarrow L^1_{loc}(G)$$

is called an **$\mathcal{A}$-representation** if it satisfies the following properties:

- $N(C_{\mathsf{c}}(G)) \subset BC^2_{\mathcal{A}}(G)$.
- $M_{\mathcal{A}}(f\overline{g})$ exists for all $f, g \in N(C_{\mathsf{c}}(G))$.
- $N(\tau_t(\varphi)) = \tau_t N(\varphi)$ for all $t \in G$ and $\varphi \in C_{\mathsf{c}}(G)$.

Any $\mathcal{A}$-representation N gives naturally rise to a $\mathcal{N}$-representation on a Hilbert space with dense range. Specifically, we define

$$\mathcal{H} := \text{Closure of } \{[N(\varphi)],\, \varphi \in C_{\mathsf{c}}(G)\}, \text{ in } BC^2_{\mathcal{A}}(G)/\equiv\ .$$

Note that

$$\langle [f],[g]\rangle := M_{\mathcal{A}}(f\overline{g})$$

is well defined and gives an inner product on $\mathcal{H}$ whose associated norm $\|\cdot\|$ satisfies

$$\|[f]\| = M_{\mathcal{A}}(|f|^2)^{1/2} = \|f\|_{b,2,\mathcal{A}}\,.$$

By construction, $\mathcal{H}$ is an Hilbert space. By the defining properties of N and Proposition 1.21 this Hilbert space is invariant under the translation action and the map

$$\overline{N} : C_{\mathsf{c}}(G) \longrightarrow \mathcal{H}\,, \qquad \varphi \mapsto [N(\varphi)]\,,$$

is G-invariant with dense range. Then, $(\overline{N}, \mathcal{H}, \langle\cdot,\cdot\rangle, T)$ is an $\mathcal{N}$-representation on a Hilbert space with dense range. If this $\mathcal{N}$-representation is intertwining, then we say that the $\mathcal{A}$-representation N is **intertwining**.

We will be mostly interested in $\mathcal{A}$-representations induced by measures. More specifically, for a measure μ on G we consider

$$N_\mu : C_{\mathsf{c}}(G) \longrightarrow L^1_{loc}(G) \quad \text{ defined by } \quad N_\mu(\varphi) := \mu * \varphi\,.$$

In order for this to give an $\mathcal{A}$-representation, $\mu*\varphi$ must belong to $BC^2_{\mathcal{A}}(G)$ for all $\varphi \in C_{\mathsf{c}}(G)$ and $M_{\mathcal{A}}(\mu*\varphi\cdot\overline{\mu*\psi})$ must exist for all $\varphi,\psi \in C_{\mathsf{c}}(G)$. The property $\tau_t N(\varphi) = N(\tau_t\varphi)$ for $t \in G$ and $\varphi \in C_{\mathsf{c}}(G)$ is then automatically satisfied (see Proposition 1.20). Finally, in order to apply the above theorems we also need that N_μ is intertwining. We next gather some classes of measures for which all these assumptions are satisfied.

Proposition 1.31 (Translation bounded measures with autocorrelation)**.** *Let μ be a translation bounded measure whose autocorrelation γ exists with respect to $\mathcal{A}$. Then, N_μ is an intertwining $\mathcal{A}$-representation.*

Proof. As μ is translation bounded, $\mu*\varphi$ belongs to $C_{\mathsf{u}}(G) \subset BC^2_{\mathcal{A}}(G)$ for all $\varphi \in C_{\mathsf{c}}(G)$. Moreover, by Proposition 1.3, we have $M_{\mathcal{A}}(\mu*\varphi\cdot\overline{\mu*\psi}) = (\gamma*\varphi*\widetilde{\psi})(0)$ for all $\varphi,\psi \in C_{\mathsf{c}}(G)$. So, N_μ is indeed an $\mathcal{A}$-representation. Clearly, N_μ possesses the autocorrelation γ and, hence, is intertwining by Lemma 1.28. ☐

Remark 1.32. From Proposition 1.3 we see that there is a converse of sorts to this proposition: If μ is translation bounded such that N_μ is an intertwining $\mathcal{A}$-representation; then μ possess an autocorrelation. Let us also note that in this case N_μ is continuous (as a short computation shows).

Proposition 1.33 ($\mathcal{A}$-representation derived from continuity)**.** *Let μ be a measure with $\mu*\varphi \in BC^2_{\mathcal{A}}(G)$ for all $\varphi \in C_{\mathsf{c}}(G)$ such that $M_{\mathcal{A}}(\mu*\varphi\cdot\overline{\mu*\psi})$ exists for all $\varphi,\psi \in C_{\mathsf{c}}(G)$. If $C_{\mathsf{c}}(G) \longrightarrow BC^2_{\mathcal{A}}(G)$, $\varphi \mapsto \mu*\varphi$, is continuous, then N_μ is an intertwining $\mathcal{A}$-representation. In particular, N_μ possesses a semi-autocorrelation.*

Proof. By Proposition 1.26 N_μ is an intertwining $\mathcal{A}$-representation. The last statement follows from Lemma 1.28. $\square$

Proposition 1.34 ($\mathcal{A}$-representation derived from positive measures)**.** *Let μ be a positive measure with $\mu * \varphi \in BC^2_{\mathcal{A}}(G)$ for all $\varphi \in C_{\mathsf{c}}(G)$ such that $M_{\mathcal{A}}(\mu * \varphi \cdot \overline{\mu * \psi})$ exists for all $\varphi, \psi \in C_{\mathsf{c}}(G)$. Then, N_μ is an intertwining $\mathcal{A}$-representation. In particular, N_μ possesses a semi-autocorrelation.*

Proof. This follows from the previous proposition as the map $C_{\mathsf{c}}(G) \longrightarrow BC^2_{\mathcal{A}}(G)$, $\varphi \mapsto \mu * \varphi$, is continuous. To see this consider a sequence (φ_n) in $C_{\mathsf{c}}(G)$ converging to $\varphi \in C_{\mathsf{c}}(G)$ in the inductive limit topology on $C_{\mathsf{c}}(G)$. Then, there exists a compact set $K \subset G$ such that the support of all φ_n (and then the support of φ as well) is contained in K. Let $\psi \in C_{\mathsf{c}}(G)$ be a nonnegative function with $\psi = 1$ on K. Then, $|\varphi - \varphi_n| \leq \|\varphi - \varphi_n\|_\infty \psi$ and

$$|\mu * \varphi - \mu * \varphi_n| = |\mu * (\varphi - \varphi_n)| \leq \|\varphi - \varphi_n\|_\infty \, (\mu * \psi)$$

follows from positivity of μ. This easily gives the desired continuity. $\square$

2. Mean almost periodicity and characterization of pure point diffraction

In this section, we introduce a notion of almost periodicity for functions and measures based on the (semi)norms arising from averaging. This form of almost periodicity seems not to have been considered before. We use it to characterize pure point diffraction.

2.1. Mean almost periodic functions and measures.

In this section, we introduce mean almost periodicity.

Definition 2.1 (Mean almost periodic functions in $C_{\mathsf{u}}(G)$)**.** Let $\mathcal{A} = (A_n)$ be a van Hove sequence. A function $f \in C_{\mathsf{u}}(G)$ is called **mean almost periodic** with respect to $\mathcal{A}$ if, for each $\varepsilon > 0$, the set

$$\{t \in G \, : \, \|f - \tau_t f\|_{b,1,\mathcal{A}} < \varepsilon\}$$

is relatively dense. The set of all mean almost periodic $f \in C_{\mathsf{u}}(G)$ will be denoted by $MAP_{\mathcal{A}}(G)$.

The previous definition features $\|\cdot\|_{b,1,\mathcal{A}}$. However, we could also work with $\|\cdot\|_{b,p,\mathcal{A}}$ for any $p \geq 1$ as shown next.

Proposition 2.2 (Independence of $p \geq 1$)**.** *For $f \in C_{\mathsf{u}}(G)$ the following assertions are equivalent:*

(i) *The function f is mean almost periodic.*

(ii) *There exists a $p \geq 1$ such that the set*
$$\{t \in G \, : \, \|f - \tau_t f\|_{b,p,\mathcal{A}} < \varepsilon\}$$
is relatively dense for each $\varepsilon > 0$.

(iii) *For all $p \geq 1$, the set*
$$\{t \in G \, : \, \|f - \tau_t f\|_{b,p,\mathcal{A}} < \varepsilon\}$$
is relatively dense for each $\varepsilon > 0$.

Proof. This is a direct consequence of Lemma 1.16 and Lemma 1.17. □

Proposition 2.3 (Mean almost periodic functions as subspace of $C_{\mathsf{u}}(G)$)**.** *Let $\mathcal{A}$ be a van Hove sequence. Then, the set of mean almost periodic $f \in C_{\mathsf{u}}(G)$ is a subspace of $C_{\mathsf{u}}(G)$ closed under uniform convergence and invariant under translations.*

Proof. As $\|\cdot\|_{b,1,\mathcal{A}} \le \|\cdot\|_\infty$, the set in question is closed under uniform convergence. It is invariant under translations as $\|\cdot\|_{b,1,\mathcal{A}}$ is invariant under translations when applied to bounded functions. It remains to show that it is a vector space. This follows by standard arguments. We sketch the proof for the convenience of the reader. The set is clearly closed under multiplication with scalars. To show that it is closed under addition of functions we proceed as follows: Whenever $\varepsilon > 0$ is given, we call a $t \in G$ an ε-almost period of f if $\|f - \tau_t f\|_{b,1,\mathcal{A}} < \varepsilon$ holds. Denote the set of all ε-almost periods of f by $A(f, \varepsilon)$.

Whenever t, s are ε-almost periods of f then $t - s$ is a 2ε-almost period of f as

$$\|f - \tau_{t-s} f\|_{b,1,\mathcal{A}} \le \|f - \tau_t f\|_{b,1,\mathcal{A}} + \|\tau_t f - \tau_{t-s} f\|_{b,1,\mathcal{A}} < \varepsilon + \|\tau_s f - f\|_{b,1,\mathcal{A}} < 2\varepsilon.$$

Note that this uses invariance of the norm under translations in a crucial way. This invariance holds for bounded functions.

Let now mean almost periodic $f, g \in C_{\mathsf{u}}(G)$ be given and $\varepsilon > 0$ be given. Then, we can find an open neighborhood U of $0 \in G$ with $\|f - \tau_t f\|_\infty, \|g - \tau_t g\|_\infty < \varepsilon$ for all $t \in U$.

The preceding considerations easily give that the $(A(f,\varepsilon) - A(f,\varepsilon)) + U$ is contained in $A(f, 3\varepsilon)$ and $(A(g,\varepsilon) - A(g,\varepsilon)) + U$ is contained in $A(g, 3\varepsilon)$. Moreover, as both $A(f,\varepsilon)$ and $A(g,\varepsilon)$ are relatively dense by assumption, the set

$$((A(f,\varepsilon) - A(f,\varepsilon)) + U) \cap ((A(g,\varepsilon) - A(g,\varepsilon)) + U)$$

is relatively dense by standard arguments (see e.g. appendix of [33]). Hence, there is a relatively dense set of 3ε-almost periods of both f and g and this easily gives that $f + g$ has a relatively dense set of 6ε- periods. As $\varepsilon > 0$ was arbitrary, we infer that $f + g$ is mean almost periodic. □

Definition 2.4 (Mean almost periodic functions)**.** Let $\mathcal{A} = (A_n)$ be a van Hove sequence and let $1 \le p < \infty$. We define $Map^p_{\mathcal{A}}(G)$ to be the closure of $MAP_{\mathcal{A}}(G)$ in $BC^p_{\mathcal{A}}(G)$ with respect to $\|\cdot\|_{b,p,\mathcal{A}}$. The elements of $Map^p_{\mathcal{A}}(G)$ are called p-mean almost periodic functions.

Remark 2.5. We have defined $Map^p_{\mathcal{A}}(G)$ as the closure of $MAP_{\mathcal{A}}(G)$ in $BC^p_{\mathcal{A}}(G)$. Of course, we could also have taken the closure in $BL^p_{\mathcal{A}}(G)$ (as $MAP_{\mathcal{A}}(G) \subset C_{\mathsf{u}}(G)$ holds).

We now turn to an intrinsic characterization of $Map^p_{\mathcal{A}}(G)$.

Lemma 2.6 (Intrinsic characterization of mean almost periodic functions)**.** *An $f \in BC^p_{\mathcal{A}}(G)$ is p-mean almost periodic if and only if for each $\varepsilon > 0$ the set*

$$\{t \in G : \|T_t[f] - [f]\|_{b,p,\mathcal{A}} < \varepsilon\}$$

is relatively dense in G.

Proof. The 'only if' statement follows easily from the definition and a short approximation argument. It remains to show the 'if' statement. So, let $f \in BC^p_{\mathcal{A}}(G)$ be given such that for each $\varepsilon > 0$ the set

$$\{t \in G : \|T_t[f] - [f]\|_{b,p,\mathcal{A}} < \varepsilon\}$$

is relatively dense in G. Without loss of generality we can assume that f is bounded as otherwise we could use Proposition 1.19 to replace it by a cut-off version of it. Now, we have $T(\varphi)[f] = [f * \varphi]$ for any $\varphi \in C_{\mathsf{c}}(G)$ and $f * \varphi$ belongs to $C_{\mathsf{u}}(G)$. Hence, a short computation gives

$$\|\tau_t(f * \varphi) - f * \varphi\|_{b,p,\mathcal{A}} = \|T(\varphi)(T_t[f] - [f])\|_{b,p,\mathcal{A}} \leq \|\varphi\|_1 \, \|T_t[f] - [f]\|_{b,p,\mathcal{A}}$$

by the results above. From the assumption on f we then easily infer that $f * \varphi$ is mean almost periodic. The desired statement now follows by choosing an approximate identity (φ_α) in $C_{\mathsf{c}}(G)$ and noting that $T(\varphi_\alpha)[f] = [f * \varphi_\alpha] \to [f]$. $\square$

Proposition 2.7 (Inclusion of spaces)**.** *$Map^p_{\mathcal{A}}(G)$ is contained in $Map^1_{\mathcal{A}}(G)$ with continuous inclusion for any $p \geq 1$. Moreover, for any $p \geq 1$, we have*

$$Map^p_{\mathcal{A}}(G) \cap L^\infty(G) = Map^1_{\mathcal{A}}(G) \cap L^\infty(G) \, .$$

Proof. The first statement as well as the inclusion $\subseteq$ in the second statement follow immediately from Lemma 1.17. The inclusion $\supseteq$ in the second statement follows from Lemma 1.16 and a short approximation argument. $\square$

Theorem 2.8 (Completeness of $Map^p_{\mathcal{A}}(G)$)**.** *Let $\mathcal{A} = (A_n)$ be a van Hove sequence and let $1 \leq p < \infty$ be given. Then, $(Map^p_{\mathcal{A}}(G), \|\cdot\|_{b,p,\mathcal{A}})$ is a complete vector space and $Map^p_{\mathcal{A}}(G)/\equiv$ is invariant under translations.*

Proof. By construction $Map^p_{\mathcal{A}}(G)$ is the closure of a vector space in the ambient space $BL^p_{\mathcal{A}}(G)$. Hence, it closed in this ambient space. As the ambient space $BL^p_{\mathcal{A}}(G)$ is complete by Theorem 1.18, so is then $Map^p_{\mathcal{A}}(G)$. Invariance under translations follows as $Map^p_{\mathcal{A}}(G)$ is invariant under translations. $\square$

We next show that taking the closure does not introduce additional mean almost periodic functions in $C_{\mathsf{u}}(G)$.

Proposition 2.9 (Consistency)**.** *Let $\mathcal{A} = (A_n)$ be a van Hove sequence and let $1 \leq p < \infty$. Let $f \in C_{\mathsf{u}}(G)$ be given. Then, the following assertions are equivalent:*

(i) *The function f is mean almost periodic.*
(ii) *The function f belongs to $Map^p_{\mathcal{A}}(G)$ for some $p \geq 1$.*
(iii) *The function f belongs to $Map^p_{\mathcal{A}}(G)$ for all $p \geq 1$.*

Proof. (i)$\Longrightarrow$(iii) and (iii)$\Longrightarrow$(ii) are obvious. We show (ii)$\Longrightarrow$(i): Let $\varepsilon > 0$ be arbitrary. By definition there exists a mean almost periodic $h \in C_{\mathsf{u}}(G)$ with $\|f - h\|_{b,p,\mathcal{A}} < \varepsilon/3$. As h is mean almost periodic, there exist a relatively dense set $\varLambda \subset G$ with

$$\|h - \tau_t h\|_{b,p,\mathcal{A}} < \varepsilon/3$$

for all $t \in \mathit{\Lambda}$. Then, we find for all $t \in \mathit{\Lambda}$

$$\|f - \tau_t f\|_{b,p,\mathcal{A}} \leq \|f - h\|_{b,p,\mathcal{A}} + \|h - \tau_t h\|_{b,p,\mathcal{A}} + \|\tau_t h - \tau_t f\|_{b,p,\mathcal{A}} < \varepsilon,$$

where we used $\|\tau_t h - \tau_t f\|_{b,p,\mathcal{A}} = \|h - f\|_{b,p,\mathcal{A}}$. Now, the statement follows from Proposition 2.2. □

Definition 2.10 (Mean almost periodic measures)**.** Let $\mathcal{A} = (A_n)$ be a van Hove sequence, and let $1 \leq p < \infty$. A measure μ is called **mean p-almost periodic** with respect to $\mathcal{A}$ if, for all $\varphi \in C_{\mathsf{c}}(G)$ we have $\mu * \varphi \in Map^p_{\mathcal{A}}(G)$. The space of mean p-almost periodic measures is denoted by $\mathcal{M}\mathsf{ap}^p_{\mathcal{A}}(G)$.

As noted in Proposition 2.3, $Map_{\mathcal{A}}(G)$ contains $Map^p_{\mathcal{A}}(G)$ for all $1 \leq p$ by Lemma 1.17 and then also $\mathcal{M}\mathsf{ap}^1_{\mathcal{A}}(G)$ contains $\mathcal{M}\mathsf{ap}^p_{\mathcal{A}}(G)$ for all $1 \leq p$. For this reason we often drop the superscript 1 when referring to $\mathcal{M}\mathsf{ap}^1_{\mathcal{A}}(G)$ and call its elements just **mean almost periodic measures**. From Proposition 2.3 we also see that a translation bounded measure belongs to $\mathcal{M}\mathsf{ap}^p_{\mathcal{A}}(G)$ for some $p \geq 1$ if and only if it belongs to $\mathcal{M}\mathsf{ap}^1_{\mathcal{A}}(G)$.

For $f \in C_{\mathsf{u}}(G)$ mean almost periodicity as a function and as a measure are equivalent. In fact, even the following holds.

Proposition 2.11. *Let $\mathcal{A}$ be a van Hove sequence on G and $1 < p < \infty$ be given. Let f belong to the closure of $C_{\mathsf{u}}(G)$ in $BL^1_{\mathcal{A}}(G)$ and assume that $f\theta_G$ is translation bounded. Then, f is mean almost periodic if and only if $f\theta_G \in \mathcal{M}\mathsf{ap}_{\mathcal{A}}(G)$. In particular, $f \in C_{\mathsf{u}}(G)$ is mean almost periodic if and only if $f\theta_G$ is mean almost periodic.*

Proof. This follows from Proposition 1.23 applied to $\mathcal{S}' = Map_{\mathcal{A}}(G)$. The assumption of that proposition are satisfied by Proposition 2.3. □

2.2. **Characterization of pure point diffraction.** In this subsection, we show that a translation bounded measure is pure point diffractive if and only if it is mean almost periodic. This generalizes partial results in this direction obtained earlier by [7, 19].

Theorem 2.12 (Characterization $\mathcal{A}$-representation with pure point spectrum)**.** *Let $N : C_{\mathsf{c}}(G) \longrightarrow BC^2_{\mathcal{A}}(G)$ be an $\mathcal{A}$-representation with semi-autocorrelation η and diffraction measure σ. Then, the following assertions are equivalent:*

(i) *The diffraction measure σ is a pure point measure.*
(ii) $N(C_{\mathsf{c}}(G)) \subset Map^2_{\mathcal{A}}(G)$.

Proof. The measure σ is a pure point measure if and only if $|\widehat{\varphi}|^2\sigma$ is a pure point measure for all $\varphi \in C_{\mathsf{c}}(G)$. By Lemma 1.28, we have $|\widehat{\varphi}|^2\,\sigma = \sigma_\varphi$ for all $\varphi \in C_{\mathsf{c}}(G)$. So, from Proposition 1.25 we see that σ is a pure point measure if and only if the set

$$\{t \in G : \|T_t[N(\varphi)] - [N(\varphi)]\| < \varepsilon\}$$

is relatively dense for all $\varepsilon > 0$ and all $\varphi \in C_{\mathsf{c}}(G)$. This in turn is equivalent to $N(\varphi) \in Map^2_{\mathcal{A}}(G)$. □

As a consequence of the previous considerations, we obtain a characterization of pure point diffraction.

Theorem 2.13 (Characterization pure point diffraction)**.** *Let μ be a translation bounded measure and let γ be its autocorrelation with respect to some van Hove sequence $\mathcal{A}$. Then, $\widehat{\gamma}$ is a pure point measure if and only if μ is mean almost periodic with respect to $\mathcal{A}$.*

Proof. As discussed in Proposition 1.31 a translation bounded measure μ with autocorrelation γ gives rise to the $\mathcal{A}$-representation N_μ (defined by $N_\mu(\varphi) = \mu * \varphi$) with autocorrelation γ. Now, the claim is a direct consequence of the previous theorem and the definition of mean almost periodicity for measures. $\square$

Remark 2.14. In Theorem 2.13, one can relate explicitly the mean almost periods of $\mu * \varphi$ to the Bohr almost periods of $\gamma * \varphi * \widetilde{\varphi}$. Indeed, it is not hard to see that

(a) every $\frac{\varepsilon^2}{2}$-Bohr almost period of $\gamma * \varphi * \widetilde{\varphi}$ is an ε-mean almost period for $\mu * \varphi$.
(b) every $\frac{\varepsilon^2}{1+(\gamma*\varphi*\widetilde{\varphi})(0)\|\mu*\varphi\|_\infty}$-mean almost period for $\mu * \varphi$ is an ε-Bohr almost period for $\gamma * \varphi * \widetilde{\varphi}$.

2.3. **Delone and Meyer sets with pure point diffraction.** In the context of aperiodic order, a particular case of interest are translation bounded measures arising from Delone sets and Meyer sets. Specifically, for a uniformly discrete set $\varLambda \subseteq G$ we define its **Dirac comb** to be the measure

$$\delta_\varLambda := \sum_{t\in\varLambda} \delta_t$$

with δ_s being the unit point mass at $s \in G$. For such measures, sufficient conditions for pure point diffraction have been given earlier. Here, we show how mean almost periodicity of Dirac combs of Delone sets and Meyer sets can be characterized. This is then used to exhibit earlier results as particular cases of Theorem 2.13.

Recall that a subset Λ of G is a Delone set if it is relatively dense and uniformly discrete. First, in the spirit of [19], given two Delone sets Λ and Γ and an open set $0 \in U \subseteq G$, we denote by $\Lambda \,\Delta_U\, \Gamma$ the set

$$\Lambda \,\Delta_U\, \Gamma := (\Lambda \backslash (\Gamma + U)) \cup (\Gamma \backslash (\Lambda + U)) \ .$$

Theorem 2.15 (Characterizing mean almost periodicity for Delone sets)**.** *If Λ is a Delone set, then δ_Λ is mean almost periodic if and only if, for each open neighbourhood $U \subset G$ of 0 and each $\varepsilon > 0$, the set*

$$\Big\{ t \in G \,:\, \limsup_{n\to\infty} \frac{\sharp\,(\Lambda \,\Delta_U\, (t+\Lambda)) \cap A_n}{|A_n|} < \varepsilon \Big\}$$

is relatively dense.

Remark 2.16. The result shows that mean almost periodicity for Delone sets is equivalent to the conditions of [19, Thm. 3.3(5)]. Hence, it follows that [19, Thm. 3.3] is a special case of Theorem 2.13.

Proof. $\Longrightarrow$: Let U be an open neighbourhood of 0, and let $\varepsilon > 0$. Pick some open set V such that $0 \in V$ and $\overline{V} \subset U$. Let $\varphi \in C_{\mathsf{c}}(G)$ be such that $\varphi \geq 1_V$ and $\operatorname{supp}(\varphi) \subset U$. A simple computation shows that

$$|(\delta_\Lambda * \varphi)(x) - \tau_t(\delta_\Lambda * \varphi)(x)| \geq 1$$

for all $t \in G$ and for all $x \in (\Lambda \,\Delta_U\, (t+\Lambda)) + V$. Therefore, for all $t \in G$ we have

$$\theta_G(V) \limsup_{n\to\infty} \frac{\sharp\,(\Lambda \,\Delta_U\, (t+\Lambda)) \cap A_n}{|A_n|} \leq \overline{M}_{\mathcal{A}}(\delta_\Lambda * \varphi - \tau_t(\delta_\Lambda * \varphi))\,.$$

The claim follows.

$\Longleftarrow$: Fix some open neighbourhood $U = -U$ of 0 such that Λ is $U - U$ uniformly discrete. Let $\varphi \in C_{\mathsf{c}}(G)$, $\varepsilon > 0$, and let $K := \operatorname{supp}(\varphi)$. As φ is uniformly continuous, there exists some open neighbourhood $V \subset U$ of 0 such that, for all $x, y \in G$ with $x - y \in V$, we have

$$|\varphi(x) - \varphi(y)| < \frac{\varepsilon}{2\,|K|\,\overline{\operatorname{dens}}(\Lambda) + 1}\,.$$

Now, for each $x \in \Lambda \backslash (\Lambda \,\Delta_V\, (t+\Lambda))$, there exists a unique $y_x \in (t+\Lambda) \backslash (\Lambda \,\Delta_V\, (t+\Lambda))$ such that $x - y_x \in V$. We know that the set

$$P := \Big\{ t \in G \,:\, \limsup_{n\to\infty} \frac{\sharp\,(\Lambda \,\Delta_V\, (t+\Lambda)) \cap A_n}{|A_n|} < \frac{\varepsilon}{2\int_G |\varphi(t)|\, \mathrm{d}t + 1} \Big\}$$

is relatively dense. It follows from a standard Fubini and van Hove type argument that, for all $t \in P$, we have

$$\begin{aligned}
\overline{M}_{\mathcal{A}}(\delta_\Lambda * \varphi - \tau_t \delta_\Lambda * \varphi) &\leq \limsup_{n\to\infty} \frac{\sharp\,(\Lambda \,\Delta_V\, (t+\Lambda)) \cap A_n}{|A_n|} \int_G |\varphi(t)|\, \mathrm{d}t \\
&\qquad + \limsup_{n\to\infty} \frac{1}{|A_n|} \sum_{x \in \Lambda \backslash (\Lambda \Delta_V (t+\Lambda)) \cap A_n} \int_G |\varphi(x-t) - \varphi(y_x - t)|\, \mathrm{d}t \\
&\leq \frac{\varepsilon}{2} + \limsup_{n\to\infty} \frac{1}{|A_n|} \sum_{x \in \Lambda \backslash (\Lambda \Delta_V (t+\Lambda)) \cap A_n} \frac{\varepsilon}{2\,|K|\,\overline{\operatorname{dens}}(\Lambda) + 1} |K| \\
&\leq \frac{\varepsilon}{2} + \overline{\operatorname{dens}}(\Lambda) \frac{\varepsilon}{2\,|K|\,\overline{\operatorname{dens}}(\Lambda) + 1}\, |K| \,<\, \varepsilon\,.
\end{aligned}$$

This finishes the proof. $\square$

Remark 2.17. If G is metrisable, it is easy to show that the condition in Theorem 2.15 can be replaced by the following statement: For each $\varepsilon > 0$, the set

$$P_\varepsilon := \Big\{ t \in G \,:\, \limsup_{n\to\infty} \frac{\sharp\,\big(\Lambda \,\Delta_{B_\varepsilon(0)}\, (t+\Lambda)\big) \cap A_n}{|A_n|} < \varepsilon \Big\}$$

is relatively dense, compare [19]. Here, $B_r(0)$ denotes the ball around $0 \in G$ with radius $r \geq 0$.

We now turn to Meyer sets. Here, a Delone set $\Lambda \subseteq G$ is **Meyer** if $\Lambda - \Lambda - \Lambda$ is uniformly discrete. Moreover, if G is compactly generated, this is equivalent to $\Lambda - \Lambda$ being uniformly discrete (and even weaker conditions [8, 54]).

Theorem 2.18 (Characterizing mean almost periodicity for Meyer sets)**.** *If Λ is a Meyer set, then δ_Λ is mean almost periodic if and only if, for each $\varepsilon > 0$, the set*

$$\Big\{ t \in G \,:\, \limsup_{n\to\infty} \frac{\sharp\,(\Lambda \,\Delta\, (t+\Lambda)) \cap A_n}{|A_n|} < \varepsilon \Big\}$$

is relatively dense.

Remark 2.19. The result shows that mean almost periodicity for Meyer sets is equivalent to the condition given in [7, Thm. 5]. Hence, it follows that [7, Thm. 5] is a special case of Theorem 2.13.

Proof. $\Longrightarrow$: If $\mathrm{dens}(\Lambda)=0$, the claim is trivial. So, without loss of generality, we can assume that $\mathrm{dens}(\Lambda)>0$.

Fix an open and precompact neighbourhood $U=-U$ of 0 such that $\Lambda-\Lambda$ is U uniformly discrete. Let $0<\varepsilon<\mathrm{dens}(\Lambda)$. We know that the set

$$P:=\left\{t\in G\,:\,\limsup_{n\to\infty}\frac{\sharp\left(\Lambda\,\Delta_U\,(t+\Lambda)\right)\cap A_n}{|A_n|}<\frac{\varepsilon}{2}\right\}$$

is relatively dense. As

$$\limsup_{n\to\infty}\frac{\sharp\left(\Lambda\,\Delta_U\,(t+\Lambda)\right)\cap A_n}{|A_n|}<\mathrm{dens}(\Lambda)\,,$$

there exists some $x\in\Lambda\cap\big((t+\Lambda)+U\big)$. Since $x\in t+\Lambda+U$, there exists some $s\in U$ such that $x\in(t-u)+\Lambda$. Then, $x=(t-u)+x'$ for some $x'\in\Lambda$ and hence $t-u=x-x'\in\Lambda-\Lambda$. We claim that

$$\begin{aligned}&\Lambda\backslash((t-u)+\Lambda)\subseteq\left(\Lambda\,\Delta_U\,(t+\Lambda)\right)\,,\\&((t-u)+\Lambda)\backslash\Lambda\subseteq-u+\left(\Lambda\,\Delta_U\,(t+\Lambda)\right)\,.\end{aligned}$$

First, let $y\in\Lambda\backslash((t-u)+\Lambda)$. Assume by contradiction that $y\in(t+\Lambda)+U$. Then, there exist some $z\in\Lambda, v\in U$ such that $y=t+z+v$. Then, one has

$$(y-z)-u=(t-u)+v\,.$$

Since $y-z,t-u\in\Lambda-\Lambda$, $-u,v\in U$ and $\Lambda-\Lambda$ is U uniformly discrete, we get that $y-z=t-u$ and hence $y\in(t-u)+\Lambda$ a contradiction. Therefore, we can conclude that $y\in\Lambda\backslash\left((t+\Lambda)+U\right)\subseteq(\Lambda\Delta_U(t+\Lambda))$.

Next, let $y\in((t-u)+\Lambda)\backslash\Lambda+U$. We will show that

$$y\in-u+(t+\Lambda)\backslash\big(-u+\Lambda+U\big)\subseteq(-u+\Lambda)\,\Delta_U\,(t+\Lambda-u)=-u+\left(\Lambda\,\Delta_U\,(t+\Lambda)\right)\,.$$

Assume by contradiction that $y\in-u+\Lambda+U$. Then, there exist some $z\in\Lambda, v\in U$ such that $y=-u+z+v$. Moreover, as $y\in((t-u)+\Lambda)$, there exists some $z'\in\Lambda$ so that $y=t-u+z'$. Therefore, we have

$$(t-u)+u=(z-z')+v\,.$$

Since $t-u,z-z'\in\Lambda-\Lambda,u,v\in U$ and $\Lambda-\Lambda$ is U uniformly discrete, we get $t-u=z-z'$ and hence $y=t-u+z'=z\in\Lambda\subset\Lambda+U$, which is a contradiction. This shows that, for each $t\in P$, there exists some $u\in U$ such that

$$\begin{aligned}\limsup_{n\to\infty}\frac{\sharp\left(\Lambda\,\Delta\,(t-u+\Lambda)\right)\cap A_n}{|A_n|}&\leq\limsup_{n\to\infty}\frac{\sharp\left(\Lambda\backslash((t-u)+\Lambda)\right)\cap A_n}{|A_n|}\\&\quad+\limsup_{n\to\infty}\frac{\sharp\left(((t-u)+\Lambda)\backslash\Lambda\right)\cap A_n}{|A_n|}\end{aligned}$$

$$\leq 2 \limsup_{n\to\infty} \frac{\sharp\left(\Lambda\, \Delta_U\, (t+\Lambda)\right)\cap A_n}{|A_n|} < \varepsilon\,.$$

Since $\bar{U}$ is compact, the claim follows.

$\Longleftarrow$: This follows from Theorem 2.15 and $\Lambda\, \Delta_U\, (t+\Lambda) \subseteq \Lambda\, \Delta\, (t+\Lambda)$. □

3. Besicovitch almost periodicity and the phase problem

In this section, we first study Besicovitch almost periodic functions. This is done in two steps. In the first step, we develop some general theory and in the second step we restrict attention to a certain Hilbert space of Besicovitch almost periodic functions. This Hilbert space structure will allow us to set up a Fourier expansion type theory. Having this theory at hand we can then turn to $\mathcal{A}$-representations with values in the Besicovitch almost periodic and characterize them by pure point diffraction together with existence of the Fourier coefficients. As a consequence, we obtain our solution of the phase problem.

3.1. **Besicovitch almost periodic functions: general theory.** In this section, we study Besicovitch almost periodic functions. Part of the subsequent considerations are known and we indicate relevant literature along the way.

Definition 3.1 (Besicovitch almost periodic functions)**.** Let $\mathcal{A} = (A_n)$ be a van Hove sequence, and let $1 \leq p < \infty$. A function $f \in L^p_{loc}(G)$ is called **Besicovitch p-almost periodic** with respect to $\mathcal{A}$ if, for each $\varepsilon > 0$, there exists a trigonometric polynomial $P = \sum_{k=1}^n c_k \chi_k$ with $c_k \in \mathbb{C}$ and $\chi_k \in \widehat{G}$ such that

$$\|f - P\|_{b,p,\mathcal{A}} < \varepsilon\,.$$

We denote the space of Besicovitch p-almost periodic functions $Bap^p_{\mathcal{A}}(G)$. When $p = 1$ we will simply write $Bap_{\mathcal{A}}(G) := Bap^1_{\mathcal{A}}(G)$.

Remark 3.2. A function is Besicovitch p-almost periodic if and only if, for each $\varepsilon > 0$, there exists a Bohr almost periodic function g such that $\|f - g\|_{b,p,\mathcal{A}} < \varepsilon$. In particular, all trigonometric polynomials and all Bohr almost periodic functions are Besicovitch almost periodic (for any $p \geq 1$). In fact, it is not hard to see that every weakly almost periodic function is Besicovitch p-almost periodic (for any p).

From the considerations in Section 1, we obtain the following.

Proposition 3.3 (Inclusions of spaces)**.** (a) *For each $1 \leq p < \infty$, we have*

$$Bap^p_{\mathcal{A}}(G) \subset Map^p_{\mathcal{A}}(G)$$

with continuous inclusion map.

(b) *For each $1 \leq p \leq q < \infty$, we have*

$$Bap^q_{\mathcal{A}}(G) \subseteq Bap^p_{\mathcal{A}}(G) \subseteq Bap_{\mathcal{A}}(G)$$

with continuous inclusion map.

Proof. (a) Let $f \in Bap_{\mathcal{A}}^p(G)$ be arbitrary. Let $\varepsilon > 0$ be arbitrary. Then, for each $\varepsilon > 0$ there exists an trigonometric polynomial P with $\|f - P\|_{b,p,\mathcal{A}} < \varepsilon$. Now, any trigonometric polynomial clearly is Bohr almost periodic and, hence, is mean almost periodic. Hence, we can approximate f arbitrarily well by mean almost periodic functions and $f \in Map_{\mathcal{A}}^p(G)$ follows.

(b) This follows from Lemma 1.17. □

Remark 3.4. The space $Map_{\mathcal{A}}^p(G)$ is in general strictly bigger than $Bap_{\mathcal{A}}^p(G)$. To see this, we consider $G = \mathbb{R}$ with the van Hove sequence $A_n = [-n, n]$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function

$$f(x) := \begin{cases} 1 & \text{if } x > 1 \\ x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x < 0 \end{cases}$$

Then, clearly f is uniformly continuous and $\|f - \tau_t f\|_{b,1,\mathcal{A}} = 0$ for all $t \in \mathbb{R}$. So, f is mean almost periodic. However, f is not Besicovitch p-almost periodic. Indeed, any trigonometric polynomial sufficiently close to f in Besicovitch norm must essentially be close to 1 on $x > 0$ and close to 0 on $x < 0$. This, however, is not possible for a Bohr almost periodic function (see Remark 6.12 below for a similar reasoning).

We next show that for bounded functions Besicovitch p-almost periodicity is independent of p. We start with the following preliminary lemma.

Lemma 3.5. *Let $f \in L^\infty(G) \cap Bap_{\mathcal{A}}^p(G)$. Then, for each $\varepsilon > 0$ there exists some trigonometric polynomial P such that*

$$\|f - P\|_{b,p,\mathcal{A}} < \varepsilon \qquad \text{and} \qquad \|P\|_\infty < \|f\|_\infty + 1\,.$$

Proof. It suffices to prove the claim for $0 < \varepsilon < 1$. Fix such an ε and pick some trigonometric polynomial Q such that $\|f - Q\|_{b,p,\mathcal{A}} < \frac{\varepsilon}{2}$. Set $L := \|f\|_\infty + \frac{1}{2}$, and define $g := c_L(Q)$ where c_L is the cutoff function from (2). Since $|f(x)| \leq L$ for all $x \in G$, we have $\|f - g\|_{b,p,\mathcal{A}} = \|c_L(f) - c_L(g)\|_{b,p,\mathcal{A}} \leq \|f - Q\|_{b,p,\mathcal{A}}$ and $\|g\|_\infty \leq L$. It is easy to see that $|\tau_t g(x) - g(x)| \leq |\tau_t Q(x) - Q(x)|$ for all $t, x \in G$. Since Q is a trigonometric polynomial, it is Bohr almost periodic, and hence so is g. Therefore, there exists a trigonometric polynomial P such that $\|g - P\|_\infty < \frac{\varepsilon}{2}$. We then have

$$\|P\|_\infty \leq \|P - g\|_\infty + \|g\|_\infty < L + \frac{\varepsilon}{2} < \|f\|_\infty + 1$$

and

$$\|f - P\|_{b,p,\mathcal{A}} \leq \|f - g\|_{b,p,\mathcal{A}} + \|g - P\|_{b,p,\mathcal{A}} \leq \|f - Q\|_{b,p,\mathcal{A}} + \|g - P\|_\infty < \varepsilon\,.$$

This finishes the proof. □

Remark 3.6 (Normal contractions)**.** The construction of g by cutting of Q in the preceding proof points to a general feature of the spaces of almost periodic functions: A map $c : \mathbb{C} \longrightarrow \mathbb{C}$ with $c(0) = 0$ and $|c(z) - c(w)| \leq |z - w|$ is called a normal contraction. Then, we clearly have $\|c(f) - c(g)\|_\infty \leq \|f - g\|_\infty$ for all $f, g \in C_{\mathsf{u}}(G)$ as well as $\|c(f) - c(g)\|_{b,p,\mathcal{A}} \leq \|f - g\|_{b,p,\mathcal{A}}$ for all $f, g \in BL_{\mathcal{A}}^p(G)$. This easily shows that the set of Bohr almost periodic functions as well as $Map_{\mathcal{A}}^p(G)$ and $Bap_{\mathcal{A}}^p(G)$ are closed under taking normal contractions.

Now, we can show that Besicovitch-p-almost periodicity does not depend on p for bounded functions (see also [36, Thm. 2.1]).

Proposition 3.7. *For each $1 \le p < \infty$ we have $Bap^p_{\mathcal{A}}(G) \cap L^\infty(G) = Bap_{\mathcal{A}}(G) \cap L^\infty(G)$.*

Proof. The inclusion $\subseteq$ follows from Proposition 3.3. To show the inclusion $\supseteq$ let $f \in Bap_{\mathcal{A}}(G) \cap L^\infty(G)$, and let $\varepsilon > 0$. Then, by Lemma 3.5 we can find some trigonometric polynomial P such that

$$\|f - P\|_{b,1,\mathcal{A}} < \frac{\varepsilon^p}{(2\|f\|_\infty)^{p-1} + 1} \qquad \text{and} \qquad \|P\|_\infty < \|f\|_\infty + 1 .$$

Therefore, by Lemma 1.16 we have

$$\|f - P\|^p_{b,p,\mathcal{A}} \le \|f - P\|^{p-1}_\infty \, \|f\|_{b,1,\mathcal{A}} \le (\|f\|_\infty + \|P\|_\infty)^{p-1} \frac{\varepsilon^p}{(2\|f\|_\infty)^{p-1} + 1} < \varepsilon^p .$$

This finishes the proof. □

We next review the basic properties of $Bap^p_{\mathcal{A}}(G)$. Since $Bap_{\mathcal{A}}(G) \cap L^\infty(G) = Bap^p_{\mathcal{A}}(G) \cap L^\infty(G)$ and $Bap^p_{\mathcal{A}}(G) \subset Bap_{\mathcal{A}}(G)$, some of the properties below need only be proven for $Bap_{\mathcal{A}}(G)$. Recall from Remark 3.2 that the subspace $SAP(G)$ of Bohr almost periodic functions is dense in $Bap^p_{\mathcal{A}}(G)$ as every trigonometric polynomial is Bohr almost periodic. We will often use this fact.

Property (a) below can be found in [36, Thm. 2.4].

Proposition 3.8. *Let $\mathcal{A}$ be a van Hove sequence on G.*

(a) *If $f \in Bap_{\mathcal{A}}(G)$, then the mean of f*

$$M_{\mathcal{A}}(f) = \lim_{m\to\infty} \frac{1}{|A_m|} \int_{A_m} f(t)\, dt$$

exists with respect to (A_m) and $|M_{\mathcal{A}}(f)| \le \|f\|_{b,1,\mathcal{A}}$ holds.

(b) *If $f, g \in Bap^p_{\mathcal{A}}(G)$ for some $1 \le p$ and $\chi \in \widehat{G}, c \in \mathbb{C}$, then $f \pm g, cf, \overline{f}, \chi f \in Bap^p_{\mathcal{A}}(G)$.*

(c) *If $f, g \in Bap_{\mathcal{A}}(G) \cap L^\infty(G)$, then $fg \in Bap_{\mathcal{A}}(G) \cap L^\infty(G)$ and $g(\cdot - t) \in Bap_{\mathcal{A}}(G)$ for all $t \in G$.*

Proof. (a) Define M_n on $L^1_{loc}(G)$ by $M_n(f) := \frac{1}{|A_n|} \int_{A_n} f(t)\, \mathrm{d}t$. Then, a short computation shows

$$\limsup_{n\to\infty} |M_n(f) - M_n(g)| \le \limsup_{n\to\infty} M_n(|f-g|) = \|f-g\|_{b,1,\mathcal{A}} .$$

Moreover, $\lim_{n\to\infty} M_n(f) = M_{\mathcal{A}}(f)$ exists for all $f \in SAP(G)$. As $Bap^1_{\mathcal{A}}(G)$ is the closure of $SAP(G)$ with respect to $\|\cdot\|_{b,1,\mathcal{A}}$, we easily infer that $M_{\mathcal{A}}(f)$ exists for $f \in Bap^1_{\mathcal{A}}(G)$ and satisfies $|M_{\mathcal{A}}(f)| \le \|f\|_{b,1,\mathcal{A}}$ for all $f \in Bap^1_{\mathcal{A}}(G)$.

(b) For each trigonometric polynomials P, Q we have

$$\begin{aligned}
\|f + g - (P+Q)\|_{b,p,\mathcal{A}} &\le \|f - P\|_{b,p,\mathcal{A}} + \|g - Q\|_{b,p,\mathcal{A}} ,\\
\|cf - cP\|_{b,p,\mathcal{A}} &= |c|\, \|f - P\|_{b,p,\mathcal{A}} ,\\
\|\chi f - \chi P\|_{b,p,\mathcal{A}} &= \|\overline{f} - \overline{P}\|_{b,p,\mathcal{A}} = \|f - P\|_{b,p,\mathcal{A}} .
\end{aligned}$$

(c) Let $\varepsilon > 0$. Pick some trigonometric polynomial P such that

$$\|f - P\|_{b,1,\mathcal{A}} < \frac{\varepsilon}{2\|g\|_\infty + 1} \qquad \text{and} \qquad \|P\|_\infty < \|f\|_\infty + 1\,.$$

Next, pick some trigonometric polynomial Q such that $\|g - Q\|_{b,1,\mathcal{A}} < \frac{\varepsilon}{2\|f\|_\infty + 3}$. Then, we have

$$\begin{aligned} \|fg - PQ\|_{b,1,\mathcal{A}} &\le \|fg - Pg\|_{b,1,\mathcal{A}} + \|Pg - PQ\|_{b,1,\mathcal{A}} \\ &\le \|f - P\|_{b,1,\mathcal{A}}\,\|g\|_\infty + \|g - Q\|_{b,1,\mathcal{A}}\,\|P\|_\infty < \varepsilon\,. \end{aligned}$$

This finishes the proof. □

We can talk about Fourier coefficients on $Bap_{\mathcal{A}}(G)$, see [36, Thm. 2.5] for existence of the corresponding limits as well.

Corollary 3.9 (Existence and continuity of the Fourier coefficients)**.** *For any $\chi \in \widehat{G}$, the map*

$$a_\chi^{\mathcal{A}} : Bap_{\mathcal{A}}(G) \longrightarrow \mathbb{C}\,, \qquad f \mapsto \lim_{m\to\infty} \frac{1}{|A_m|} \int_{A_m} \overline{\chi(t)}\, f(t)\, dt\,,$$

is well defined and continuous. In particular, $a_\chi^{\mathcal{A}}(f) = a_\chi^{\mathcal{A}}(g)$ whenever $\|f - g\|_{b,1,\mathcal{A}} = 0$. Moreover,

$$a_\chi^{\mathcal{A}}(f * \varphi) = \widehat{\varphi}(\chi)\, a_\chi^{\mathcal{A}}(f)$$

holds for all $f \in Bap_{\mathcal{A}}$ with $f\theta_G \in \mathcal{M}^\infty(G)$ and all $\varphi \in C_{\mathsf{c}}(G)$.

Proof. By (b) of the previous proposition, χf belongs to $Bap_{\mathcal{A}}$ for any $f \in Bap_{\mathcal{A}}$. Moreover, the map $m_\chi : Bap_{\mathcal{A}} \longrightarrow Bap_{\mathcal{A}}$, $f \mapsto \chi f$, is clearly continuous. By (a) of the previous proposition, $M_{\mathcal{A}}$ is a continuous map on $Bap_{\mathcal{A}}$. Hence, $a_\chi^{\mathcal{A}} = M_{\mathcal{A}} \circ m_\chi$ exists and is continuous on $Bap_{\mathcal{A}}$. Continuity directly gives that $a_\chi^{\mathcal{A}}$ agrees on f and g with $\|f - g\|_{b,p,\mathcal{A}} = 0$.

It remains to prove the last statement. If f is a trigonometric polynomial, the statement follows from a simple direct computation. The general case now follows from the continuity of $a_\chi^{\mathcal{A}}$, the continuity of convolutions (see (b) of Proposition 1.23 with $\mathcal{S}' = SAP(G)$) and the denseness of trigonometric polynomials in $Bap_{\mathcal{A}}$. □

Next, we turn $Bap_{\mathcal{A}}^p(G)$ into a normed space, and show that it is a Banach space. To do this, we need to factor out all the elements of norm 0. We define an equivalence relation $\equiv$ on $Bap_{\mathcal{A}}^p(G)$ via

$$f \equiv g \Leftrightarrow \|f - g\|_{b,p,\mathcal{A}} = 0\,.$$

Moreover, if $h \in L^p_{loc}(G)$ satisfies $\|h\|_{b,p,\mathcal{A}} = 0$, then $h \in Bap_{\mathcal{A}}^p(G)$ and $h \equiv 0$. As usual we denote by $[f]_p$ the equivalence class of f. When there is no possibility of confusion, we will use the shorter notation $[f]$. Then, $\|\cdot\|_{b,p,\mathcal{A}}$ becomes a norm on the space

$$\left(Bap_{\mathcal{A}}^p(G)/\equiv\right) := \{[f] : f \in Bap_{\mathcal{A}}^p(G)\}\,,$$

of equivalence classes, and $(Bap_{\mathcal{A}}^p(G)/\equiv, \|\cdot\|_{b,p,\mathcal{A}})$ is a normed space.

Note that Corollary 3.9 allows us to define the Fourier–Bohr coefficient of $[f] \in Bap_{\mathcal{A}}(G)/\equiv$ in $\chi \in \widehat{G}$ via

$$a_\chi^{\mathcal{A}}([f]) := a_\chi^{\mathcal{A}}(f).$$

We show now that $Bap_{\mathcal{A}}(G)/\equiv$ is a Banach space. The following result gives the crucial completeness of the Besicovitch spaces (as spaces of functions on the group). The result is certainly known, see e.g. [13, Rem. 2].

Theorem 3.10 (Completeness of Besicovitch spaces)**.** *For each $1 \leq p < \infty$ the space $(Bap^p_{\mathcal{A}}(G)/\equiv, \|\cdot\|_{b,p,\mathcal{A}})$ is a Banach space.*

Proof. From Theorem 1.18, we know that $(BL^p_{\mathcal{A}}(G), \|\cdot\|_{b,p,\mathcal{A}})$ is complete. Thus, it suffices to show that $Bap^p_{\mathcal{A}}(G)$ is closed in $BL^p_{\mathcal{A}}(G)$. To do so, consider a sequence (f_n) in $Bap^p_{\mathcal{A}}(G)$ with $f_n \to f$ in $(BL^p_{\mathcal{A}}(A), \|\cdot\|_{b,p,\mathcal{A}})$. Let $\varepsilon > 0$. Pick some n such that $\|f_n - f\|_{b,p,\mathcal{A}} < \frac{\varepsilon}{2}$. Since $f_n \in Bap^p_{\mathcal{A}}(G)$, there exists a trigonometric polynomial P such that $\|f_n - P\|_{b,p,\mathcal{A}} < \frac{\varepsilon}{2}$. Then, $\|f - P\|_{b,p,\mathcal{A}} < \varepsilon$. As $\varepsilon > 0$ was arbitrary, the desired statement follows. $\square$

It is possible to extend the translation action from $SAP(G)$ as well as the convolution on $SAP(G)$ to all of $Bap^p_{\mathcal{A}}(G)/\equiv$. Indeed, with $\mathcal{S}' = SAP(G)$ and $\mathcal{S} = Bap^p_{\mathcal{A}}(G)$ we are exactly in the situation discussed in Section 1. In particular, from Proposition 1.21 we find that for each $t \in G$ there is a unique isometric map $T_t : Bap^p_{\mathcal{A}}(G)/\equiv \longrightarrow Bap^p_{\mathcal{A}}(G)/\equiv$ with $T_t[f] = [\tau_t f]$ for all $f \in SAP(G)$. Moreover, Proposition 1.21 gives the following:

Proposition 3.11 (Translation)**.** (a) *For each $s, t \in G$, we have*

$$T_t \circ T_s = T_{t+s} \qquad \text{and} \qquad T_0 = Id .$$

(b) *For each $[f] \in Bap^p_{\mathcal{A}}(G)/\equiv$, the function $G \longrightarrow Bap^p_{\mathcal{A}}(G)/\equiv$, $t \to T_t[f]$, is continuous with $\|T_t[f]\|_{b,p,\mathcal{A}} = \|[f]\|_{b,p,\mathcal{A}}$ for all $t \in G$.*
(c) *If $f \in (Bap^p_{\mathcal{A}}(G)/\equiv) \cap L^\infty(G)$, then we have $T_t[f] = [\tau_t f]$.*

We finish this section by providing an alternative view on $Bap^p_{\mathcal{A}}(G)/\equiv$ via the Bohr compactification of G. In particular, this will show that $(Bap^p_{\mathcal{A}}(G)/\equiv, \|\cdot\|_{b,p,\mathcal{A}})$ is isometrically isomorphic to $(L^p(G_{\mathsf{b}}), \|\cdot\|_p)$ [44] (compare [12, p.45]). In the case $p = 2$ we get a Hilbert space isometric isomorphism, which will yield some strong consequences.

Recall first that we have a natural embedding $i_{\mathsf{b}} : G \to G_{\mathsf{b}}$ of G into its Bohr compactification G_{b}. Under this embedding, a function $f \in C_{\mathsf{u}}(G)$ on is Bohr almost periodic if and only if there exists $f_{\mathsf{b}} \in C(G_{\mathsf{b}})$ such that $f = f_{\mathsf{b}} \circ i_{\mathsf{b}}$. In this case the function f_{b} is unique, and the mapping $f \to f_{\mathsf{b}}$ is called the **Bohr mapping** (see [43] for the details). Moreover, we have

$$M_{\mathcal{A}}(f) = \int_{G_{\mathsf{b}}} f_{\mathsf{b}}(t)\,\mathrm{d}t\,,$$

where on the right hand side we use the probability Haar measure on G_{b} (see again [43]). This immediately yields the following.

Lemma 3.12. *Fix a van Hove sequence $\mathcal{A}$. Then, for all $f \in SAP(G)$ and all $1 \leq p < \infty$, we have*

$$\|f\|_{b,p,\mathcal{A}} = \left(\int_{G_{\mathsf{b}}} |f_{\mathsf{b}}(t)|^p \, dt \right)^{\frac{1}{p}} .$$

$\square$

Therefore, we obtain the next result (compare [16] for $G = \mathbb{R}^d$).

Theorem 3.13. [44, p.12] *Fix a van Hove sequence $\mathcal{A}$. Then, for each $1 \leq p < \infty$ the Bohr mapping $(\cdot)_{\mathsf{b}} : SAP(G) \to C(G_{\mathsf{b}}) \subseteq L^p(G_{\mathsf{b}})$ extends uniquely to an isometric isomorphism*

$$(\cdot)_{\mathsf{b},p} : (Bap^p_{\mathcal{A}}(G)/\equiv, \| \cdot \|_{b,p,\mathcal{A}}) \to (L^p(G_{\mathsf{b}}), \| \cdot \|_p) \,.$$

Proof. Let us think of $SAP(G)$ as a subspace of $Bap^p_{\mathcal{A}}(G)/\equiv$.

For each $1 \leq p < \infty$, by Lemma 3.12, the Bohr mapping is a norm preserving isometry from $(SAP(G), \| \cdot \|_{b,p,\mathcal{A}})$ into the Banach space $L^p(G_{\mathsf{b}})$. Since $(SAP(G), \| \cdot \|_{b,p,\mathcal{A}})$ is dense in $(Bap^p_{\mathcal{A}}(G)/\equiv, \| \cdot \|_{b,p,\mathcal{A}})$, the Bohr mapping has a unique extension to an isometry $(\cdot)_{\mathsf{b},p}$: $(Bap^p_{\mathcal{A}}(G)/\equiv, \| \cdot \|_{b,p,\mathcal{A}}) \to (L^p(G_{\mathsf{b}}), \| \cdot \|_p)$. Since the range contains $C(G_{\mathsf{b}})$, as the image of $SAP(G)$, the extension $(\cdot)_{\mathsf{b},p}$ has dense range, and hence, as an isometry, is onto. □

Remark 3.14. (a) If $P = \sum_{k=1}^n c_k \, \chi_k$, then $P_{\mathsf{b}} = \sum_{k=1}^n c_k \, (\chi_k)_{\mathsf{b}}$ where χ_{b} denotes the character $\chi \in \widehat{G} = \widehat{G_{\mathsf{b}}}$ viewed as a character on G_{b}.

(b) Let $f \in Bap^p_{\mathcal{A}}(G)$ and let (P_n) be a sequence of trigonometric polynomials such that $\lim_{n\to\infty} \|f - P_n\|_{b,p,\mathcal{A}} = 0$. Then, it follows from Theorem 3.13 that in $(L^p(G_{\mathsf{b}}), \| \cdot \|_p)$ we have

$$[f]_{\mathsf{b},p} = \lim_{n\to\infty} (P_n)_{\mathsf{b}} \,.$$

In particular, for all $1 \leq p < q < \infty$ and all $f \in Bap^q_{\mathcal{A}}(G) \subseteq Bap^p_{\mathcal{A}}(G)/\equiv$, we have $[f]_{\mathsf{b},p} = [f]_{\mathsf{b},q}$ in $L^p(G_{\mathsf{b}})$. (Indeed, if we pick some trigonometric polynomials P_n such that $\|f - P_n\|_{b,q,\mathcal{A}} \to 0$, then $\|f - P_n\|_{b,p,\mathcal{A}} \to 0$. This gives

$$\lim_{n\to\infty} \|[f]_{\mathsf{b},q} - (P_n)_{\mathsf{b}}\|_q = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \|[f]_{\mathsf{b},p} - (P_n)_{\mathsf{b}}\|_p = 0 \,.$$

Since $L^q(G_{\mathsf{b}}) \subseteq L^p(G_{\mathsf{b}})$ and on $L^q(G)$ we have $\| \cdot \|_q \leq \| \cdot \|_p$, the claim follows.)

We now discuss how Theorem 3.13 gives a complementary view on averaging and taking Fourier–Bohr coefficients on $Bap_{\mathcal{A}}(G)$.

Lemma 3.15. *Let $\mathcal{A}$ be a van Hove sequence. Then, for all $f \in Bap_{\mathcal{A}}(G)$ we have*

(a) $\lim_{m\to\infty} \frac{1}{|A_m|} \int_{A_m} f(t) \, dt = \int_{G_{\mathsf{b}}} ([f])_{\mathsf{b},1}(t) \, dt$.

(b) $a^{\mathcal{A}}_{\chi}(f) = \widehat{[f]_{\mathsf{b},1}}(\chi_{\mathsf{b}})$.

Proof. Both sides are continuous functionals on $Bap_{\mathcal{A}}(G)$ which agree on the dense subspace $SAP(G)$. □

If $1 < p, q < \infty$ are conjugates, then $L^p(G_{\mathsf{b}})$ and $L^q(G_{\mathsf{b}})$ are dual spaces, with the duality given by $(f, g) := \int_{G_{\mathsf{b}}} f(t) \, g(t) \, \mathrm{d}t$. This leads to the following observation.

Theorem 3.16. *Let $1 < p < \infty$ and let q be the conjugate of p. Then $Bap^q(G)/\equiv$ is the dual space of $Bap^p_{\mathcal{A}}(G)/\equiv$, with the duality given by*

$$([g], [f]) = M_{\mathcal{A}}(gf)$$

for all $[g] \in Bap^q(G)/\equiv$ and $[f] \in Bap^p_{\mathcal{A}}(G)/\equiv$.

Finally, we note that one can also understand the translation action via the Bohr map. Let $f \in Bap^p_{\mathcal{A}}(G)$ and $t \in G$. Then, $T_t[f]$ is the only class $[g] \in Bap^p_{\mathcal{A}}(G)$ such that

$$([g])_{\mathsf{b},p} = ([f])_{\mathsf{b},p} \left(\cdot - i_{\mathsf{b}}(t) \right) .$$

3.2. **Besicovitch almost periodic functions: Fourier expansion.** In this section, we turn to $Bap^2_{\mathcal{A}}(G)$. This space has a natural Hilbert space structure and we use it to develop a Fourier expansion theory.

Recall from Proposition 3.8 that the mean

$$M_{\mathcal{A}}(f) = \lim_{m\to\infty} \frac{1}{|A_m|} \int_{A_m} f(t)\, \mathrm{d}t$$

exists for all $f \in Bap_{\mathcal{A}}(G)$. Together with the subsequent proposition this will allow us to introduce an inner product on $Bap^2_{\mathcal{A}}(G)$. This inner product structure is the crucial tool in our Fourier analysis.

Proposition 3.17. *Let $\mathcal{A}$ be a van Hove sequence. If $f, g \in Bap^2_{\mathcal{A}}(G)$, then $fg \in Bap_{\mathcal{A}}(G)$.*

Proof. Let $0 < \varepsilon < 1$. Pick trigonometric polynomials P, Q such that $\|f - P\|_{b,2,\mathcal{A}} < \frac{\varepsilon}{2\,\|g\|_{b,2,\mathcal{A}}+1}$ and $\|g - Q\|_{b,2,\mathcal{A}} < \frac{\varepsilon}{2\|f\|_{b,2,\mathcal{A}}+3}$. Note first that

$$\|P\|_{b,2,\mathcal{A}} \leq \|f\|_{b,2,\mathcal{A}} + \|f - P\|_{b,2,\mathcal{A}} < \|f\|_{b,2,\mathcal{A}} + 1\,.$$

By a standard application of Cauchy-Schwarz inequality for $\int_{A_n} \mathrm{d}t$, we have

$$\begin{aligned}
\|fg - PQ\|_{b,1,\mathcal{A}} &= \limsup_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} |f(t)g(t) - P(t)Q(t)|\, \mathrm{d}t \\
&\leq \limsup_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} |(f(t) - P(t))(g(t))|\, \mathrm{d}t \\
&\qquad + \limsup_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} |(g(t) - Q(t))(P(t))|\, \mathrm{d}t \\
&\leq \limsup_{n\to\infty} \frac{1}{|A_n|} \sqrt{\int_{A_n} |f(t) - P(t)|^2\, \mathrm{d}t \int_{A_n} |g(t)|^2\, \mathrm{d}t} \\
&\qquad + \limsup_{n\to\infty} \frac{1}{|A_n|} \sqrt{\int_{A_n} |g(t) - Q(t)|^2\, \mathrm{d}t \int_{A_n} |P(t)|^2\, \mathrm{d}t} \\
&\leq \|f - P\|_{b,2,\mathcal{A}}\, \|g\|_{b,2,\mathcal{A}} + \|g - Q\|_{b,2,\mathcal{A}}\, \|P\|_{b,2,\mathcal{A}} < \varepsilon\,.
\end{aligned}$$

This finishes the proof. $\square$

As a consequence, we obtain that $Bap^2_{\mathcal{A}}(G)/\equiv$ is a Hilbert space.

Remark 3.18. Note that Prop 3.17 is also proven in [36, Thm. 2.7]. Note also that the proof can easily be generalized to give that $fg \in Bap_{\mathcal{A}}(G)$ whenever $f \in Bap^p_{\mathcal{A}}(G)$ and $g \in Bap^q_{\mathcal{A}}(G)$ with $1/p + 1/q = 1$.

Theorem 3.19 ($Bap^2_{\mathcal{A}}(G)/\equiv$ as Hilbert space)**.** *Let $\mathcal{A}$ be a van Hove sequence.*

(a) *The map $\langle\cdot,\cdot\rangle : Bap^2_{\mathcal{A}}(G)/\equiv \times Bap^2_{\mathcal{A}}(G)/\equiv \longrightarrow \mathbb{C}$ defined by*

$$\langle [f], [g]\rangle_{\mathcal{A}} = \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} f(t)\, \overline{g(t)}\, dt$$

is an inner product on $Bap^2_{\mathcal{A}}(G)/\equiv$. The norm defined by this inner product is $\|\cdot\|_{b,2,\mathcal{A}}$.

(b) $(Bap^2_{\mathcal{A}}(G)/\equiv, \langle\cdot,\cdot\rangle_{\mathcal{A}})$ *is a Hilbert space.*
(c) $\widehat{G}$ *is an orthogonal basis in* $Bap^2_{\mathcal{A}}(G)/\equiv$*.*
(d) *For all* $f \in Bap^2_{\mathcal{A}}(G)$ *and* $\chi \in \widehat{G}$*, we have*

$$a_\chi^{\mathcal{A}}([f]) = \langle [f], [\chi] \rangle .$$

(e) *For all* $f \in Bap^2_{\mathcal{A}}(G)$*, one has* $a_\chi^{\mathcal{A}}(f) \neq 0$ *for at most a countable set of characters, and we have the* ***Parseval identity***

$$\|f\|_{b,2,\mathcal{A}}^2 = \sum_{\chi\in\widehat{G}} \left| a_\chi^{\mathcal{A}}(f) \right|^2$$

i.e. $f = \sum_{\chi\in\widehat{G}} a_\chi^{\mathcal{A}}(f)\chi$ *in* $(Bap^2_{\mathcal{A}}(G)/\equiv, \|\cdot\|_{b,2,\mathcal{A}})$*.*

Proof. (a) For $f, g \in Bap^2_{\mathcal{A}}(G)$, Proposition 3.8 gives $f\overline{g} \in Bap_{\mathcal{A}}(G)$ and hence $M_{\mathcal{A}}(f\overline{g})$ exists. It follows immediately from Cauchy-Schwarz' inequality that $\langle [f], [g] \rangle_{\mathcal{A}}$ does not depend on the choice of the representative, and hence is well defined. Clearly, the associated norm is just $\|\cdot\|_{b,2,\mathcal{A}}$.

(b) Follows from Theorem 3.10.

(c) It is well-known that $M_{\mathcal{A}}(\chi\overline{\xi}) = 0$ whenever $\chi, \xi \in \widehat{G}$ do not agree. This shows that the characters form an orthonormal system in $Bap^2_{\mathcal{A}}(G)/\equiv$. Moreover, linear combinations of characters are dense in $Bap^2_{\mathcal{A}}(G)/\equiv$ by the very definition of Besicovitch space and as the norm on $Bap^2_{\mathcal{A}}(G)/\equiv$ agrees with $\|\cdot\|_{b,2,\mathcal{A}}$ due to (a). This gives (c).

(d) This follows directly from the definition of the inner product and the Fourier–Bohr coefficient.

(e) is immediate from (b) and (c). □

Corollary 3.20 (Riesz–Fischer Property)**.** *Let* $a : \widehat{G} \to \mathbb{C}$*, and let* $\mathcal{A}$ *be a van Hove sequence. Then, there is some* $f \in Bap^2_{\mathcal{A}}(G)$ *such that* $a_\chi^{\mathcal{A}}(f) = a(\chi)$ *if and only if* $\sum_{\chi\in\widehat{G}} |a(\chi)|^2 < \infty$*. Moreover, in this case,* $[f]$ *is unique.*

The preceding results allow us to give an intrinsic direct characterization of $Bap^2_{\mathcal{A}}(G)$.

Corollary 3.21 (Characterization $Bap^2_{\mathcal{A}}(G)$)**.** *Let* $f \in L^2_{loc}(G)$ *and* $\mathcal{A}$ *a van Hove sequence. Then* $f \in Bap^2_{\mathcal{A}}(G)$ *if and only if the following three conditions hold:*

(a) *For each* $\chi \in \widehat{G}$ *the Fourier–Bohr coefficient* $a_\chi^{\mathcal{A}}(f)$ *exists.*
(b) $M_{\mathcal{A}}(|f|^2)$ *exists.*
(c) *The Parseval equality*

$$M_{\mathcal{A}}(|f|^2) = \sum_{\chi\in\widehat{G}} \left| a_\chi^{\mathcal{A}}(f) \right|^2$$

holds.

Proof. This follows from Theorem 3.19. Indeed, the 'only if' part is immediate from Theorem 3.19. As for the 'if' statement, let $\varepsilon > 0$ be given. Then, we can find characters

$\chi_1, \ldots, \chi_N \in \widehat{G}$ such that

$$\sum_{k=1}^{N} \left| a_{\chi_k}^{\mathcal{A}}(f) \right|^2 \geq M_{\mathcal{A}}(|f|^2) - \varepsilon \,.$$

Let $P := \sum_{k=1}^{N} a_{\chi_k}^{\mathcal{A}}(f)\chi_k$. Using that $M_{\mathcal{A}}(|f|^2)$ exists and that $a_{\chi}^{\mathcal{A}} = M_{\mathcal{A}}(f\overline{\chi})$ exist for all $\chi \in \widehat{G}$ and that $M_{\mathcal{A}}(\chi) = 1$ for $\chi = 1$ and $M_{\mathcal{A}}(\chi) = 0$ we easily compute

$$\begin{aligned} M_{\mathcal{A}}(|f-P|^2) &= M(|f|^2) - \sum_{k=1}^{N} \left| a_{\chi_k}^{\mathcal{A}}(f) \right|^2 - \sum_{k=1}^{N} \left| a_{\chi_k}^{\mathcal{A}}(f) \right|^2 + \sum_{k=1}^{N} \left| a_{\chi_k}^{\mathcal{A}}(f) \right|^2 \\ &= M_{\mathcal{A}}(|f|^2) - \sum_{k=1}^{N} \left| a_{\chi_k}^{\mathcal{A}}(f) \right|^2 \,. \end{aligned}$$

Putting this together, we see that $\|f-P\|_{b,2,\mathcal{A}}^2 \leq \varepsilon$. As $\varepsilon > 0$ was arbitrary, this finishes the proof. $\square$

Next, as consequence of Proposition 3.11, we obtain that the translation action of G on $Bap_{\mathcal{A}}^2(G)/\equiv$ is a strong unitary representation.

Proposition 3.22 (Translation)**.** *Let $\mathcal{A}$ be a van Hove sequence.*

(a) *For each $t \in G$, the map $T_t : Bap_{\mathcal{A}}^2(G)/\equiv \longrightarrow Bap_{\mathcal{A}}^2(G)/\equiv$ is a unitary map.*
(b) *For each $[g] \in Bap_{\mathcal{A}}^2(G)/\equiv$, the function $t \mapsto T_t[g]$ is continuous and so is then $t \mapsto \langle [g], T_t[g] \rangle$ as well.*
(c) *For each $s, t \in G$, we have $T_t \circ T_s = T_{t+s}$ and $T_0 = Id$.*

Remark 3.23. It is instructive to consider the Bohr completion in this situation as well. Theorem 3.13 gives immediately that the Bohr mapping $(\cdot)_{\mathsf{b}} : SAP(G) \to C(G_{\mathsf{b}}) \subseteq L^2(G_{\mathsf{b}})$ extends uniquely to an inner product preserving isomorphism

$$(\cdot)_{\mathsf{b},2} : (Bap_{\mathcal{A}}^2(G)/\equiv, \langle \cdot, \cdot \rangle_{\mathcal{A}}) \to (L^2(G_{\mathsf{b}}), \langle \cdot, \cdot \rangle) \,.$$

We complete the section by defining an Eberlein convolution for $Bap_{\mathcal{A}}^2(G)/\equiv$ and discussing some of its properties.

Proposition 3.24 (Involution)**.** *There exists a unique isometric involution $\widetilde{\cdot}$ on $Bap_{\mathcal{A}}^2(G)/\equiv$ satisfying*

$$\widetilde{[f]} = [\widetilde{f}] \qquad \text{for all } f \in SAP(G)$$

and

$$\langle [\widetilde{f}], [\widetilde{g}] \rangle_{\mathcal{A}} = \overline{\langle [f], [g] \rangle_{\mathcal{A}}} \qquad \text{for all } f, g \in Bap_{\mathcal{A}}^2(G)/\equiv \,.$$

Proof. The mapping $f \to \widetilde{f}$ is an involution on $SAP(G)$. Moreover, for all $f, g \in SAP(G)$, we have

$$\begin{aligned} \langle [\widetilde{f}], [\widetilde{g}] \rangle_{\mathcal{A}} &= \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} \widetilde{f}(t)\, \overline{\widetilde{g}(t)} \, \mathrm{d}t = \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} \overline{f(-t)\, g(-t)} \, \mathrm{d}t \\ &= \lim_{n\to\infty} \frac{1}{|A_n|} \int_{-A_n} \overline{f(t)\, g(t)} \, \mathrm{d}t = \overline{\langle [f], [g] \rangle_{\mathcal{A}}} \end{aligned}$$

with the last equality following from the fact that for the Bohr almost periodic functions the mean is independent of the Følner sequence [15, 43].

In particular, for all $f \in SAP(G)$ we have $\|[\widetilde{f}]\|_{b,2,\mathcal{A}} = \|[f]\|_{b,2,\mathcal{A}}$. The claim follows immediately from the denseness of $SAP(G)$ in $Bap^2_{\mathcal{A}}(G)/\equiv$. □

Remark 3.25. (a) It is easy to see that $(\widetilde{f})_{\mathsf{b}} = \widetilde{f_{\mathsf{b}}}$ for all $f \in SAP(G)$. It follows immediately that, for all $f \in Bap^2_{\mathcal{A}}(G)/\equiv$, we have

$$\widetilde{[f]_{\mathsf{b},p}} = \left(\widetilde{[f]}\right)_{\mathsf{b},p} .$$

(b) In the same way as in Proposition 3.24, it can be shown that the involution $\widetilde{\cdot}$ on SAP can be uniquely extended to an isometric involution $\widetilde{\cdot}$ on $(Bap^2_{\mathcal{A}}(G)/\equiv, \|\cdot\|_{b,p,\mathcal{A}})$.

Definition 3.26 (Abstract Eberlein convolution)**.** Let $[f],[g] \in Bap^2_{\mathcal{A}}(G)/\equiv$. We define the **abstract Eberlein convolution** $[f] \circledast \widetilde{[g]} : G \to \mathbb{C}$ via

$$([f] \circledast \widetilde{[g]})(t) = \langle [f], T_t[g] \rangle .$$

By the properties of the inner product on $Bap^2_{\mathcal{A}}(G)/\equiv$, we have for $[f],[g] \in Bap^2_{\mathcal{A}}(G)/\equiv$, $t \in G$, and $h \in T_t[g]$,

$$([f] \circledast \widetilde{[g]})(t) = \lim_{m\to\infty} \frac{1}{|A_m|} \int_{A_m} f(s)\, \widetilde{h}(s)\, \mathrm{d}s \,. \tag{4}$$

In particular, whenever f and g are bounded functions with $[f],[g] \in Bap^2_{\mathcal{A}}(G)/\equiv$, we have by (c) of Proposition 3.11,

$$([f] \circledast \widetilde{[g]})(t) = \lim_{m\to\infty} \frac{1}{|A_m|} \int_{A_m} f(s)\, \overline{g}(s-t)\, \mathrm{d}s.$$

So, in this case we just recover the usual definition of the Eberlein convolution between f and $\widetilde{g}$ (see Section 1). This is the reason for our notation. Also note that

$$([f] \circledast \widetilde{[g]})(t) = \langle [f], T_t[g] \rangle = \langle T_{-t}[f], g \rangle = \overline{([g] \circledast \widetilde{[f]})(-t)} \,. \tag{5}$$

We note the following continuity property of the Eberlein convolution.

Proposition 3.27 (Continuity of Eberlein convolution)**.** *Let $\mathcal{A}$ be a van Hove sequence. Let (f_n), (g_n) be sequences in $Bap^2_{\mathcal{A}}(G)$ converging to f and g respectively. Then, $[f_n] \circledast \widetilde{[g_n]} \to [f] \circledast \widetilde{[g]}$ with respect to $\|\cdot\|_\infty$.*

Proof. This follows by a straightforward computation: For each $t \in G$, we find

$$\begin{aligned}
|([f_n] \circledast \widetilde{[g_n]})(t) - ([f] \circledast \widetilde{[g]})(t)| &= |\langle [f_n], T_t[g_n] \rangle - \langle [f], T_t[g] \rangle| \\
&\leq \|f_n\|\, \|T_t([g_n]-[g])\| + \|[f]-[f_n]\|\, \|T_t[g]\| \\
&= \|f_n\|\, \|[g_n]-[g]\| + \|[f]-[f_n]\|\, \|[g]\| \to 0 \,.
\end{aligned}$$

Here, we used that T_t is an isometry. As the convergence to zero in the last line is clearly independent of $t \in G$ the proof is finished. □

Now, we can list the properties of the Eberlein convolution.

Theorem 3.28 (Properties abstract Eberlein convolution)**.** *Let $f, g \in Bap^2_{\mathcal{A}}(G)$. Then,*

(a) $[f] \circledast_{\mathcal{A}} \widetilde{[g]} \in SAP(G)$.

(b) *For all $\chi \in \widehat{G}$, we have*

$$a_\chi([f] \circledast_{\mathcal{A}} \widetilde{[g]}) = a^{\mathcal{A}}_\chi(f)\, \overline{a^{\mathcal{A}}_\chi(g)}\,.$$

(c) *If $t \in G$ is such that $T_t[g] = [\tau_t g]$, then*

$$([f] \circledast \widetilde{[g]})(t) = \lim_{m\to\infty} \frac{1}{|A_m|} \int_{A_m} f(s)\, \overline{g(s-t)}\, ds\,.$$

(d) *If $g \in Bap^2_{\mathcal{A}}(G) \cap L^\infty(G)$, then $f \circledast_{\mathcal{A}} g$ exists and*

$$[f] \circledast \widetilde{[g]} = f \circledast \widetilde{g}\,.$$

Remark 3.29. We note that the Eberlein convolution of $[f]$ and $\widetilde{[g]}$ from $Bap^2_{\mathcal{A}}(G)/\equiv$ can also be understood as the usual convolution of the functions $([f])_b$ and $(\widetilde{[g]})_b$ on the Bohr compactification. Indeed, this is clear if f and g are trigonometric polynomials. It then follows for general $f, g \in Bap^2_{\mathcal{A}}(G)$ by continuity of Eberlein convolution (Proposition 3.27).

Proof. (a) This is easy to see when f and g are trigonometric polynomials. The general case follows from the denseness of trigonometric polynomials in $Bap^2_{\mathcal{A}}(G)$ and the continuity of the Eberlein convolution given in Proposition 3.27.

(b) As in (a) this is easy to see when f and g are trigonometric polynomials. The general case follows from the denseness of trigonometric polynomials in $Bap^2_{\mathcal{A}}(G)$, the continuity of the Eberlein convolution given in Proposition 3.27 and the continuity of the Fourier–Bohr coefficients given in Corollary 3.9.

(c) follows immediately from Eqn. (4) by setting $h = \tau_t g \in T_t[g]$.

(d) By Proposition 3.11, we have $T_t[g] = [\tau_t g]$ and hence the claim follows from (c). ☐

3.3. **Besicovitch almost periodic measures.** Having studied Besicovitch almost periodic functions in the last section we now turn to Besicovitch almost periodic measures.

Definition 3.30 (Besicovitch almost periodic measures)**.** Let a van Hove sequence $\mathcal{A}$ on G and $1 \leq p < \infty$ be given. A measure μ on G is called **Besicovitch p-almost periodic** (with respect to $\mathcal{A}$) if the function $\varphi * \mu$ is Besicovitch p-almost periodic for all $\varphi \in C_{\mathsf{c}}(G)$. The space of Besicovitch p-almost periodic measures is denoted by $\mathcal{B}\mathsf{ap}^p_{\mathcal{A}}(G)$. In the case $p = 1$ we drop the superscript 1.

Remark 3.31 (Independence of p for translation bounded measures)**.** It follows from Proposition 3.7 that a measure in $\mathcal{M}^\infty(G)$ is Besicovitch p-almost periodic if and only if it is Besicovitch 1-almost periodic.

Remark 3.32 (Inclusion of spaces)**.** From the definition and Proposition 3.3, we immediately obtain the following:

(a) For each $1 \leq p < \infty$, we have

$$\mathcal{B}\mathsf{ap}^p_{\mathcal{A}}(G) \subsetneq \mathcal{M}\mathsf{ap}^p_{\mathcal{A}}(G)\,.$$

(b) For each $1 \leq p \leq q < \infty$, we have

$$\mathcal{B}\mathsf{ap}^q_{\mathcal{A}}(G) \subseteq \mathcal{B}\mathsf{ap}^p_{\mathcal{A}}(G) \subseteq \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G)\,.$$

As in the case of mean almost periodic measures we can use (c) of Proposition 1.23 (with $\mathcal{S}' = SAP(G)$ instead of $\mathcal{S}' = MAP_{\mathcal{A}}(G)$) to show that for a function $f \in C_{\mathsf{u}}(G)$ almost periodicity as function and as a measure coincide:

Proposition 3.33. *Let $\mathcal{A}$ be a van Hove sequence on G and $1 \leq p < \infty$ be given. Let $f \in L^1_{loc}(G)$ be given such that $f\theta_G$ is a translation bounded measure, and assume that there exists a sequence (f_n) in $C_{\mathsf{u}}(G)$ with $f_n \to f$ with respect to $\| \cdot \|_{b,p,\mathcal{A}}$. Then, $f\theta_G$ belongs to $\mathcal{B}\mathsf{ap}^p_{\mathcal{A}}(G)$ if and only if f belongs to $Bap^p_{\mathcal{A}}(G)$. In particular, $f \in C_{\mathsf{u}}(G)$ belongs to $Bap^p_{\mathcal{A}}(G)$ if and only if $f\theta_G \in \mathcal{B}\mathsf{ap}^p_{\mathcal{A}}(G)$.*

3.4. **Pure point diffraction with Fourier–Bohr coefficients.** In this section, we characterize when an $\mathcal{A}$-representation has pure point diffraction and at the same time possesses Fourier coefficients. As a consequence, we obtain a solution to the phase problem.

For $\chi \in \widehat{G}$ we denote throughout the section the characteristic function of $\{\chi\}$ by 1_χ.

Theorem 3.34 (Characterization of $\mathcal{A}$-representation with Fourier coefficients)**.** *Let N be an $\mathcal{A}$-representation which possesses a semi-autocorrelation. Let $\mathcal{H}$ be the associated Hilbert space. Then, the following assertions are equivalent:*

(i) *The Fourier transform of the semi-autocorrelation of N is a pure point measure σ and there exist (necessarily unique) complex numbers A_χ for $\chi \in \widehat{G}$ satisfying $M_{\mathcal{A}}(N(\varphi)\overline{\chi}) = A_\chi \widehat{\varphi}(\chi)$ for all $\varphi \in C_{\mathsf{c}}(G)$ as well as $|A_\chi|^2 = \sigma(\{\chi\})$.*
(ii) *$N(C_{\mathsf{c}}(G)) \subseteq Bap^2_{\mathcal{A}}(G)$.*
(iii) *The space $\mathcal{H}$ has a dense subspace consisting of trigonometric polynomials.*

If these equivalent conditions hold, then

$$[N(\varphi)] = \sum_{\chi \in \widehat{G}} A_\chi \widehat{\varphi}(\chi)[\chi]$$

in $Bap^2_{\mathcal{A}}(G)/\equiv$ holds for all $\varphi \in C_{\mathsf{c}}(G)$ and the (unique) unitary map

$$U : L^2(\widehat{G}, \sigma) \longrightarrow \mathcal{H}$$

with $\widehat{\varphi} \mapsto N(\varphi)$ for all $\varphi \in C_{\mathsf{c}}(G)$ satisfies $U(1_\chi) = A_\chi \chi$. Moreover, in this case the trigonometric polynomials in (iii) are exactly the linear span of the $\chi \in \widehat{G}$ with $\sigma(\{\chi\}) > 0$.

Remark 3.35 (Fourier–Bohr coefficients A_χ)**.** (a) The theorem gives that the A_χ, $\chi \in \widehat{G}$, are exactly what we called the Fourier–Bohr coefficients of the $\mathcal{N}$-representation associated to N with respect to the orthonormal basis given by the characters.

(b) From (i) of the theorem we see that the A_χ, $\chi \in \widehat{G}$, have the property that

$$\sum_{\chi \in \widehat{G}} |\widehat{\varphi}(\chi)|^2\, |A_\chi|^2 = \sum_{\chi \in \widehat{G}} |\widehat{\varphi}(\chi)|^2\, \sigma(\{\chi\}) = \|\widehat{\varphi}\|_{L^2(\sigma)} < \infty$$

for all $\varphi \in C_{\mathsf{c}}(G)$. Conversely, when $A_\chi \in \mathbb{C}$, $\chi \in \widehat{G}$, are given with the summability property $\sum_{\chi\in\widehat{G}} |\widehat{\varphi}(\chi)|^2 |A_\chi|^2 < \infty$ for all $\varphi \in C_{\mathsf{c}}(G)$, we can define an $\mathcal{N}$-representation

$$N : C_{\mathsf{c}}(G) \longrightarrow Bap^2_{\mathcal{A}}(G)/\equiv\,, \qquad N(\varphi) := \sum_{\chi\in\widehat{G}} \widehat{\varphi}(\chi)\, A_\chi[\chi]\,.$$

This $\mathcal{N}$-representation has a semi-autocorrelation, whose Fourier transform is a pure point measure σ with $\sigma(\{\chi\}) = |A_\chi|^2$ for all $\chi \in \widehat{G}$. In this sense, there is a one-to-one correspondence between A_χ, $\chi \in \widehat{G}$ satisfying this summability property and $\mathcal{N}$-representations with Fourier coefficients.

Proof. (iii)$\Longrightarrow$(ii): Recall that the norm on $\mathcal{H}$ agrees with $\|\cdot\|_{b,2,\mathcal{A}}$. Given this, (iii) and the definition of Besicovitch almost periodicity clearly imply (ii).

(ii)$\Longrightarrow$(i): As $Bap^2_{\mathcal{A}}(G) \subset Map^2_{\mathcal{A}}(G)$ and N possesses a semi-autocorrelation, we infer from (ii) and Theorem 2.12 that the Fourier transform of the semi-autocorrelation is a pure point measure σ. Theorem 1.29 then gives that there exists a (unique) unitary G-map

$$U : L^2(\widehat{G}, \sigma) \longrightarrow \mathcal{H}$$

with $\widehat{\varphi} \mapsto N(\varphi)$ for all $\varphi \in C_{\mathsf{c}}(G)$. As $1_\chi \in L^2(\widehat{G}, \sigma)$ is an eigenfunction for each $\chi \in \widehat{G}$ with $\sigma(\{\chi\}) > 0$, we obtain for each such χ an eigenfunction $U(1_\chi)$ in $\mathcal{H}$. As the Besicovitch space has an orthonormal basis consisting of characters and these characters are eigenfunctions to different eigenvalues, we obtain that an eigenfunction in the Besicovitch space must be a multiple of the character. Hence, each $U(1_\chi)$ must be a (multiple of a) character, i.e. there exist A_χ, $\chi \in \widehat{G}$ with $U(1_\chi) = A_\chi[\chi]$. Moreover, we then have

$$[N(\varphi)] = U(\widehat{\varphi})\, U\Big(\sum_{\chi\in\widehat{G}} \widehat{\varphi}(\chi) 1_\chi\Big) = \sum_{\chi\in\widehat{G}} \widehat{\varphi}(\chi)\, A_\chi\, [\chi]$$

for all $\varphi \in C_{\mathsf{c}}(G)$. This gives

$$\widehat{\varphi}(\chi)\, A_\chi = \langle [N(\varphi), [\chi] \rangle = M_{\mathcal{A}}(N(\varphi)\overline{\chi})$$

for all $\chi \in \widehat{G}$ and $\varphi \in C_{\mathsf{c}}(G)$.

Finally, since $|\widehat{\varphi}|^2 \sigma = \sigma_{N(\varphi)}$ we get for all $\chi \in \widehat{G}$ and $\varphi \in C_{\mathsf{c}}(G)$.

$$|\widehat{\varphi}(\chi)|^2\, \sigma(\{\chi\}) = \sigma_{N(\varphi)}(\{\chi\}) = |\widehat{\varphi}(\chi) A_\chi|^2$$

This finishes the proof of (i).

(i)$\Longrightarrow$(iii): By (i), N possesses a semi-autocorrelation whose Fourier transform σ is a pure point measure. Hence, we infer from Theorem 1.29 that there exists a (unique) unitary G-map

$$U : L^2(\widehat{G}, \sigma) \longrightarrow \mathcal{H}$$

with $U(\widehat{\varphi}) = N(\varphi)$ for all $\varphi \in C_{\mathsf{c}}(G)$. As U is unitary, we infer from (i)

$$\|N(\varphi)\|^2 = \|\widehat{\varphi}\|^2 = \sum_{\chi\in\widehat{G}} |\widehat{\varphi}(\chi)|^2\, \sigma(\{\chi\}) = \sum_{\chi\in\widehat{G}} |\widehat{\varphi}(\chi)|^2\, |A_\chi|^2 = \sum_{\chi\in\widehat{G}} |M_{\mathcal{A}}(N(\varphi)\overline{\chi})|^2\,.$$

Moreover, a short direct computation invoking $M_{\mathcal{A}}(\chi\overline{\varrho}) = 0$ for $\chi, \varrho \in \widehat{G}$ with $\chi \neq \varrho$ gives

$$M_{\mathcal{A}}(|N(\varphi) - \sum_{\chi \in F} M_{\mathcal{A}}(N(\varphi)\chi)|^2) = \|N(\varphi)\|^2 - \sum_{\chi \in F} |M_{\mathcal{A}}(N(\varphi)\overline{\chi})|^2$$

for all finite sets $F \subset \widehat{G}$. Putting these together, we arrive at (iii).

The last statements of the theorem have been shown along the proof of the equivalence. □

From the previous theorem, we obtain easily the solution to the phase problem.

Theorem 3.36 (Solution to the phase problem). *Let μ be a translation bounded measure. Then, $\mu \in \mathcal{B}\mathsf{ap}^2_{\mathcal{A}}(G)$ if and only if the following three conditions hold true:*

(a) *The autocorrelation γ of μ exists with respect to $\mathcal{A}$ and $\widehat{\gamma}$ is a pure point measure.*
(b) *The Fourier–Bohr coefficients $a^{\mathcal{A}}_{\chi}(\mu)$ exist for all $\chi \in \widehat{G}$.*
(c) *The consistent phase property*

$$\widehat{\gamma}(\{\chi\}) = \left|a^{\mathcal{A}}_{\chi}(\mu)\right|^2$$

holds for all $\chi \in \widehat{G}$.

Proof. We first note that for $\mu \in \mathcal{B}\mathsf{ap}^2_{\mathcal{A}}(G)$ the limit $M_{\mathcal{A}}(\mu*\varphi\cdot\overline{\mu*\psi})$ exists for all $\varphi, \psi \in C_{\mathsf{c}}(G)$ due to the existence of means for products of functions from $Bap^2_{\mathcal{A}}(G)$. Hence, for such μ the autocorrelation exists by Proposition 1.3. Given this, we infer from Proposition 1.31 that the map

$$N_{\mu} : C_{\mathsf{c}}(G) \longrightarrow L^1_{loc}(G)\,, \qquad N_{\mu}(\varphi) = \mu * \varphi$$

is an intertwining $\mathcal{A}$-representation.

We also note that, for a translation bounded measure μ, the existence of the Fourier–Bohr coefficients $a^{\mathcal{A}}_{\chi}(\mu)$ is equivalent to the existence of the Fourier–Bohr coefficients $a^{\mathcal{A}}_{\chi}(\mu * \varphi) = M_{\mathcal{A}}((\mu * \varphi)\cdot\overline{\chi})$ for all $\varphi \in C_{\mathsf{c}}(G)$ due to Corollary 1.12.

Now, we can easily infer the statement of the theorem by an application of Theorem 3.34 to the $\mathcal{A}$-representation N_{μ} with the Fourier–Bohr coefficients A_{χ} appearing in Theorem 3.34 given by the Fourier–Bohr coefficients $a^{\mathcal{A}}_{\chi}(\mu)$ of the measure μ. □

Remark 3.37. (a) We could replace the assumption that μ is translation bounded by the assumption that μ is positive. The proof would proceed along the very same lines with Proposition 1.3 and Proposition 1.31 replaced by Proposition 1.34.

(b) In the situation of the theorem the Fourier–Bohr coefficients $a^{\mathcal{A}}_{\chi}(\mu)$ are exactly the abstract Fourier–Bohr coefficients appearing in Theorem 3.34. Hence, they satisfy the following square summability type condition: For all $\varphi \in C_{\mathsf{c}}(G)$, we have

$$\sum_{\chi \in \widehat{G}} \left|\widehat{\varphi}(\chi)a^{\mathcal{A}}_{\chi}(\mu)\right|^2 = \|\mu * \varphi\|^2_{b,2,\mathcal{A}}\,.$$

In the particular case $\mu \in \mathcal{SAP}(G)$, this was proven in [17, Prop. 8.3].

Remark 3.38 (Example). As it is instructive, we discuss here an example to see the difference between mean almost periodic measures and Besicovitch almost periodic measures. Consider in $G = \mathbb{R}$ and with $A_n = [-n, n]$ the function $f : \mathbb{R} \longrightarrow [0, 1]$ with $f(x) = 0$ for $x \leq 0$ and

$f(x) = 1$ for $x \geq 1$ and $f(x) = x$ for $0 \leq x \leq 1$. Then, f belongs to $C_u(\mathbb{R})$. It is clearly mean almost periodic (as $\|f - \tau_t f\|_{b,1,\mathcal{A}} = 0$ for all $t \in \mathbb{R}$). Let $\mu = f\boldsymbol{\lambda}$ (with the Lebesgue measure $\boldsymbol{\lambda}$). In this case, everything can be computed explicitly:

$$\gamma = \frac{1}{2}\,\boldsymbol{\lambda}\,,$$

i.e. γ has a density function $h = \frac{1}{2}$. So,

$$\widehat{\gamma} = \frac{1}{2}\delta_{\underline{1}},$$

where we write $\underline{1}$ for the character which maps everything to 1. (This character could also be denoted as 0 if we identify $\widehat{\mathbb{R}}$ with $\mathbb{R}$.)

Now, let us consider $\mathcal{H}_\mu$: Clearly, the $\mathcal{H}_\mu$ norm is just the Besicovitch 2-norm. (This is always true in the mean almost periodic case.) Moreover, we note that

$$\mu * \varphi = f * \varphi = \left(\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x \right) \cdot f \in \mathcal{H}_\mu$$

for all $\varphi \in C_c(\mathbb{R})$. (Here the first equality holds in the sense of honest functions and the last equality holds in $BL^1_{\mathcal{A}}$ i.e. after factoring out things which vanish in Besicovitch norm). Therefore, $\mathcal{H}_\mu$ is one dimensional

$$\mathcal{H}_\mu = \{cf : c \in \mathbb{C}\}.$$

In particular, f is an eigenfunction to the eigenvalue 1. (This can also directly be seen as clearly $T_t f = f$ in the sense of Besicovitch space). Now, the character $\underline{1}$ does *not* belong to $\mathcal{H}_\mu$. However,

$$M_{\mathcal{A}}(f \cdot \overline{\chi}) = \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} f(s)\,\overline{\chi(s)}\,\mathrm{d}s$$

exists for all χ in the dual group of $\mathbb{R}$ and it is zero for $\chi \neq \underline{1}$. For $\chi = \underline{1}$, we find

$$a_{\underline{1}}(f) = \frac{1}{2}\,.$$

So in this example all Fourier coefficients exist but the characters are not in the Hilbert space $\mathcal{H}_\mu$ and, moreover,

$$|a_{\underline{1}}|^2 = \frac{1}{4} \neq \frac{1}{2} = \widehat{\gamma}(\{0\})\,.$$

3.5. **Weak model sets of maximal density.** In this section, we apply the preceding considerations to the study of weak model sets of maximal density. This will allow us to recover various recent results. For generalities on cut and project schemes we refer to Appendix B.

Whenever a CPS $(G, H, \mathcal{L})$ and a compact set $W \subset H$ is given, we say that $\curlywedge(W)$ is a **weak model set of maximal density** with respect to $\mathcal{A}$ (see Definition B.3) if

$$\mathrm{dens}(\curlywedge(W)) = \mathrm{dens}(\mathcal{L})\,\theta_H(W)\,.$$

Proposition 3.39. *Let Λ be a weak model set of maximal density with respect to $\mathcal{A}$. Then, $\delta_\Lambda \in \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G) \cap \mathcal{M}^\infty(G)$.*

Proof. First, it is well known that $\delta_\Lambda \in \mathcal{M}^\infty(G)$.

Let $\varphi \in C_{\mathsf{c}}(G)$ be non-negative. Let $(G, H, \mathcal{L})$ be the CPS and W the window which gives $\Lambda = \curlywedge(W)$ as a maximal density model set.

Pick $g \in C_{\mathsf{c}}(H)$ such that $1_{W'} \geq g$ and

$$\int_H (g(t) - 1_W(t))\, \mathrm{d}t < \frac{\varepsilon}{2\left(\int_G \varphi(t)\, \mathrm{d}t\right) \mathrm{dens}(\mathcal{L}) + 1}\,.$$

For simplicity, we will set $C_1 := 2\left(\int_G \varphi(t)\, \mathrm{d}t\right) \mathrm{dens}(\mathcal{L}) + 1$.

Then, since $\varphi \geq 0$, we have $\delta_\Lambda * \varphi \leq \omega_g * \varphi$ and, by [47],

$$\lim_{n\to\infty} \frac{1}{|A_n|}\, \omega_g(A_n) = \mathrm{dens}(\mathcal{L}) \int_H g(t)\, \mathrm{d}t\,.$$

Also, by the maximal density condition, we have

$$\lim_{n\to\infty} \frac{1}{|A_n|}\, \delta_\Lambda(A_n) = \mathrm{dens}(\mathcal{L}) \int_H 1_W(t)\, \mathrm{d}t\,.$$

Therefore,

$$\lim_{n\to\infty} \frac{1}{|A_n|} (\omega_g - \delta_\Lambda)(A_n) = \mathrm{dens}(\mathcal{L}) \int_H (g(t) - 1_W(t))\, \mathrm{d}t < \mathrm{dens}(\mathcal{L})\, \frac{\varepsilon}{C_1}\,.$$

By the van Hove condition and positivity, we get

$$\|(\omega_g - \delta_\Lambda) * \varphi\|_{b,1,\mathcal{A}} = \int_G \varphi(t)\, \mathrm{d}t\, \lim_{n\to\infty} \frac{1}{|A_n|} (\omega_g - \delta_\Lambda)(A_n) < \left(\int_G \varphi(t)\, \mathrm{d}t\right) \mathrm{dens}(\mathcal{L}) \frac{\varepsilon}{C_1} < \frac{\varepsilon}{2}\,.$$

Finally, as $\omega_g \in \mathcal{SAP}(G)$ [32, Thm. 3.1], we can find a trigonometric polynomial P such that

$$\|\omega_g * \varphi - P\|_\infty < \frac{\varepsilon}{2}\,.$$

Therefore,

$$\begin{aligned}\|\delta_\Lambda * \varphi - P\|_{b,1,\mathcal{A}} &\leq \|(\omega_g - \delta_\Lambda) * \varphi\|_{b,1,\mathcal{A}} + \|\omega_g * \varphi - P\|_{b,1,\mathcal{A}} \\ &< \frac{\varepsilon}{2} + \|\omega_g * \varphi - P\|_\infty < \varepsilon\,.\end{aligned}$$

This shows that $\delta_\Lambda * \varphi \in \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G)$ for all non-negative $\varphi \in C_{\mathsf{c}}(G)$. The claim now follows via linearity. $\square$

Now, we can give an alternative proof to the following result first shown in [5].

Corollary 3.40. *Let $(G, H, \mathcal{L})$ be a CPS, $W \subset H$ a compact set such that $\Lambda = \curlywedge(W)$ is a weak model set of maximal density with respect to $\mathcal{A}$. Then,*

(a) *The autocorrelation γ of Λ exists with respect to $\mathcal{A}$.*

(b) *$\widehat{\gamma}$ is pure point.*

(c) *For each $\chi \in \widehat{G}$, the Fourier–Bohr coefficient exists with respect to $\mathcal{A}$ and satisfies*

$$a_\chi^{\mathcal{A}}(\delta_\Lambda) = \begin{cases} \mathrm{dens}(\mathcal{L})\, \widecheck{1_W}(\chi^\star), & \text{if } \chi \in \pi_{\widehat{G}}(\mathcal{L}^0)\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

(d) *For all $\chi \in \widehat{G}$, we have $\widehat{\gamma}(\{\chi\}) = \left|a_\chi^{\mathcal{A}}(\delta_\Lambda)\right|^2$.*

(e) *We have*

$$\gamma = \operatorname{dens}(\mathcal{L})\omega_{1_W * \widetilde{1_W}} \qquad \textit{and} \qquad \widehat{\gamma} = (\operatorname{dens}(\mathcal{L}))^2 \, \omega_{|\check{1_W}|^2} \, .$$

Proof. (a) and (b) are obvious.

(c) The Fourier–Bohr coefficients exist by Besicovitch almost periodicity.

Now, fix some $\chi \in \widehat{G}$ and $\varphi \in C_{\mathsf{c}}(G)$ so that $\widehat{\varphi}(\chi) = 1$. With the notations of Proposition 3.39, pick some $1_{W'} \geq g_n \geq 1_W$ and

$$\int_H (g_n(t) - 1_W(t)) \ \mathrm{d}t < \frac{\varepsilon}{2\left(\int_G \varphi(t) \ \mathrm{d}t\right) \operatorname{dens}(\mathcal{L}) + 1} \, .$$

Then, exactly as in the proof of Proposition 3.39, we have

$$\overline{M}_{\mathcal{A}}(\omega_{g_n} * \varphi - \delta_\Lambda * \varphi) < \frac{1}{n}$$

and hence, by Corollary 3.9, we have

$$a_\chi^{\mathcal{A}}(\delta_\Lambda) = \lim_{n\to\infty} a_\chi(\omega_{g_n}) \, .$$

Now, since (by [54, 3, 55])

$$a_\chi(\omega_{g_n}) = \begin{cases} \operatorname{dens}(\mathcal{L}) \, \widetilde{g_n}(\chi^\star), & \text{if } \chi \in \pi_{\widehat{G}}(\mathcal{L}^0) \, , \\ 0, & \text{otherwise } \ , \end{cases}$$

the claim follows.

(d) follows from Theorem 3.34.

(e) is now immediate. Indeed, by the above, we have

$$\widehat{\gamma} = (\operatorname{dens}(\mathcal{L}))^2 \sum_{\chi \in \pi_{\widehat{G}}(\mathcal{L}^0)} \left|\widetilde{1_W}(\chi^\star)\right|^2 \delta_\chi = (\operatorname{dens}(\mathcal{L}))^2 \, \omega_{\left|\widetilde{1_W}\right|^2} \, .$$

Moreover, $\operatorname{dens}(\mathcal{L}) \, \omega_{1_W * \widetilde{1_W}}$ is positive definite, thus Fourier transformable and, by [47],

$$\begin{aligned} \widehat{\operatorname{dens}(\mathcal{L}) \, \omega_{1_W * \widetilde{1_W}}} &= (\operatorname{dens}(\mathcal{L}))^2 \sum_{\chi \in \pi_{\widehat{G}}(\mathcal{L}^0)} \widetilde{1_W * \widetilde{1_W}}(\chi^\star) \, \delta_\chi \\ &= (\operatorname{dens}(\mathcal{L}))^2 \sum_{\chi \in \pi_{\widehat{G}}(\mathcal{L}^0)} \left|\widetilde{1_W}(\chi^\star)\right|^2 \delta_\chi = \widehat{\gamma} \, , \end{aligned}$$

which yields

$$\operatorname{dens}(\mathcal{L}) \, \omega_{1_W * \widetilde{1_W}} = \gamma \, .$$

□

Remark 3.41. (a) The corollary contains the main results of [5]. The only result from [5] which is missing above, namely that every maximal density weak model set is generic for an ergodic measure, follows from Theorem 6.13, which we prove in Section 6.

(b) Recall that given a CPS $(G, H, \mathcal{L})$, a compact set $W \subset H$ and a tempered van Hove sequence $\mathcal{A}$, for almost all $(s,t) + \mathcal{L} \in \mathbb{T} = (G \times H)/\mathcal{L}$, the set $-s + \curlywedge(t + W)$ has maximal density with respect to $\mathcal{A}$, see [41], and is therefore Besicovitch almost periodic. This explains the pure point spectrum of the extended hull of weak model sets [26].

4. Weyl almost periodicity and the uniform phase problem

In this section, we study Weyl almost periodicity. By its very definition this is a very uniform form of almost periodicity. This uniformity is first formulated by allowing arbitrary translations of a fixed van Hove sequence. It turns out, however, that this amounts to allowing arbitrary van Hove sequences (under suitable conditions). So, we will see that (under suitable boundedness assumptions) Weyl almost periodicity is the same as Besicovitch almost periodicity with respect to any van Hove sequence. This will allow us to use Weyl almost periodicity to solve the uniform phase problem.

4.1. **Weyl almost periodic functions and measures.** In this section, we discuss Weyl almost periodic functions and measures.

Definition 4.1 (Weyl almost periodic functions and measures)**.** Let $\mathcal{A} = (A_n)$ be a van Hove sequence, and let $1 \leq p < \infty$. A function $f \in L^p_{loc}(G)$ is called **Weyl p-almost periodic** with respect to $\mathcal{A}$ if, for each $\varepsilon > 0$, there exists a trigonometric polynomial $P = \sum_{k=1}^{n} c_k \chi_k$ with $c_k \in \mathbb{C}$ and $\chi_k \in \widehat{G}$ such that

$$\|f - P\|_{w,p,\mathcal{A}} < \varepsilon \,.$$

We denote the space of Weyl p-almost periodic functions by $Wap^p_{\mathcal{A}}(G)$. A measure μ on G is called **Weyl p-almost periodic** if the function $\varphi * \mu$ is Weyl p-almost periodic for all $\varphi \in C_{\mathsf{c}}(G)$. The space Weyl p-almost periodic measures is denoted by $\mathcal{W}\mathsf{ap}^p_{\mathcal{A}}(G)$. When $p = 1$ we will simply denote this spaces by $Wap_{\mathcal{A}}(G) := Wap^1_{\mathcal{A}}(G)$ and $\mathcal{W}\mathsf{ap}_{\mathcal{A}}(G) := \mathcal{W}\mathsf{ap}^1_{\mathcal{A}}(G)$

Remark 4.2.
- (a) A function is Weyl p-almost periodic if and only if, for each $\varepsilon > 0$, there exists a Bohr almost periodic function g such that $\|f - g\|_{w,p,\mathcal{A}} < \varepsilon$.
- (b) As is clear from (a), all Bohr almost periodic functions are Weyl almost periodic. In fact, it is not hard to see that every weakly almost periodic function is Weyl almost periodic. Indeed, any such f can be decomposed as $f = g + h$ with g Bohr almost periodic and h with uniform vanishing mean (see above) and the statement follows easily.
- (c) Whenever $f \in Wap_{\mathcal{A}}(G)$ and $h : G \longrightarrow \mathbb{C}$ is measurable with $\overline{uM}_{\mathcal{A}}(|h|) = 0$ then $f + h \in Wap_{\mathcal{A}}(G)$ (with the same seminorm).
- (d) $(Wap^p_{\mathcal{A}}(G), \|\cdot\|_{w,p,\mathcal{A}})$ is not complete [12].

Proposition 4.3 (Inclusions of spaces)**.**
- (a) *For each $1 \leq p < \infty$, we have*
$$Wap^p_{\mathcal{A}}(G) \subset Bap^p_{\mathcal{A}}(G)$$
with continuous inclusion map.
- (b) *For each $1 \leq p \leq q < \infty$, we have*
$$Wap^q_{\mathcal{A}}(G) \subseteq Wap^p_{\mathcal{A}}(G) \subseteq Wap_{\mathcal{A}}(G)$$
with continuous inclusion map.

Proof. These statements follow from Lemma 1.13, Lemma 1.16 and Lemma 1.17. □

Remark 4.4. Example A.5 shows $Wap^p_{\mathcal{A}}(G) \neq Bap^p_{\mathcal{A}}(G)$.

Remark 4.5. From the definition and Proposition 4.3 we immediately obtain the following:

(a) For each $1 \le p < \infty$, we have $\mathcal{W}\mathsf{ap}_{\mathcal{A}}^p(G) \subsetneq \mathcal{B}\mathsf{ap}_{\mathcal{A}}^p(G)$.
(b) For each $1 \le p \le q < \infty$, we have $\mathcal{W}\mathsf{ap}_{\mathcal{A}}^q(G) \subseteq \mathcal{W}\mathsf{ap}_{\mathcal{A}}^p(G) \subseteq \mathcal{W}\mathsf{ap}_{\mathcal{A}}(G)$.

We now recollect a few results for Weyl almost periodic functions that follow easily when one replaces $\|\cdot\|_{b,p,\mathcal{A}}$ by $\|\cdot\|_{w,p,\mathcal{A}}$ in the corresponding statements and proofs for Besicovitch almost periodic functions. We begin with an analogue of Proposition 3.33.

Proposition 4.6. *Let $\mathcal{A}$ be a van Hove sequence on G and $1 \le p < \infty$ be given. Let $f \in C_{\mathsf{u}}(G)$ be arbitrary. Then $f \in Wap_{\mathcal{A}}^p(G)$ if and only if $f\theta_G \in \mathcal{W}\mathsf{ap}_{\mathcal{A}}^p(G)$.*

The next result is the analogue of Proposition 3.8 (compare [52] as well).

Proposition 4.7 (Basic properties Weyl almost periodic functions)**.** *Let $\mathcal{A}$ be a van Hove sequence.*

(a) *Any $f \in Wap_{\mathcal{A}}(G)$ is amenable, i.e.*
$$\lim_{m\to\infty} \frac{1}{|A_m|} \int_{s+A_m} f(t)\, dt$$
exists uniformly in $s \in G$.
(b) *Whenever f, g belong to $Wap_{\mathcal{A}}^p(G)$ for some $p \ge 1$, so do $f \pm g$ and cf and χf for all $c \in \mathbb{C}$ and $\chi \in \widehat{G}$.*
(c) *If $f, g \in Wap_{\mathcal{A}}(G)$ are bounded, then fg belongs to $Wap_{\mathcal{A}}(G)$ as well.*

Note that the previous proposition gives that for any $p \ge 1$, any $f \in Wap_{\mathcal{A}}^p(G) \subset Wap_{\mathcal{A}}^1(G)$ and any $\chi \in \widehat{G}$, the Fourier–Bohr coefficient
$$a_\chi(f) = \lim_{n\to\infty} \frac{1}{|A_n|} \int_{s+A_n} \overline{\chi(t)}\, f(t)\, \mathrm{d}t$$
exists uniformly in $s \in G$.

Analogously to Proposition 3.7, one gets the following.

Proposition 4.8. *For each $1 \le p < \infty$, we have $Wap_{\mathcal{A}}^p(G) \cap L^\infty(G) = Wap_{\mathcal{A}}(G) \cap L^\infty(G)$.*

Remark 4.9. From the preceding proposition, we obtain easily that a measure in $\mathcal{M}^\infty(G)$ is Weyl p-almost periodic if and only if is Weyl 1-almost periodic.

The subsequent statement does not have an analogue for Besicovitch almost periodic functions.

Proposition 4.10. *Whenever f belongs to $Wap_{\mathcal{A}}^p(G)$ for some $p \ge 1$, then so does $\tau_t f$ for any $t \in G$ and $\|f\|_{w,p,\mathcal{A}} = \|\tau_t f\|_{w,p,\mathcal{A}}$ holds.*

Proof. This is immediate from the definition of the Weyl seminorm. □

We next show that for bounded Weyl almost periodic functions the van Hove sequence does not matter.

Proposition 4.11. *Let $f: G \longrightarrow \mathbb{C}$ be a bounded and measurable function. Let $\mathcal{A}$ and $\mathcal{B}$ be van Hove sequences. Then, f belongs to $Wap_{\mathcal{A}}^p(G)$ if and only if it belongs to $Wap_{\mathcal{B}}^p(G)$. If f belongs to $Wap_{\mathcal{A}}^p(G)$ and $Wap_{\mathcal{B}}^p(G)$, then $\|f\|_{w,p,\mathcal{A}} = \|f\|_{w,p,\mathcal{B}}$ holds.*

Proof. This follows from Proposition D.1: Assume $f \in Wap_{\mathcal{A}}^p(G)$. Let $\varepsilon > 0$ be given. Then, there exists a natural number N and a trigonometric polynomial P with

$$\frac{1}{|A_N|} \int_{A_N+s} |f(t) - P(t)|^p \, \mathrm{d}t < \varepsilon$$

for all $s \in G$. With $h = |f - P|$, $A = A_N$ and $r = \varepsilon$, we infer then from Proposition D.1

$$\frac{1}{|B_n|} \int_{B_n+u} |f(t) - P(t)|^p \, \mathrm{d}t < 2\varepsilon$$

for all $u \in G$ and n sufficiently large. As $\varepsilon > 0$ was arbitrary, we infer that $f \in Wap_{\mathcal{B}}^p(G)$.

Similarly, there exist a natural number N with

$$\frac{1}{|A_N|} \int_{A_N+s} |f(t)|^p \, \mathrm{d}t < \|f\|_{w,p,\mathcal{A}}^p + \varepsilon$$

for all $s \in G$. With $h = |f|^p$, $A = A_N$ and $r = \|f\|_{w,p,\mathcal{A}}^p + \varepsilon$ we then infer, again from Proposition D.1,

$$\frac{1}{|B_n|} \int_{B_n+s} |f(t)| \, \mathrm{d}t < \|f\|_{w,p,\mathcal{A}} + 2\,\varepsilon$$

for all $s \in G$ and n sufficiently large. As $\varepsilon > 0$ was arbitrary this gives

$$\|f\|_{w,p,\mathcal{B}} \leq \|f\|_{w,p,\mathcal{A}} \, .$$

Reversing the roles of $\mathcal{A}$ and $\mathcal{B}$ we obtain the remaining statement. □

In fact, it is even possible to think about bounded Weyl almost periodic functions as functions, which are Besicovitch almost periodic for every van Hove sequence.

Proposition 4.12 (Characterization of bounded elements of $Wap_{\mathcal{A}}(G)$)**.** *Let $f: G \longrightarrow \mathbb{C}$ be a bounded measurable function and $\mathcal{A}$ a van Hove sequence. Then, the following assertions are equivalent:*

(i) *The function f belongs to $Wap_{\mathcal{A}}(G)$.*

(ii) *The Fourier–Bohr coefficient*

$$a_\chi^{\mathcal{A}}(f) = \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n+s} f(t)\, \chi(t)\, dt$$

exist uniformly in $s \in G$, and so does $\lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n+s} |f(t)|^2 \, dt = M(|f|^2)$ and $\sum_{\chi \in \widehat{G}} |a_\chi^{\mathcal{A}}|^2 = M(|f|^2)$ holds.

(iii) *The Fourier–Bohr coefficient*

$$a_\chi^{\mathcal{B}}(f) = \lim_{n\to\infty} \frac{1}{|B_n|} \int_{B_n} f(t)\, \overline{\chi(t)}\, dt$$

exist for each Hove sequence $\mathcal{B}$, and so does $\lim_{n\to\infty} \frac{1}{|B_n|} \int_{B_n} |f(t)|^2 \, dt = M_{\mathcal{B}}(|f|^2)$ and $\sum_{\chi \in \widehat{G}} |a_\chi^{\mathcal{B}}|^2 = M_{\mathcal{B}}(|f|^2)$ holds.

(iv) *The function f belongs to $Bap_{\mathcal{B}}^2(G)$ for every van Hove sequence $\mathcal{B}$.*

In particular, any bounded function in $Wap_{\mathcal{A}}(G)$ is amenable.

Proof. By Proposition 3.7, a bounded measurable function f belongs to $Bap_{\mathcal{B}}(G)$ if and only if it belongs to $Bap^2_{\mathcal{B}}(G)$. This will be used throughout the proof.

The equivalence between (iii) and (iv) follows from Corollary 3.21.

The equivalence between (iii) and (ii) follows easily from Proposition 1.2.

The equivalence between (i) and (ii) is just a uniform (in $s \in G$) version of the characterization of $Bap^2_{\mathcal{A}}(G)$ in Corollary 3.21. It can be shown by mimicking the proof of that corollary.

Finally, we turn to the last statement: Note that by (ii) the mean $M(f) = c_1(f)$ exists uniformly in translates. □

Next, for Weyl almost periodic functions we show a stronger version of Theorem 3.28.

Proposition 4.13. *Let $\mathcal{A}$ be a van Hove sequence. Let f, g be functions in $Wap^2_{\mathcal{A}}(G)$. Then, the Eberlein convolution $f \circledast_{\mathcal{A}} g$ exists and belongs to $SAP(G)$.*

Proof. The space $Wap^2_{\mathcal{A}}(G)$ is contained in $Bap^2_{\mathcal{A}}(G)$. Moreover, for any $f \in Wap^2_{\mathcal{A}}(G)$ and $t \in G$ its translate $\tau_t f$ clearly belongs to $Bap^2_{\mathcal{A}}(G)$ as well and, hence, is a representative of $T_t[f]$. Hence, $f \circledast_{\mathcal{A}} g$ exists and belongs to $SAP(G)$ by Theorem 3.28. □

4.2. **Uniform pure point diffraction with Fourier–Bohr coefficients.** The standard examples of aperiodic order do exhibit not only pure point diffraction and existence of phases but rather a uniform version of existence of phases and consistent phase property. In this section, we characterize the validity of these properties by Weyl almost periodicity.

First, let us look at the Fourier–Bohr coefficients of a Weyl almost periodic measure. There, we get as an immediate consequence of Lemma 4.7 and Corollary 1.11 the following.

Lemma 4.14. *Let $\mu \in \mathcal{W}\mathsf{ap}(G) \cap \mathcal{M}^\infty(G)$. Then, for each $\chi \in \widehat{G}$, the Fourier–Bohr coefficient*

$$a_\chi(\mu) = \lim_{n\to\infty} \frac{1}{|A_n|} \int_{s+A_n} \overline{\chi(t)}\, d\mu(t)$$

exists uniformly in $s \in G$, and does not depend on the choice of the van Hove sequence. Moreover, for all $\varphi \in C_{\mathsf{c}}(G)$, we have

$$a_\chi(\mu * \varphi) = a_\chi(\mu)\, \widehat{\varphi}(\chi)\,.$$

We can now characterize the space $\mathcal{W}\mathsf{ap}^2(G)$.

Theorem 4.15. *Let $\mu \in \mathcal{M}^\infty(G)$ be given and $\mathcal{A}$ a van Hove sequence. Then, the following assertions are equivalent:*

(i) *The measure μ belongs to $\mathcal{W}\mathsf{ap}^2_{\mathcal{A}}(G)$.*

(ii) *The measure μ belongs to $\mathcal{B}\mathsf{ap}^2_{\mathcal{A}}(G)$ and the following hold:*

- *For all $\varphi \in C_{\mathsf{c}}(G)$, the function $|\mu * \varphi|^2$ is amenable.*

- *For each $\chi \in \widehat{G}$, the Fourier–Bohr coefficient*
$$a_\chi(\mu) = \lim_{n\to\infty} \frac{1}{|A_n|} \int_{s+A_n} \overline{\chi(t)}\, d\mu(t)$$
exist uniformly in $s \in G$.

(iii) *The measure μ belongs to $\mathcal{B}\mathsf{ap}^2_{\mathcal{B}}(G)$ for all van Hove sequences $\mathcal{B}$. Moreover, in this case, any finite product of functions from the set $\{\mu * \varphi, \overline{\mu * \varphi} : \varphi \in C_{\mathsf{c}}(G)\}$ is amenable.*

Proof. We note that, by Corollary 1.12, existence of the Fourier–Bohr coefficients for a translation bounded measure μ is equivalent to existence of the Fourier–Bohr coefficients for all $\mu*\varphi$, $\varphi \in C_{\mathsf{c}}(G)$. Clearly, $\mu*\varphi$ is bounded (and even belongs to $C_{\mathsf{u}}(G)$) for every $\varphi \in C_{\mathsf{c}}(G)$. Now, the characterization follows easily from Proposition 4.12. The last claim is immediate. Indeed, for each $\varphi \in C_{\mathsf{c}}(G)$ the bounded functions $\mu*\varphi, \overline{\mu*\varphi}$ belong to $Wap^2_{\mathcal{A}}(G) \subseteq Wap_{\mathcal{A}}(G)$. Therefore, by Proposition 4.7(c), any product of such functions belongs to $Wap_{\mathcal{A}}(G)$ and hence is amenable. □

At the end of this section, let us discuss the solution to the uniform phase problem. Recall from Proposition 1.2 that the existence of means for each van Hove sequence actually implies independence of the mean of the van Hove sequence. For this reason we do not state independence of the van Hove sequence in the condition below.

Theorem 4.16 (Solution to the uniform phase problem)**.** *Let $\mu \in \mathcal{M}^\infty(G)$. Then, $\mu \in \mathcal{W}\mathsf{ap}^2(G)$ if and only if the following three conditions hold:*

(a) *The autocorrelation γ of μ exists for each van Hove sequence $\mathcal{B}$ and $\widehat{\gamma}$ is a pure point measure.*
(b) *The Fourier–Bohr coefficients $a_\chi(\mu)$ exist for all $\chi \in \widehat{G}$ and for each Hove sequence $\mathcal{B}$.*
(c) *The consistent phase property*
$$\widehat{\gamma}(\{\chi\}) = |a_\chi(\mu)|^2$$
holds for all $\chi \in \widehat{G}$.

Proof. This is a direct consequence of the characterization of Weyl almost periodic measures in Theorem 4.15 and the solution to the phase problem in Theorem 3.36. □

4.3. **Meyer almost periodic functions and measures.** In this part, we look at a generalisation of almost periodicity which was introduced by Yves Meyer [39]. We show that elements of this large class of measures are Weyl almost periodic.

Recall that we denote the mean of amenable functions by M and that all Bohr almost periodic functions are amenable.

Definition 4.17 (Generalized almost periodicity)**.** [39, Def. 2.1]. A function $f : G \to \mathbb{R}$ is called **generalized almost periodic (g-a-p)** if it is a measurable and, for each $\varepsilon > 0$, there exist Bohr almost periodic functions g and h such that $g \le f \le h$ and $M(h-g) < \varepsilon$. A complex valued function $f : G \longrightarrow \mathbb{C}$ is called **generalized almost periodic** if both its real and its imaginary part are generalized almost periodic. A real valued Borel measure μ on

G is called **generalized almost periodic (g-a-p) measure** if, for each $\varepsilon > 0$, there exist strongly almost periodic measures ν and ω such that $\nu \le \mu \le \omega$ and $M(\omega - \nu) < \varepsilon$.

Remark 4.18. (a) Note that any g-a-p function must be bounded as Bohr almost periodic functions are bounded and the g-a-p function is bounded by Bohr almost periodic functions from above and below.

(b) The article of Meyer deals with functions on $\mathbb{R}^n$ and, accordingly, gives the definition for $\mathbb{R}^n$.

We note the following immediate consequence of the definition.

Proposition 4.19 (g-a-p implies Weyl almost periodicity)**.** (a) *If $f \in L^1_{loc}(G)$ is a g-a-p function, then $f \in Wap(G)$.*

(b) *If $\mu \in \mathcal{M}^\infty(G)$ is a g-a-p measure, then $\mu \in \mathcal{W}\mathsf{ap}(G)$.*

Proof. It suffices to show (a). To prove (a), it suffices to consider real valued functions f. Let $\varepsilon > 0$ and $h, g \in SAP(G)$ with $g \le f \le h$ and $M(h-g) \le \varepsilon$ be given. Now, clearly $|f - g| \le (h - g)$, and $\|f - h\|_{w,1} \le M(h-g) \le \varepsilon$ follows. As $\varepsilon > 0$ was arbitrary, the desired statement holds. □

A main merit of generalized almost periodicity is that the class of regular model sets can be seen to have this property. So, this class includes the arguably most important examples of aperiodic order. This is already shown by Meyer in the Euclidean setting (compare [39, Thm. 3.3] for $G = \mathbb{R}^d$). Our more general situation can be treated along similar lines. We include a discussion for completeness reasons. For the definition of regular model sets see Appendix B. For further details on (regular) model sets we refer the reader to [3, 4, 40, 38, 51, 54].

Lemma 4.20. *If Λ is a regular model set on G, then δ_Λ is a g-a-p measure.*

Proof. Let $\varphi \in C_{\mathsf{c}}(G)$ be non-negative. Let $(G, H, \mathcal{L})$ be the CPS and W the regular window which produces the regular model set. Pick $h, g \in C_{\mathsf{c}}(H)$ such that $h \le 1_W \le g$ and

$$\int_H (g(t) - h(t))\,\mathrm{d}t < \frac{\varepsilon}{1 + \mathrm{dens}(\mathcal{L})\ \int_G \varphi(t)\,\mathrm{d}t}\,.$$

Then, since $\varphi \ge 0$, we have $\omega_h * \varphi \le \delta_\Lambda * \varphi \le \omega_g * \varphi$ and $M(\omega_g * \varphi - \omega_h * \varphi) < \varepsilon$. Here, ω_g and ω_h are the measures defined via

$$\omega_g := \sum_{(x,x^\star)\in\mathcal{L}} g(x^\star)\,\delta_x \qquad \text{and} \qquad \omega_h := \sum_{(x,x^\star)\in\mathcal{L}} h(x^\star)\,\delta_x\,.$$

Since $\omega_g, \omega_h \in \mathcal{SAP}(G)$ (see for example [7, 32, 46, 54]), it follows that $\delta_\Lambda * \varphi$ is a g-a-p function for all non-negative $\varphi \in C_{\mathsf{c}}(G)$. The claim follows because every $\psi \in C_{\mathsf{c}}(G)$ can be decomposed as a linear combination of positive functions $\varphi \in C_{\mathsf{c}}(G)$. □

It is possible to give a characterization of g-a-p functions via the Bohr compactification. This is already hinted at in Meyer's work but details are not given. For this reason we include the proof.

Lemma 4.21 (Characterization of g-a-p)**.** *Let $f : G \longrightarrow \mathbb{R}$ be a bounded and measurable function. Then, the following assertions are equivalent:*

(i) *f is g-a-p.*

(ii) *There exist two Riemann integrable functions $k'_<, k'_>$ on G_b with $k'_< \leq k'_>$ and $\int_{G_b}(k'_>(x) - k'_<(x))(x)\, dx = 0$ such that $g' \circ i_b \leq f \leq h' \circ i_b$ holds for all $g', h' \in C(G_b)$ with $g' \leq k'_<$ and $k'_> \leq h'$.*

If the equivalent conditions (i) and (ii) hold, then f belongs to $Bap^1_{\mathcal{A}}G$ for every van Hove sequence $\mathcal{A}$ and $(f)_{\mathsf{b},1} = [k'_<]$ holds.

Proof. (i)$\Longrightarrow$(ii): By (i), there exist $g_n, h_n \in SAP(G)$, $n \in \mathbb{N}$, with $g_n \leq f \leq h_n$ and $M(h_n - g_n) \to 0, n \to \infty$. Without loss of generality we can assume that $g_n \leq g_{n+1}$ and $h_{n+1} \leq h_n$ for all $n \in \mathbb{N}$ (as otherwise we could replace g_n by $\max\{g_1, \ldots, g_n\}$ and similarly h_n by $\min\{h_1, \ldots, h_n\}$). By the defining property of the Bohr compactification, there exist then unique $g'_n, h'_n \in C(G_b)$ with $h_n = h'_n \circ i_b$ and $g_n = g'_n \circ i_b$ for each $n \in \mathbb{N}$. As the Bohr map preserves positivity, the sequences (g'_n) and (h'_n) are increasing and decreasing respectively and $g'_n \leq h'_n$. Define $k'_>$ to be the pointwise limit of the h'_n and $k'_<$ to be the pointwise limit of the g'_n. Then, $k'_<$ and $k'_>$ are Riemann integrable with $\int_{G_b}(k'_> - k'_<)(x)\, \mathrm{d}x = 0$ by construction.

Now, let $h' \in C(G_b)$ with $k'_> \leq h'$ be given. Then, for each $\varepsilon > 0$ and each $s \in G$, we have

$$f(s) \leq h_n(s) = h'_n \circ i_b(s) \leq k'_> \circ i_b(s) + \varepsilon \leq h' \circ i_b(s) + \varepsilon$$

for all sufficiently large n. As $\varepsilon > 0$ and $s \in G$ was arbitrary, this shows $f \leq h' \circ i_b$.

The statement for $g' \in C(G_b)$ with $g' \leq k'_<$ can be shown similarly. This shows (ii).

(ii)$\Longrightarrow$(i): Let $\varepsilon > 0$ be given. By definition of Riemann integrability, we can find $h', g' \in C(G_b)$ with $h' \geq k'_>$ and $k'_< \geq g'$ and $\int_{G_b}(h' - g')(x)\, \mathrm{d}x < \varepsilon$. Then, $g' \circ i_b, h' \circ i_b$ belong to $SAP(G)$. Due to (ii) they satisfy $g' \circ i_b \leq f \leq h' \circ i_b$. Moreover, $M(h' \circ i_b - g' \circ i_b) = \int_{G_b}(h' - g')(x)\, \mathrm{d}x < \varepsilon$ holds. This proves (i).

We have already shown in Lemma 4.19 that any g-a-p function belongs to $Wap(G)$ and, hence, to $Bap_{\mathcal{A}}(G)$. The last part of the statement follows from the construction in the proof of the equivalence. □

We have already noted in (c) of Remark 4.2 that any perturbation of a Weyl almost periodic function by a function with vanishing uniform absolute mean is also Weyl almost periodic. This suggests to consider the class of functions which are g-a-p up to such a perturbation. It will give a natural class of functions and measures on arbitrary locally compact Abelian groups. This class will contain all g-a-p functions and measures. At the same time it will also contain all weakly almost periodic functions and measures. So, it seems fair to say that this class contains all 'smooth' examples studied so far in the context of aperiodic order. We will refer to this type of almost periodicity as Meyer almost periodicity. A precise definition is provided next.

Definition 4.22 (Meyer almost periodicity)**.** A real valued function $f : G \to \mathbb{R}$ is called **Meyer almost periodic** if

$$f = g + h$$

with a g-a-p function g and a bounded function h with $\overline{uM}(|h|) = 0$. A complex valued function $f : G \to \mathbb{C}$ is called **Meyer almost periodic** if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are Meyer almost periodic functions.

A measure $\omega \in \mathcal{M}^\infty(G)$ is called **Meyer almost periodic** if, for all $\varphi \in C_\mathsf{c}(G)$, the function $\omega * \varphi$ is a Meyer almost periodic function.

Note here that since h is bounded $\overline{uM}(|h|) = 0$ holds independently of the choice of the van Hove sequence. In particular, Meyer almost periodicity is independent of the choice of van Hove sequence.

Remark 4.23. (a) If f is Meyer almost periodic, and $\overline{uM}(|h|) = 0$ then $f + h$ is Meyer almost periodic.
(b) It is obvious that g-a-p functions are Meyer almost periodic functions. Similarly, g-a-p measures are Meyer almost periodic measures.
(c) A weakly almost periodic function or measure, respectively, is a Meyer almost periodic function, or Meyer almost periodic measure, respectively.
(d) It is easy to see that a linear combination of Meyer almost periodic functions or measures, respectively, is a Meyer almost periodic function or measure, respectively.

From the definition and Proposition 4.19, we immediately infer the following result.

Corollary 4.24 (Meyer almost periodicity entails Weyl almost periodicity)**.** (a) *If $f \in L^1_{loc}(G)$ is a Meyer almost periodic function, then $f \in Wap(G)$.*
(b) *If $\mu \in \mathcal{M}^\infty(G)$ is a Meyer almost periodic measure, then $\mu \in \mathcal{W}\mathsf{ap}(G)$.*

Indeed, we can characterize Meyer almost periodicity within Weyl almost periodicity as follows.

Theorem 4.25 (Characterization of Meyer almost periodicity)**.** *Let $f \in L^1_{loc}(G)$. Then, the following are equivalent:*
(i) *f is Meyer almost periodic.*
(ii) *$f \in Wap(G)$ and the class of $(f)_{\mathsf{b},1} \in L^1(G_\mathsf{b})$ contains a Riemann integrable function.*

Proof. (i)$\Longrightarrow$(ii). The preceding corollary gives that any Meyer almost periodic function is Weyl almost periodic. Moreover, if $f = g+h$ is Meyer almost periodic with g being g-a-p and h having uniform mean zero, then $(f)_{\mathsf{b},1} = (g)_{\mathsf{b},1}$ contains an Riemann integrable representative by Lemma 4.21.

(ii)$\Longrightarrow$(i):

Note first that it suffices to prove the claim for real valued functions.

Let $k \in L^1(G_\mathsf{b})$ be Riemann integrable such that $(f)_{\mathsf{b},1} = k$ in $L^1(G_\mathsf{b})$. By Riemann integrability, we can then find functions

$$g_1' \leq g_2' \leq \ldots \leq g_n' \leq \ldots \leq k \leq \ldots h_n' \leq \ldots \leq h_2' \leq h_1' \,.$$

such that $\int_{G_\mathsf{b}} (h_n' - g_n') \,\mathrm{d}\theta_{G_\mathsf{b}} \leq \frac{1}{n}$. Then, the restrictions g_n, h_n to G are Bohr almost periodic and satisfy

$$g_1 \leq g_2 \leq \ldots \leq g_n \leq \ldots \leq \ldots h_n \leq \ldots \leq h_2 \leq h_1$$

as well as

$$(g_n)_{\mathsf{b}} = g_n' \qquad \text{and} \qquad (h_n)_{\mathsf{b}} = h_n' \,.$$

In particular, one has

$$M(h_n - g_n) = \int_{G_{\mathsf{b}}} (h_n' - g_n')\, \mathrm{d}\theta_{G_{\mathsf{b}}} \leq \frac{1}{n} \,.$$

Define

$$g(x) = \sup\{g_n(x) : n \in \mathbb{N}\} \,.$$

Then, we have

$$g_1 \leq g_2 \leq \ldots \leq g_n \leq \ldots \leq g \leq \ldots h_n \leq \ldots \leq h_2 \leq h_1$$

from where we get that g is a g-a-p function.

Let $h = f - g$. Note first that $|h| = |f - g| \leq |f| + |g| \leq |f| + |h_1|$ is bounded. We show that $\overline{uM}(|h|) = 0$, which together with $f = g + h$ completes the proof.

Note here that since $g_n \to g$ in $Wap(G)$, we have $[g_n]_{\mathsf{b}} \to [g]_{\mathsf{b},1}$ in $L^1(G_{\mathsf{b}})$ and hence $[g]_{\mathsf{b},1} = k = [f]_{\mathsf{b},1}$ in $L^1(G_{\mathsf{b}})$. In particular,

$$\int_{G_{\mathsf{b}}} |g_{\mathsf{b},1}(t) - f_{\mathsf{b},1}(t)| \ \mathrm{d}\theta_{G_{\mathsf{b}}}(t) = 0 \,.$$

Finally, since $f, g \in Wap(G)$, we have $g - f \in Wap(G)$ and hence so is $|g - f|$. Indeed, for each $\varepsilon > 0$ we can find some $u \in SAP(G)$ such that $\overline{uM}(|(g - f) - u|) < \varepsilon$. Then, $|u| \in SAP(G)$ and

$$||g - f| - u| \leq |(g - f) - u| \implies \overline{uM}(||g - f| - |u||) < \varepsilon$$

gives that $|g - f| \in Wap(G)$.

Next, if $u_n \in SAP(G)$ is such that $\overline{uM}((g - f) - u_n) < \frac{1}{n}$, for all $n \in \mathbb{N}$, then we get $\overline{uM}(||g - f| - |u_n||) < \frac{1}{n}$ and hence, by the definition of $(\cdot)_{\mathsf{b},1}$, we have in $L^1(G_{\mathsf{b}})$

$$|g_{\mathsf{b},1} - f_{\mathsf{b},1}| = |(g - f)_{\mathsf{b},1}| = \big| \lim_{n\to\infty} (u_n)_{\mathsf{b}} \big| = \lim_{n\to\infty} |(u_n)_{\mathsf{b}}| = \lim_{n\to\infty} (|u_n|)_{\mathsf{b}} = (|g - f|)_{\mathsf{b},1} \,.$$

Therefore,

$$\overline{uM}(|h|) = \overline{uM}(|g - f|) = \int_{G_{\mathsf{b}}} (|g - f|)_{\mathsf{b},1}\, \mathrm{d}\theta_{G_{\mathsf{b}}}(t) = \int_{G_{\mathsf{b}}} |g_{\mathsf{b},1}(t) - f_{\mathsf{b},1}(t)|\, \mathrm{d}\theta_{G_{\mathsf{b}}}(t) = 0 \,.$$

This shows that $\overline{uM}(|h|) = 0$. Since $f = g + h$ and g is a g-a-p function, we get the claim. $\square$

As a consequence of the preceding results, we can show that $\mathcal{W}\mathsf{ap}(G)$ contains an ample supply of examples.

Corollary 4.26. *The set $\mathcal{W}\mathsf{ap}(G)$ contains all Dirac combs of regular model sets as well as all weakly almost periodic measures.*

It is well known that weakly almost periodic measures and model sets are uniquely ergodic, have pure point spectrum and continuous eigenfunctions [35, 51]. When combining our previous corollary with Theorem 6.15 below, we obtain an alternative proof for this.

5. Unavoidability of Besicovitch and Weyl almost periodicity

In the preceding sections, we have discussed how Besicovitch and Weyl almost periodic functions allow one to solve the phase problem and the uniform phase problem. In this section, we discuss how - under a mild additional regularity condition - these are actually the only solutions.

In the article [27], Lagarias outlines some conditions that a vector space C of almost periodic functions should satisfy in order to give a good theory. These conditions include the following conditions:

- Expansion in Fourier series, i.e. each $f \in C$ has a formal Fourier series $f \sim \sum c_\chi \chi$.
- Riesz–Fischer property holds, i.e. for each square summable (c_χ) there is an element $f \in C$ with $f \sim \sum c_\chi \chi$.
- Parseval equality holds, i.e. $\|f\|^2 = \sum |c_\chi|^2$ for all $f \in C$.

While it is not explicitly stated, two further requirements seem to be natural. First, the Fourier expansion is linear. Second, with the choice that the coefficients c_χ vanish for all but one χ, one obtains that the space C contains the characters. Now, the basic idea is that the measures μ (or distributions) with $\mu * \varphi \in C$ for all $\varphi \in C_{\mathsf{c}}(G)$ have the desired diffraction properties. This suggests to add another assumption viz that the elements of C themselves also have the desired diffraction properties. Making this additional assumption one ends up with Besicovitch almost periodic functions as the next lemma shows.

Lemma 5.1 (Appearance of $Bap^2_{\mathcal{A}}(G)$). *Let C be a subspace of $L^2_{loc}(G)$ with a seminorm $\|\cdot\|$, and $\mathcal{A}$ be a van Hove sequence with the following properties:*

(a) $\widehat{G} \subset C$.

(b) *For all $f \in C$ there exist $c_\chi \in \mathbb{C}$ and a formal expansion $f \sim \sum_\chi c_\chi \chi$ such that*

$$\|f\|^2 = \sum_{\chi \in \widehat{G}} |c_\chi|^2 .$$

(c) *The formal Fourier expansion in (b) is linear.*

(d) *For all $f \in C$ the Eberlein convolution $f \circledast \widetilde{f}$ exists with respect to $\mathcal{A}$, is continuous, and the measure $\gamma_f := (f \circledast \widetilde{f})\theta_G$ satisfies*

$$\widehat{\gamma_f} = \sum_{\chi \in \widehat{G}} |c_\chi|^2 \, \delta_\chi .$$

Then, $C \subseteq Bap^2_{\mathcal{A}}(G)$ and for each $f \in C$ we have $\|f\| = \|f\|_{b,2}$ and c_χ is the Fourier–Bohr coefficient of f for each $\chi \in \widehat{G}$, that is $c_\chi = a^{\mathcal{A}}_\chi(f)$. Moreover, if the Riesz–Fischer condition holds in C, then $C = \mathcal{B}\mathsf{ap}^2_{\mathcal{A}}(G)$.

Proof. Let $f \in C$ be arbitrary. Let $\{\chi_n : n \in \mathbb{N}\}$ be an enumeration such that

$$f \sim \sum_{n=1}^{\infty} c_{\chi_n} \chi_n .$$

Note that this is possible, since $\{\chi : c_\chi \neq 0\}$ is at most countable, and, if finite we can pick some χ_n such that $c_{\chi_n} = 0$.

For each $N \in \mathbb{N}$, set $P_N := \sum_{n=1}^{N} c_{\chi_n} \chi_n$. Let $g_N = f - P_N \in C$. Then, by (c), g_N has the formal Fourier series

$$g_N \sim \sum_{n=N+1}^{\infty} c_{\chi_n} \chi_n$$

and

$$\|g_N\|^2 = \sum_{n=N+1}^{\infty} |c_{\chi_n}|^2 .$$

We also know that

$$\widehat{\gamma_{g_N}} = \sum_{n=N+1}^{\infty} |c_{\chi_n}|^2 \delta_{\chi_n}$$

is a finite measure. Let h be the inverse Fourier transform of this finite measure. Then, we have by [1, 43]

$$\widehat{\gamma_{g_N}} = \widehat{h\theta_G}$$

and hence $\gamma_{g_N} = h\theta_G$. We also have $\gamma = (g_N \circledast \widetilde{g_N})\theta_G$ [9, Rem. 2.3]. This shows that

$$h\theta_G = (g_N \circledast \widetilde{g_N})\theta_G .$$

Since h and $g_N \circledast \widetilde{g_N}$ are continuous, they are equal everywhere. In particular

$$h(0) = (g_N \circledast \widetilde{g_N})(0) = M_{\mathcal{A}}(|g_N|^2) .$$

Therefore,

$$\|g_N\|^2 = \sum_{n=N+1}^{\infty} |c_{\chi_n}|^2 = h(0) = M_{\mathcal{A}}(|g_N|^2) .$$

Note that in the case $N = 0$ this yields

$$\|f\|^2 = M_{\mathcal{A}}(|f|^2) = \|f\|_{2,b} .$$

We also have

$$\lim_{N\to\infty} M_{\mathcal{A}}(|f - P_N|^2) = \lim_{N\to\infty} \sum_{n=N+1}^{\infty} |c_{\chi_n}|^2 = 0 .$$

This shows that $f \in Bap^2_{\mathcal{A}}(G)$ and that $P_N \to f$ in $Bap^2_{\mathcal{A}}(G)$. Therefore,

$$f = \sum_{\chi} c_\chi \, \chi \qquad \text{holds in } Bap^2_{\mathcal{A}}(G) ,$$

which implies $c_\chi = a^{\mathcal{A}}_\chi(f)$.

The last claim is obvious. □

If one assumes uniform existence of the autocorrelation one ends up with Weyl almost periodic functions as follows by a variant of the preceding considerations.

Lemma 5.2 (Appearance of $Wap^2(G)$). *Let C be a subspace of $L^2_{loc}(G)$ with a seminorm $\|\cdot\|$. Assume that the following properties hold:*

(a) $\widehat{G} \subset C$.

(b) *For all $f \in C$ there exist $c_\chi \in \mathbb{C}$ and a formal expansion $f \sim \sum_\chi c_\chi \chi$ with*

$$\|f\|^2 = \sum_{\chi \in \widehat{G}} |c_\chi|^2 .$$

(c) *The formal Fourier expansion in (b) is linear.*

(d) *For all $f \in C$ the Eberlein convolution $f \circledast_{\mathcal{A}} \widetilde{f}$ exists, is independent of the choice of the van Hove sequence, is continuous, and the measure $\gamma_f := (f \circledast \widetilde{f})\theta_G$ satisfies*

$$\widehat{\gamma_f} = \sum_\chi |c_\chi|^2 \delta_\chi .$$

Then, $C \subseteq Wap^2(G)$ and for each $f \in C$ we have $\|f\| = \|f\|_{2,w}$ and c_χ is the Fourier–Bohr coefficient of f for each $\chi \in \widehat{G}$.

Proof. Follow the lines of the previous proof until the line: In particular

$$h(0) = (g_N \circledast \widetilde{g_N})(0) = M(|g_N|^2) .$$

Note here that $(g_N \circledast \widetilde{g_N})(0)$ is independent of the choice of the van Hove sequence, and so is $M(|g_N|^2)$. In particular, the mean exists uniformly in translates. Therefore,

$$\|g_N\|^2 = \sum_{n=N+1}^{\infty} |c_{\chi_n}|^2 = h(0) = M(|g_N|^2)$$

Note that in the case $N = 0$ this yields

$$\|f\|^2 = M(|f|^2) = \|f\|_{2,w}$$

(since the mean exists uniformly in translates). We also have

$$\lim_{N\to\infty} M(|f - P_N|^2) = \lim_{N\to\infty} \sum_{n=N+1}^{\infty} |c_{\chi_n}|^2 = 0 .$$

Since, by the above observations, the mean exists uniformly in translates, we get $f \in Wap^2(G)$ and $P_N \to f$ in $Wap^2(G)$. Therefore,

$$f = \sum_\chi c_\chi \chi \qquad \text{holds in } Wap^2(G) ,$$

which implies $c_\chi = M(f\bar{\chi}) = a_\chi(f)$. □

Remark 5.3. The preceding two lemmas contain in (d) the requirement of continuity of the Eberlein convolution. This may seem like an extra condition. However, we note that for a translation bounded measures μ the existence of the Eberlein convolution $h := (\mu*\varphi)\circledast(\widetilde{\mu * \psi})$, for $\varphi, \psi \in C_{\mathsf{c}}(G)$, automatically entails that h is continuous and even uniformly continuous (see Proposition 1.4). So, as far as our application goes this is not a restriction.

6. Pure point diffraction, almost periodicity and TMDS

In this section, we have a look at pure point diffraction and almost periodicity from the point of dynamical systems. As discussed in the introduction, dynamical systems play a key role in the investigation of aperiodic order. In a companion article [33], we study related aspects for general dynamical systems.

6.1. Dynamical systems of translation bounded measures (TMDS).

Suitable dynamical systems provide a convenient (and heavily used) way to deal with diffraction. The necessary background is discussed in this section. We follow [6] to which we refer for further details and background.

Recall that, given a relatively compact open set $V \subset G$ and some $C > 0$, the set

$$\mathcal{M}_{C,V} := \{\mu \in \mathcal{M}^{\infty}(G) : \|\mu\|_V \leq C\}$$

is vaguely compact [6, Thm. 2]. Moreover, if G is second countable, the vague topology is metrisable on $\mathcal{M}_{C,V}$ [6, Thm. 2]. The natural group action of G on $\mathcal{M}^{\infty}(G)$ leaves $\mathcal{M}_{C,V}$ invariant and is continuous [6, Prop. 2]. Specifically,

$$G \times \mathcal{M}_{C,V} \longrightarrow \mathcal{M}_{C,V}\,, \qquad (t, \mu) \mapsto \delta_t * \mu$$

is a continuous action on $\mathcal{M}_{C,V}$.

Let us now recall the following definition [6, Def. 2].

Definition 6.1 (Translation bounded measure dynamical system (TMDS))**.** A pair $(\mathbb{X}, G)$ is called a **dynamical system on the translation bounded measures** on G (TMDS) if there exist a constant $C > 0$ and a relatively compact and open set $V \subset G$ such that $\mathbb{X}$ is a closed subset of $\mathcal{M}_{C,V}$ that is invariant under the G-action.

Note here that a closed G-invariant subset $\mathbb{X} \subset \mathcal{M}^{\infty}(G)$ is vaguely compact if and only if it is contained in some $\mathcal{M}_{C,V}$ [53]. Therefore, $(\mathbb{X}, G)$ is a TMDS if and only if $\mathbb{X} \subset \mathcal{M}^{\infty}(G)$ is G-invariant and vaguely compact.

Any translation bounded measure μ gives rise to a TMDS $(\mathbb{X}(\mu), G)$, where the **hull** $\mathbb{X}(\mu)$ is defined as

$$\overline{\{\tau_t \mu : t \in G\}},$$

with closure taken in the vague topology.

Given any TMDS $(\mathbb{X}, G)$, each $\varphi \in C_{\mathsf{c}}(G)$ induces a continuous function $f_{\varphi} : \mathbb{X} \to \mathbb{C}$ via

$$f_{\varphi}(\omega) := (\omega * \varphi)(0) = \int_G \varphi(-s)\, \mathrm{d}\omega(s)\,,$$

which is compatible with the action from [6, Lem. 3], given by $f_{\varphi}(\tau_t \omega) = f_{\tau_t \varphi}(\omega)$ for all $t \in G, \varphi \in C_{\mathsf{c}}(G)$ and $\omega \in \mathbb{X}$.

Next, let us review the notion of an autocorrelation measure.

Theorem 6.2. [6, Prop. 6, Lem. 7] *Let $(\mathbb{X}, G)$ be a TMDS. Given any G-invariant probability measure m on $\mathbb{X}$, there exists a unique positive definite measure γ on G such that, for all*

$\varphi, \psi \in C_{\mathsf{c}}(G)$ and all $t \in G$, we have

$$(\gamma * \varphi * \widetilde{\psi})(t) = \langle f_\varphi, T_t f_\psi \rangle := \int_{\mathbb{X}} (\tau_t f_\varphi)(\omega) \overline{f_\psi(\omega)} \ dm(\omega) \,. \tag{6}$$

Definition 6.3. [6, Def. 6] Given a TMDS $(\mathbb{X}, G)$ with a G-invariant probability measure m, the measure γ from Theorem 6.2 is called the **autocorrelation** of $(\mathbb{X}, G, m)$. Its Fourier transform $\widehat{\gamma}$ is called the **diffraction** of $(\mathbb{X}, G, m)$. We say that $(\mathbb{X}, G, m)$ has **pure point diffraction spectrum** if $\widehat{\gamma}$ is a pure point measure.

Whenever $(\mathbb{X}, G, m)$ is a TMDS, we call an $f \in L^2(\mathbb{X}, m)$ with $f \neq 0$ an **eigenfunction to the eigenvalue** $\chi \in \widehat{G}$ if $T_t f = \chi(t) f$ holds for all $t \in G$. The dynamical system $(\mathbb{X}, G, m)$ is said to have **pure point spectrum** if $L^2(\mathbb{X}, m)$ possess an orthonormal basis consisting of eigenfunctions.

We will make use of the following result (see [34] for generalisations to non-translation bounded measures).

Theorem 6.4. [6, Thm. 7, Thm. 8, Thm. 9] *Let $(\mathbb{X}, G)$ be a TMDS with a G-invariant probability measure m. Then, $(\mathbb{X}, G, m)$ has pure point diffraction spectrum if and only if $(L^2(\mathbb{X}, m), G)$ has pure point dynamical spectrum.*

We now turn to unique ergodicity of (TMDS). First, let us recall the following characterization of unique ergodicity. For $G = \mathbb{Z}$ this is given e.g. [57]. The case of more general G follows by simple adaption of the argument. We refrain from giving the details.

Theorem 6.5. *Let $(\mathbb{X}, G)$ be a transitive dynamical system and let $x \in \mathbb{X}$ be any element with a dense orbit. Then, $(\mathbb{X}, G)$ is uniquely ergodic if and only if the set*

$$\mathbb{A} := \{ f \in C(\mathbb{X}) : t \mapsto f(\tau_t x) \text{ is amenable } \}$$

is dense in $C(\mathbb{X})$. Moreover, in this case, $t \mapsto f(\tau_t x)$ is amenable for all $f \in C(\mathbb{X})$.

As a consequence, we obtain the next corollary.

Corollary 6.6. *Let $\mu \in \mathcal{M}^\infty(G)$. Then, $\mathbb{X}(\mu)$ is uniquely ergodic if and only if, any (finite) product of functions in the set $\{\mu * \varphi, \overline{\mu * \varphi} : \varphi \in C_{\mathsf{c}}(G)\}$ is amenable.*

Proof. Clearly the linear span of the mentioned functions is an algebra. This algebra separates the points and does not vanish anywhere and is closed under complex conjugation. Hence, it is dense in $C(\mathbb{X})$ by Stone-Weierstraß' theorem, and the preceding theorem proves the corollary. $\square$

Next, we show that the uniform existence of the Fourier–Bohr coefficients implies the continuity of the corresponding eigenfunction (compare [30]).

Theorem 6.7. *Let $\mu \in \mathcal{M}^\infty(G)$, $\chi \in \widehat{G}$ and $\mathcal{A}$ be any van Hove sequence. Assume that*

$$a_\chi(\mu) = \lim_{n\to\infty} \frac{1}{|A_n|} \int_{s+A_n} \overline{\chi(t)} \, d\mu(t)$$

exists uniformly in $s \in G$ and satisfies $a_\chi(\mu) \neq 0$. Then, for each $\omega \in \mathbb{X}(\mu)$, the Fourier–Bohr coefficient $a_\chi^{\mathcal{A}}(\omega)$ exists, does not vanish and the function

$$a_\chi^{\mathcal{A}} : \mathbb{X}(\mu) \longrightarrow \mathbb{C}$$

is continuous with $a_\chi^{\mathcal{A}}(\tau_t \omega) = \chi(t)\, a_\chi^{\mathcal{A}}(\omega)$ for all $\omega \in \mathbb{X}(\mu)$ and $t \in G$.

Proof. Let $\varphi \in C_{\mathsf{c}}(G)$ be such that $\widehat{\varphi}(\chi) = 1$. Then, by Corollary 1.11, we have $a_\chi(\mu * \varphi) = a_\chi(\mu)\, \widehat{\varphi}(\chi) \neq 0$.

For each n, define $\mathbb{A}_n^\chi : C(\mathbb{X}(\mu)) \to C(\mathbb{X}(\mu))$ (compare [30]) via

$$\mathbb{A}_n^\chi(f)(\omega) := \frac{1}{|A_n|} \int_{A_n} \overline{\chi(s)}\, f(\tau_s \omega)\, \mathrm{d}s\, .$$

A straightforward computation indeed reveals that $\mathbb{A}_n^\chi(f) \in C(\mathbb{X}(\mu))$ for each $f \in C(\mathbb{X})$.

Let $\varepsilon > 0$. Since the Fourier–Bohr coefficient $a_\chi(\mu)$ exists uniformly in x, by Corollary 1.11, so does $a_\chi(\mu * \varphi)$. Therefore, there exists $N \in \mathbb{N}$ such that

$$\left| \int_{A_n} \overline{\chi(s)}\, (\varphi * \mu)(-t+s)\, \mathrm{d}s - a_\chi(\tau_t(\varphi * \mu)) \right| < \frac{\varepsilon}{2}$$

for all $n > N$ and all $t \in G$. Therefore, for all $m, n > N$, we have

$$\left| \frac{1}{|A_n|} \int_{A_n} \overline{\chi(s)}\, (\varphi * \mu)(-t+s)\ \mathrm{d}s - \frac{1}{|A_m|} \int_{A_m} \overline{\chi(s)}\, (\varphi * \mu)(-t+s)\ \mathrm{d}s \right| < \varepsilon\, .$$

This shows that, for each $m, n > N$, we have

$$|\mathbb{A}_n^\chi(f_\varphi)(\tau_t \mu) - \mathbb{A}_m^\chi(f_\varphi)(\tau_t \mu)| < \varepsilon \qquad \text{for all } t \in G\, .$$

Since the orbit $\{\tau_t \mu \,:\, t \in G\}$ is dense in $\mathbb{X}(\mu)$ and since $\mathbb{A}_n^\chi(f_\varphi) - \mathbb{A}_m^\chi(f_\varphi) \in C(\mathbb{X}(\mu))$, one has

$$\|\mathbb{A}_n^\chi(f_\varphi) - \mathbb{A}_m^\chi(f_\varphi)\|_\infty \leq \varepsilon\, .$$

Consequently, since $(C(\mathbb{X}(\mu), \|\cdot\|_\infty)$ is a Banach spaces, there exists some $g \in C(\mathbb{X}(\mu))$ such that $\mathbb{A}_n^\chi(f_\varphi) \to g$ in $(C(\mathbb{X}(\mu), \|\cdot\|_\infty))$. Hence, g is continuous. As soon as we can show that g is an eigenfunction for χ and that $g \not\equiv 0$, the proof is complete.

Since $\big(\mathbb{A}_n^\chi(f_\varphi)\big)$ converges uniformly to g, for all $\omega \in \mathbb{X}(\mu)$, we have

$$g(\omega) = \lim_{n\to\infty} \mathbb{A}_n^\chi(f_\varphi)(\omega) = \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} \overline{\chi(s)}\, (\varphi * \omega)(s)\, \mathrm{d}s = a_\chi^{\mathcal{A}}(\varphi * \omega)\, .$$

This show that the Fourier–Bohr coefficient of $\varphi * \omega$ exists with respect to $\mathcal{A}$. Since $\widehat{\varphi}(\chi) = 1$, by Corollary 1.11, the Fourier–Bohr coefficient $a_\chi^{\mathcal{A}}(\omega)$ exists and

$$g(\omega) = a_\chi^{\mathcal{A}}(\varphi * \omega) = a_\chi^{\mathcal{A}}(\omega) \widehat{\varphi}(\chi) = a_\chi^{\mathcal{A}}(\omega)\, .$$

This shows that $g(\omega) = a_\chi^{\mathcal{A}}(\omega)$ for all $\omega \in \mathbb{X}(\mu)$. Since g is continuous, $\omega \to a_\chi^{\mathcal{A}}(\omega)$ is a continuous function, which is trivially an eigenfunction. Moreover, this is not trivial since $a_\chi^{\mathcal{A}}(\mu) \neq 0$. $\square$

6.2. **Characterizing TMDS with pure point spectrum.** In this section, we study the connection between the pure point spectrum of a (uniquely) ergodic TMDS $(\mathbb{X}, G, m)$ and the mean/Besicovitch almost periodicity of elements $\omega \in \mathbb{X}$. We prove that $(\mathbb{X}, G, m)$ has pure point spectrum if and only if m-almost all $\omega \in \mathbb{X}$ are mean/Besicovitch almost periodic.

Theorem 6.8. *Let $(\mathbb{X}, G)$ be a TMDS. Let m be an ergodic measure on $(\mathbb{X}, G)$ and $\mathcal{A}$ a van Hove sequence along which Birkhoff's ergodic theorem holds. Then, $(\mathbb{X}, G, m)$ has pure point spectrum if and only if, for m-almost all $\omega \in \mathbb{X}$, we have $\omega \in \mathcal{M}\mathsf{ap}_{\mathcal{A}}(G)$.*

Proof. Let γ be the autocorrelation of $(\mathbb{X}, G, m)$. By [6, Thm. 5(b)], there exists a full measure set $X \subseteq \mathbb{X}$ such that, for all $\omega \in X$, γ is the is the autocorrelation of ω with respect to $\mathcal{A}$. We now show both implications:

$\Longrightarrow$: Since $\widehat{\gamma}$ is pure point, every $\omega \in X$ has pure point diffraction with respect to $\mathcal{A}$. Therefore, by Theorem 2.13, we have $X \subseteq \mathcal{M}\mathsf{ap}_{\mathcal{A}}(G)$.

$\Longleftarrow$: We know that there exists a set $Y \subseteq \mathbb{X}$ of full measure such that $Y \subseteq \mathcal{M}\mathsf{ap}_{\mathcal{A}}(G)$. Then, $X \cap Y$ has full measure in $\mathbb{X}$, and hence is not trivial.

Pick some $\omega \in X \cap Y$. Then, $\omega \in \mathcal{M}\mathsf{ap}_{\mathcal{A}}(G)$ and hence, by Theorem 2.13, its diffraction $\widehat{\gamma}$ is pure point. $\square$

As a corollary (rather from the proof than from the actual statement), we obtain the following.

Corollary 6.9. *If $(\mathbb{X}, G)$ is uniquely ergodic, then $(\mathbb{X}, G)$ has pure point spectrum if and only if $\mathbb{X} \subseteq \mathcal{M}\mathsf{ap}_{\mathcal{A}}(G)$.*

Proof. Let γ be the unique autocorrelation of $(\mathbb{X}, G)$. By [6, Thm. 5(a)], the measure γ is the autocorrelation of ω with respect to (A_n), for all $\omega \in \mathbb{X}$. The claim follows now from Theorem 2.13. $\square$

If G is second countable, it turns out that we can also work with Besicovitch almost periodic measures instead of mean almost periodic measures.

Theorem 6.10. *Let $(\mathbb{X}, G, m)$ be an ergodic TMDS with second countable G, and let $\mathcal{A}$ be a van Hove sequence along which Birkhoff's ergodic theorem holds. Then, the system $(\mathbb{X}, G, m)$ has pure point spectrum if and only if for m-almost all $\omega \in \mathbb{X}$, we have $\omega \in \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G)$.*

Moreover, in this case, for each χ such that $\widehat{\gamma}(\{\chi\}) \neq 0$, there exists a non-trivial eigenfunction $f_\chi \in L^1(\mathbb{X}, m)$ such that

$$f_\chi(\omega) = a_\chi^{\mathcal{A}}(\omega) \qquad \text{for all } \omega \in \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G) \cap \mathbb{X}\,,$$

and

$$\widehat{\gamma}(\{\chi\}) = |f_\chi(\omega)|^2 = \left|a_\chi^{\mathcal{A}}(\omega)\right|^2 \qquad \text{for } m\text{-almost all } \omega \in \mathbb{X}\,.$$

Proof. $\Longleftarrow$: This follows from $\mathcal{B}\mathsf{ap}_{\mathcal{A}}(G) \subseteq \mathcal{M}\mathsf{ap}_{\mathcal{A}}(G)$ and Theorem 6.8.

$\Longrightarrow$: Since $\mathcal{B}\mathsf{ap}_{\mathcal{A}}^2(G) \cap \mathcal{M}^\infty(G) = \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G) \cap \mathcal{M}^\infty(G)$, it suffices to show that for m-almost all $\omega \in \mathbb{X}$, we have $\omega \in \mathcal{B}\mathsf{ap}_{\mathcal{A}}^2(G)$.

Denote the set of eigenvalues by E. Then, E is a countable subgroup of $\widehat{G}$ by standard arguments. Let $\chi_1, \chi_2, \dots$ be any enumeration of E. Via a standard procedure, we can choose a family $\{f_\chi\}_{\chi\in E}$ of eigenfunctions which are normalised, such that $f_1 = 1$ and

$$f_\chi(\tau_s\omega) = \chi(s) f_\chi(\omega)$$

for all $\chi \in E$ and $s \in G$.

Since G is second countable, we can find some sequence (K_j) of compact sets such that $G = \bigcup_j K_j$ and $K_j \subset K_{j+1}^\circ$. In particular, for each $K \subset G$ compact, $\bigcup_j K_j^\circ$ is an open cover of K, and hence, there exists some j such that $K \subseteq K_j$.

Next, by the metrisability of G, for each j, there exists some $Q_j \subset C(G : K_j) := \{f \in C_{\mathsf{c}}(G) : \operatorname{supp}(f) \subseteq K_j\}$ which is dense in $C(G : K)$ and set $Q := \bigcup Q_j$. It is easy to see that Q is dense in $C_{\mathsf{c}}(G)$.

Now, for each $\varphi \in Q$, since $L^2(\mathbb{X}, m)$ has pure point spectrum, we have

$$\lim_{N\to\infty} \|f_\varphi - \sum_{k=1}^{N} \langle f_\varphi, f_{\chi_k}\rangle f_{\chi_k}\|_2 = 0\,.$$

By Birkhoff's ergodic theorem, there exists a set $X_{\varphi,N}$ such that, for all $\mu \in X_{\varphi,N}$, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{|A_n|}\int_{A_n} |(\mu*\varphi)(s) - P_N(s)|^2 \, \mathrm{d}s &= \lim_{n\to\infty} \frac{1}{|A_n|}\int_{A_n} \left|(\mu*\varphi)(s) - \sum_{k=1}^{N} \langle f_\varphi, f_{\chi_k}\rangle f_{\chi_k}(T_s\mu)\right|^2 \, \mathrm{d}s \\ &= \int_{\mathbb{X}} \left| f_\varphi(\omega) - \sum_{k=1}^{N} \langle f_\varphi, f_{\chi_k}\rangle f_{\chi_k}(\omega)\right|^2 \, \mathrm{d}m(\omega)\end{aligned}$$

where $P(s) := \sum_{k=1}^N c_k \chi_k(s)$ with $c_k := \langle f_\varphi, f_{\chi_k}\rangle f_{\chi_k}(\mu)$. This shows that, for all $\varphi \in Q$ and $\mu \in X_{\varphi,N}$, the trigonometric polynomial P satisfies

$$\|\mu*\varphi - P\|_{b,2,\mathcal{A}} = \|f_\varphi - \sum_{k=1}^{N} \langle f_\varphi, f_{\chi_k}\rangle f_{\chi_k}\|_2\,. \tag{7}$$

Since Q is countable, the set

$$\mathbb{Y} := \bigcap_{\varphi\in Q} \bigcap_{N\in\mathbb{N}} X_{\varphi,N}$$

has full measure in $\mathbb{X}$. We show that $\mathbb{Y} \subseteq \mathcal{B}\mathsf{ap}^2_{\mathcal{A}}(G)$. Let $\omega \in \mathbb{Y}$.

For each $\varphi \in Q$, we have $\omega \in \bigcap_{N\in\mathbb{N}} X_{\varphi,N}$. Since $\lim_{N\to\infty} \|f_\varphi - \sum_{k=1}^N \langle f_\varphi, f_{\chi_k}\rangle f_{\chi_k}\|_2 = 0$, we get that $\omega * \varphi \in Bap^2_{\mathcal{A}}(G)$.

Next, let $\psi \in C_{\mathsf{c}}(G)$ be arbitrary. Let $\varepsilon > 0$. By [51, Lem. 1.1(2)], the sequence $\left(\frac{|\omega|(A_n)}{|A_n|}\right)$ is bounded because $\omega \in \mathcal{M}^\infty(G)$. Let C be an upper bound of this.

Pick some j such that $\operatorname{supp}(\psi) \subset K_j$. Since Q_j is dense in $C(G : K_j)$, there exists some $\varphi \in Q_j$ such that

$$\|\varphi - \psi\|_\infty < \min\{\frac{\varepsilon}{2C|K_j|\sqrt{\|\omega\|_{K_j}} + 1}, 1\}\,.$$

Since $\mathrm{supp}(\varphi), \mathrm{supp}(\psi) \subseteq K_j$ we also have

$$\|\varphi - \psi\|_1 < \frac{\varepsilon}{2C\sqrt{\|\omega\|_{K_j}} + 1} .$$

Since $\omega*\varphi \in Bap^2_{\mathcal{A}}(G)$, there is a trigonometric polynomial P such that $\|\omega*\varphi - P\|_{b,2,\mathcal{A}} < \frac{\varepsilon}{2}$. Therefore, one has

$$\begin{aligned} \|\omega * \psi - P\|_{b,2,\mathcal{A}} &\leq \|\omega * \varphi - P\|_{b,2,\mathcal{A}} + \|\omega * \psi - \omega * \varphi\|_{,b2,\mathcal{A}} \\ &< \frac{\varepsilon}{2} + \|\omega * \psi - \omega * \varphi|\|_{b,2,\mathcal{A}} . \end{aligned}$$

Now, by Lemma 1.16, we have

$$\begin{aligned} \|\omega * \psi - \omega * \varphi\|_{b,2,\mathcal{A}} &\leq \sqrt{\|\omega * \psi - \omega * \varphi\|_\infty} \, \|\omega * \psi - \omega * \varphi\|_{b,1,\mathcal{A}} \\ &\leq \sqrt{\|\psi - \varphi\|_\infty \|\omega\|_{K_j}} \, \|\omega * \psi - \omega * \varphi\|_{b,1,\mathcal{A}} \\ &\leq \sqrt{\|\omega\|_{K_j}} \, \|\omega * \psi - \omega * \varphi\|_{b,1,\mathcal{A}} . \end{aligned}$$

Now, by a standard van Hove and Fubini type argument, we get

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} |(\omega * \psi)(t) - (\omega * \varphi)(t)| \, \mathrm{d}t &\leq \|\varphi - \psi\|_1 \limsup_{n\to\infty} \frac{|\omega|\,(A_n)}{|A_n|} \\ &\leq C\,\|\varphi - \psi\|_1 < \frac{\varepsilon}{2\sqrt{\|\omega\|_{K_j}}} . \end{aligned}$$

Therefore, we obtain

$$\|\omega * \psi - P\|_{b,2,\mathcal{A}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon ,$$

which completes the argument.

We now show the last statement.

Pick a $\chi \in \widehat{G}$ such that $\widehat{\gamma}(\{\chi\}) \neq 0$. Define

$$f_\chi(\omega) = \begin{cases} a^{\mathcal{A}}_\chi(\omega), & \text{if } \omega \in \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G) \cap \mathbb{X} , \\ 0, & \text{otherwise} . \end{cases}$$

This is well defined as the Fourier–Bohr coefficients of Besicovitch almost periodic measures exist by Theorem 3.36.

We claim that this satisfies the given condition. Note first that by Lemma 1.13(c) the set $\mathbb{X} \cap \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G)$ is G-invariant. It follows that for all $\omega \notin \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G) \cap \mathbb{X}$, we have

$$f_\chi(\tau_t\omega) = 0 = \chi(t) f_\chi(\omega) .$$

For $\omega \in \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G) \cap \mathbb{X}$, it follows immediately from the definition of the Fourier–Bohr coefficients and translation boundedness that $f_\chi(\tau_t\omega) = \chi(t) f_\chi(\omega)$.

Next, we show that $f_\chi \in L^1(\mathbb{X}, m)$.

Pick some φ such that $\widehat{\varphi}(\chi) = 1$. As in the proof of Theorem A.4, define $\mathbb{A}^\chi_n : C(\mathbb{X}(\mu)) \to C(\mathbb{X}(\mu))$ (compare [30]) via

$$\mathbb{A}^\chi_n(f)(\omega) := \frac{1}{|A_n|} \int_{A_n} \overline{\chi(s)}\, f(\tau_s\omega) \, \mathrm{d}s .$$

Then, $\mathbb{A}_n^{\chi}(f_\varphi) \in C(\mathbb{X})$ for all n. Moreover, by definition, $\|\mathbb{A}_n^{\chi}(\varphi)\|_\infty \leq \|f_\varphi\|_\infty$.

For all $\omega \in \mathbb{X} \cap \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G)$ the Fourier–Bohr coefficients of ω exist by Corollary 5.5, and hence

$$f_\chi(\omega) = a_\chi^{\mathcal{A}}(\omega) = a_\chi^{\mathcal{A}}(\omega * \varphi) = \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} f_\varphi(\tau_t \omega)\, \mathrm{d}\omega = \lim_{n\to\infty} \mathbb{A}_n^{\chi}(f_\varphi)(\omega)\,.$$

Since $m(\mathbb{X} \cap \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G)) = 1$, it follows that $\mathbb{A}_n^{\chi}(f_\varphi)$ is a sequence of functions in $C(\mathbb{X}) \subseteq L^1(\mathbb{X}, m)$, which is bounded by the constant function $\|f_\varphi\|_\infty 1_{\mathbb{X}} \in L^1(\mathbb{X}, m)$ and which converges almost everywhere to f_χ. The dominated convergence theorem then implies that $f_\chi \in L^1(\mathbb{X}, m)$ as claimed.

Finally, as $\widehat{\gamma}(\{\chi\}) \neq 0$ and as γ is almost surely the autocorrelation of $\omega \in \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G) \cap \mathbb{X}$, we have by Theorem 3.34

$$0 \neq \widehat{\gamma}(\{\chi\}) = \left|a_\chi^{\mathcal{A}}(\omega)\right|^2 = |f_\chi(\omega)|^2 \qquad \text{for } m\text{-almost all } \omega \in \mathbb{X}\,.$$

This gives that f_χ is non-trivial, as well as the last claim. □

Combining the results in this section we obtain the following.

Corollary 6.11 (Characterization of pure point spectrum via almost periodicity)**.** *Consider an ergodic TMDS $(\mathbb{X}, G, m)$ with second countable G, and let $\mathcal{A}$ be a van Hove sequence along which Birkhoff's ergodic theorem holds. Then, the following statements are equivalent:*

(i) *The system $(\mathbb{X}, G, m)$ has pure point spectrum.*
(ii) $m(\mathbb{X} \cap \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G)) = 1$.
(iii) $m(\mathbb{X} \cap \mathcal{M}\mathsf{ap}_{\mathcal{A}}(G)) = 1$.

Remark 6.12. (a) In Corollary 6.11, we can have

$$\mathbb{X} \cap \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G) \subsetneq \mathbb{X} \cap \mathcal{M}\mathsf{ap}_{\mathcal{A}}(G)\,.$$

Consider for example the hull $\mathbb{X} := \mathbb{X}(\mu)$, where μ is the a-defect of $\mathbb{Z}$ for some $a \in (0,1)\backslash\mathbb{Q}$ from Proposition A.2. Then, by Proposition A.2, $(\mathbb{X}, G)$ is uniquely ergodic, has pure point spectrum and $\mathbb{X} \subset \mathcal{M}\mathsf{ap}_{\mathcal{A}}(G)$ but $\mu \notin \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G)$.

(b) Let $\mathbb{X}$ be a unique ergodic TMDS with pure point diffraction. Then, all elements $\omega \in \mathbb{X}$ are mean almost periodic and almost all elements $\omega \in \mathbb{X}$ are Besicovitch almost periodic. It is not necessarily true that all elements $\omega \in \mathbb{X}$ are Besicovitch almost periodic. Indeed, the hull $\mathbb{X} := \mathbb{X}(\mu)$, where μ is the a-defect of $\mathbb{Z}$ for some $a \in (0,1)\backslash\mathbb{Q}$, provides again such an example.

We complete the section by proving the following result which complements Theorem 6.10.

Theorem 6.13. *Let $\mu \in \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G) \cap \mathcal{M}^\infty(G)$. Then, there exists an ergodic G-invariant probability measure m on $\mathbb{X} := \mathbb{X}(\mu)$ with the following properties:*

(a) *For all $f \in C(\mathbb{X}(\mu))$, we have*

$$\lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} f(\tau_t \mu)\, dt = \int_{\mathbb{X}} f(\omega)\, dm(\omega)\,.$$

(b) *The autocorrelation* γ *of* $(\mathbb{X}, m, G)$ *is also the autocorrelation* γ_μ *of* μ *with respect to* $\mathcal{A}$.

(c) $(\mathbb{X}, m, G)$ *has pure point dynamical spectrum, which is generated by* $\{\chi : a_\chi^{\mathcal{A}}(\mu) \neq 0\}$.

Proof. First, for all $\varphi \in C_{\mathsf{c}}(G)$, we have $f_\varphi(\tau_t\mu) = (\mu * \varphi)(t)$. Therefore, the functions $t \mapsto f_\varphi(\tau_t\mu)$ and $t \mapsto \overline{f_\varphi(\tau_t\mu)}$ belong to $Bap_{\mathcal{A}}(G)$. It follows immediately that, for any f in the algebra generated by $\{f_\varphi, \overline{f_\varphi}\}$, the function $t \mapsto f(\tau_t\mu)$ belongs to $Bap_{\mathcal{A}}(G)$. By a standard density argument, we get (compare [33]) that for all $f \in C(\mathbb{X})$ the function $t \mapsto f(\tau_t\mu)$ belongs to $Bap_{\mathcal{A}}(G)$. Therefore, by Proposition 3.8, for each $f \in C(\mathbb{X}(\mu))$, the limit

$$m(f) := \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} f(\tau_t\mu)\,\mathrm{d}t$$

exists. It is obvious that $m : C(\mathbb{X}) \to \mathbb{C}$ is linear, positive, and therefore a positive measure. Moreover, for the constant function $1_{\mathbb{X}}$ we have

$$m(\mathbb{X}) = m(1_{\mathbb{X}}) = \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} 1\,\mathrm{d}t = 1\,.$$

Finally, for all $s \in G$ and $f \in C(\mathbb{X})$, we have

$$\begin{aligned}
|m(f) - \tau_s m(f)| &= |m(f) - m(\tau_{-s} f)| \\
&= \left| \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} f(\tau_t\mu)\,\mathrm{d}t - \frac{1}{|A_n|} \int_{A_n} f(\tau_{z+s}\mu)\,\mathrm{d}z \right| \\
&= \left| \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} f(\tau_t\mu)\,\mathrm{d}t - \frac{1}{|A_n|} \int_{-s+A_m} f(\tau_t\mu)\,\mathrm{d}t \right| \\
&= \left| \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n \Delta(-s+A_n)} f(\tau_t\mu)\,\mathrm{d}t \right| \leq \|f\|_\infty \lim_{n\to\infty} \frac{|A_n \Delta(-s+A_n)|}{|A_n|} = 0\,.
\end{aligned}$$

Therefore, m is G-invariant. This proves (a).

(b) For each $\varphi, \psi \in C_{\mathsf{c}}(G)$, we have

$$\begin{aligned}
(\gamma * \varphi * \widetilde{\psi})(0) = \langle f_\varphi, f_\psi \rangle = m(f_\varphi \overline{f_\psi}) &= \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} f_\varphi(\tau_s\mu)\, \overline{f_\psi}(\tau_s\mu)\,\mathrm{d}s \\
&= ((\mu * \varphi) \circledast_{\mathcal{A}} \widetilde{(\mu * \psi)})(0) = (\gamma_\mu * \varphi * \widetilde{\psi})(0)\,.
\end{aligned}$$

The claim follows immediately.

Next we show that m is ergodic. The proof below is similar to [33, Thm. 3.4].

Recall that, for all $k \in \mathbb{N}$ and all $\varphi_1, \ldots, \varphi_k \in C_{\mathsf{c}}(G)$, we have

$$\prod_{j=1}^{k} f_{\varphi_j}(\tau_t\mu) = \prod_{j=1}^{k} (\mu * \varphi_j)(t)\,.$$

Since $\mu \in \mathcal{B}\mathsf{ap}_{\mathcal{A}}^2(G)$, we get that $\prod_{j=1}^{k} (\mu * \varphi_j) \in Bap_{\mathcal{A}}^2(G)$.

Let $\mathbb{A}$ be the complex subalgebra of $C(\mathbb{X})$ generated by

$$\{1\} \cup \left\{ \prod_{j=1}^{k} f_{\varphi_j} : k \geq 1, \varphi_1, \ldots, \varphi_k \in C_{\mathsf{c}}(G) \right\}.$$

Then, by the above, for all $f \in \mathbb{A}$, the function $t \mapsto f(\tau_t \mu)$ belongs to $Bap^2_{\mathcal{A}}(G)$ and hence it has a well defined Fourier–Bohr coefficient.

Define $F_\chi : \mathbb{A} \to \mathbb{C}$ via

$$F_\chi(f) = a_\chi^{\mathcal{A}}(t \mapsto f(\tau_t \mu)) \,.$$

Next, by the Cauchy-Schwartz inequality, we have

$$\left| \frac{1}{|A_n|} \int_{A_n} \overline{\chi(t)}\, f(\tau_t \mu)\, \mathrm{d}t \right|^2 \leq \frac{1}{|A_n|} \int_{A_n} |f(\tau_t \mu)|^2 \, \mathrm{d}t \,.$$

Moreover, by (a), we have

$$\lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} |f(\tau_t \mu)|^2 \, \mathrm{d}t = \int_{\mathbb{X}} |f(\omega)|^2 \, \mathrm{d}m(\omega) \,. \tag{8}$$

Therefore, for all $f \in \mathbb{A}$, we have

$$|F_\chi(f)| \leq \|f\|_2 \,.$$

Since $\mathbb{A}$ is separating the points of $\mathbb{X}$, it is dense in $(C(X), \|\cdot\|_\infty)$ by Stone-Weierstraß' theorem. It follows that $\mathbb{A}$ is a dense subspace of $L^2(m)$. Therefore, F_χ can be extended to a continuous functional on the Hilbert space $L^2(m)$. By Riesz' lemma, there exists some element $f_\chi \in L^2(m)$ with $\|f_\chi\| \leq 1$ such that, for all $f \in L^2(m)$, we have

$$F_\chi(f) = \int_{\mathbb{X}} f(\omega)\, \overline{f_\chi(\omega)}\, \mathrm{d}m(\omega) \,. \tag{9}$$

Next, define $E := \{\chi \in \widehat{G} : f_\chi \neq 0\}$. By construction, $\chi \in E$ if and only if there exists some $f \in \mathbb{A}$ such that $a_\chi^{\mathcal{A}}(t \mapsto f(\tau_t \mu)) \neq 0$.

A short computation shows that, for all $f \in \mathbb{A}$, we have

$$\begin{aligned}
0 &= \int_{\mathbb{X}} f(\omega)\, \overline{f_\chi(\tau_t\omega)}\, \mathrm{d}m(\omega) - \int_{\mathbb{X}} f(\omega)\, \overline{f_\chi(\tau_t\omega)}\, \mathrm{d}m(\omega) \\
&= \int_{\mathbb{X}} f(\omega)\, \overline{f_\chi(\tau_t\omega)}\, \mathrm{d}m(\omega) - \int_{\mathbb{X}} f(\tau_{-t}\omega)\, \overline{f_\chi(\omega)}\, \mathrm{d}m(\omega) \\
&= \int_{\mathbb{X}} f(\omega)\, \overline{f_\chi(\tau_t\omega)}\, \mathrm{d}m(\omega) - F_\chi(\tau_t f) \\
&= \int_{\mathbb{X}} f(\omega)\, \overline{f_\chi(\tau_t\omega)}\, \mathrm{d}m(\omega) - \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} \tau_t f(\tau_s\omega)\, \overline{\chi(s)}\, \mathrm{d}s \\
&= \int_{\mathbb{X}} f(\omega)\, \overline{f_\chi(\tau_t\omega)}\, \mathrm{d}m(\omega) - \overline{\chi(t)} \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} f(\tau_{s-t}\omega)\, \overline{\chi(s-t)}\, \mathrm{d}s \\
&= \int_{\mathbb{X}} f(\omega)\, \overline{f_\chi(\tau_t\omega)}\, \mathrm{d}m(\omega) - \overline{\chi(t)} \lim_{n\to\infty} \frac{1}{|A_n|} \int_{t+A_n} f(\tau_r\omega)\, \overline{\chi(r)}\, \mathrm{d}r \\
&= \int_{\mathbb{X}} f(\omega)\, \overline{f_\chi(\tau_t\omega)}\, \mathrm{d}m(\omega) - \overline{\chi(t)} \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} f(\tau_r\omega)\, \overline{\chi(r)}\, \mathrm{d}r \\
&= \int_{\mathbb{X}} f(\omega) \left(\overline{f_\chi(\tau_t\omega)} - \overline{\chi(t)}\, \overline{f_\chi(\omega)} \right) \mathrm{d}m(\omega)
\end{aligned}$$

where the second last equality follows as f is bounded from the van Hove condition.

By the density of $\mathbb{A}$ in $L^2(m)$, we get that

$$\overline{f_\chi(\tau_t\omega)} = \overline{\chi(t) f_\chi(\omega)}\,.$$

It follows that for all $\chi \in E, f_\chi$ is an eigenfunction. Clearly, eigenfunctions to different eigenvalues are orthogonal.

Finally, for each $f \in \mathbb{A}$, Eq. (8) and Parseval's identity for the function $t \mapsto F(\tau_t\mu)$ give

$$\begin{aligned}\int_{\mathbb{X}} |f(\omega)|^2 \, \mathrm{d}m(\omega) &= \lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} |f(\tau_t\mu)|^2 \, \mathrm{d}t = \sum_{\chi\in\widehat{G}} \left| a_\chi^{\mathcal{A}}(t \mapsto f(\tau_t\mu)) \right|^2 \\ &= \sum_{\chi\in\widehat{G}} |F_\varphi(f)|^2 = \sum_{\chi\in\widehat{G}} \left|\langle f, f_\chi\rangle\right|^2 .\end{aligned}$$

Since the elements in $\{f_\chi : \chi \in E\}$ are orthogonal, $\|f_\chi\| \leq 1$ and $\mathbb{A}$ is dense in $L^2(m)$, it follows that $\|f_\chi\| = 1$ for all $\chi \in E$ and that $\{f_\chi : \chi \in E\}$ is an orthonormal basis in $L^2(m)$. By orthogonality, for each $\chi \in E$ the corresponding eigenspace is $\operatorname{span}(f_\chi)$.

It follows that each eigenspace is one dimensional. In particular, the eigenspace to the eigenvalue 1 is one dimensional and the system is ergodic.

(c) follows from (b) and the consistent phase property. □

As a consequence, we get the following theorem.

Theorem 6.14. *Let G be a second countable group and let ω be a positive pure point measure on $\widehat{G}$. Then, the following statements are equivalent.*

(i) *There is an ergodic TMDS $(\mathbb{X}, G, m)$ with autocorrelation γ such that $(\widehat{\gamma})_{pp} = \omega$.*
(ii) *There is an ergodic TMDS $(\mathbb{X}, G, m)$ with pure point spectrum and diffraction ω.*
(iii) *There is a van Hove sequence $\mathcal{A}$ and some $\mu \in \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G) \cap \mathcal{M}^\infty(G)$ such that ω is the diffraction of μ with respect to $\mathcal{A}$.*

Proof. (i) $\Longrightarrow$ (ii) follows from [2, Thm. 4.1].

(ii) $\Longrightarrow$ (iii) Since $(\mathbb{X}, G, m)$ has pure point spectrum, we have $m(\mathbb{X} \cap \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G)) = 1$ by Corollary 6.11.

Now, let $\mathcal{A}$ be a van Hove sequence along which Birkhoff's ergodic theorem holds. By [6, Thm. 5(b)], ω is almost surely the diffraction of $\nu \in \mathbb{X}$. In particular, there exists some $\mu \in \mathbb{X} \cap \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G)$ such that ω is the diffraction of μ.

(iii) $\Longrightarrow$ (i) This follows from Theorem 6.13. □

6.3. **Characterizing Weyl almost periodic measures via TMDS.** We showed in the last section that pure point spectrum for a TMDS can be characterized via mean and Besicovitch almost periodicity. Now, we show that for a TMDS $(\mathbb{X}, G)$ Weyl almost periodicity for one/all elements is equivalent to pure point dynamical spectrum, unique ergodicity and continuous eigenfunctions.

Theorem 6.15. *Let $\mu \in \mathcal{M}^\infty(G)$. Then, the following statements are equivalent:*

(i) $\mu \in \mathcal{W}\mathsf{ap}(G)$.
(ii) $\mathbb{X}(\mu) \subseteq \mathcal{W}\mathsf{ap}(G)$.

(iii) $\mathbb{X}(\mu)$ *is uniquely ergodic, has pure point dynamical spectrum and continuous eigenfunctions.*

Moreover, in this case, for each χ *with* $a_\chi(\mu) \neq 0$ *the function* $\mathbb{X}(\mu) \longrightarrow \mathbb{C}$, $\omega \mapsto a_\chi(\omega)$, *is a continuous eigenfunction for the system.*

Proof. We show (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i).

(ii) $\Longrightarrow$ (i): This trivially holds.

(iii) $\Longrightarrow$ (ii): This is similar to the proof of Theorem 6.10, with the difference being the usage of the unique ergodic theorem instead of the ergodic theorem:

Denote the set of eigenvalues by E. Then, E is a countable subgroup of $\widehat{G}$ by standard arguments. Choose a family $\{f_\chi\}_{\chi\in E}$ of eigenfunctions which are continuous, normalized and such that $f_1 = 1$.

Now, for each $\varphi \in C_{\mathsf{c}}(G)$ and each $\varepsilon > 0$, since $L^2(\mathbb{X}, m)$ has pure point spectrum, exactly as in Theorem. 6.10, there exists some $F = \sum_{k=1}^N c_k f_{\chi_k}$ such that

$$\int_{\mathbb{X}} |f_\varphi(\omega) - F(\omega)|^2 \, \mathrm{d}m(\omega) < \varepsilon^2 .$$

Next, fix some arbitrary $\nu \in \mathbb{X}(\mu)$. Since the eigenfunctions are continuous, so is $f_\varphi - F$. Therefore, by the unique ergodic theorem,

$$\begin{aligned}\int_{\mathbb{X}} |f_\varphi(\omega) - F(\omega)|^2 \, \mathrm{d}m(\omega) &= \lim_{n\to\infty} \frac{1}{|A_n|} \int_{x+A_n} |(\nu * \varphi)(s) - F(\tau_s \nu)|^2 \, \mathrm{d}s \\ &= \lim_{n\to\infty} \frac{1}{|A_n|} \int_{x+A_n} \left| (\nu * \varphi)(s) - \sum_{k=1}^N c'_k \chi_k(s) \right|^2 \, \mathrm{d}s\end{aligned}$$

uniformly in x, where $c'_k := c_k f_{\chi_k}(\nu)$. This shows that, for all $\varphi \in C_{\mathsf{c}}(G)$ and $\varepsilon > 0$, there exists a trigonometric polynomial $P = \sum_{k=1}^N c'_k \chi_k$ such that $\|\nu * \varphi - P\|_{w,2} < \varepsilon$. Therefore, $\nu * \varphi \in Wap^2(G)$ for all $\varphi \in C_{\mathsf{c}}(G)$.

(i) $\Longrightarrow$ (iii): (1) We start by proving the unique ergodicity.

Since $\mu \in \mathcal{W}\mathsf{ap}(G)$, for all $\varphi \in C_{\mathsf{c}}(G)$, we have $\mu * \varphi \in Wap(G) \cap C_{\mathsf{u}}(G)$, and hence $\overline{\mu * \varphi} \in Wap(G) \cap C_{\mathsf{u}}(G)$.

It follows that $A := \{\mu * \varphi, \overline{\mu * \varphi} : \varphi \in C_{\mathsf{c}}(G)\} \subseteq Wap(G) \cap C_{\mathsf{u}}(G)$. By Lemma 4.7 (c), we get that $\{\prod_{j=1}^n f_j : f_j \in A\} \subseteq Wap(G)$. In particular, any product of elements in A is amenable. Unique ergodicity then follows from Corollary 6.6.

(2) Next, let us prove that $\mathbb{X}(\mu)$ has pure point dynamical spectrum.

Let m be the unique ergodic measure, and let γ be the autocorrelation of the dynamical system. Then, by the unique ergodicity, γ is the autocorrelation of μ with respect to some van Hove sequence (A_n). Since μ is Weyl almost periodic, hence mean almost periodic, $\widehat{\gamma}$ is pure point by Theorem 2.13.

(3) Finally we prove the continuity of the eigenfunctions.

Let $\chi \in \widehat{G}$ be any element such that $a_\chi(\mu) \neq 0$. Since, by Lemma 4.14 the Fourier–Bohr coefficient $a_\chi(\mu)$ exists uniformly in translates, the corresponding eigenfunction can be chosen to be continuous by Theorem 6.7.

This shows that for each χ with $a_\chi(\mu) \neq 0$ we can choose a continuous eigenfunction. Since Fourier–Bohr coefficient $a_\chi(\mu)$ exists uniformly, we have $\widehat{\gamma}(\{\chi\}) = |a_\chi(\mu)|^2$. It follows that each χ in the Bragg spectrum has a continuous eigenfunction. Since the pure point dynamical spectrum is generated as a group by the Bragg spectrum, and since the product of continuous eigenfunctions is a continuous eigenfunction, the claim follows.

The last claim follows from Theorem 6.7. □

As an immediate consequence we get the following.

Corollary 6.16. *Let μ be a Weyl almost periodic measure. Then,*

(a) *For all $\omega \in \mathbb{X}(\mu)$ and all $\chi \in \widehat{G}$, we have $\widehat{\gamma}(\{\chi\}) = |a_\chi(\omega)|^2$.*
(b) *For each $\omega \in \mathbb{X}(\mu)$, the dynamical spectrum is the group generated by $\{\chi \in \widehat{G} : a_\chi(\omega) \neq 0\}$.*
(c) *For each $\chi \in \widehat{G}$ with $\widehat{\gamma}(\{\chi\}) \neq 0$, the function $\omega \mapsto a_\chi(\omega)$ is a continuous eigenfunction.*

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Appendix A. Some (counter)examples

In this section, we consider some examples showing strictness of certain inclusions. We also show that the space $\mathcal{M}\mathsf{ap}_{\mathcal{A}}(G)$ does not answer the Lagarias question 6 [27, Problem 4.6] The first example will be relevant in various parts of the article and for this reason we give it a name.

Definition A.1 (a-defect)**.** Let $a \in (0,1)$. We define the **a-defect of** $\mathbb{Z}$ by

$$\Lambda_a := \{-n : n \in \mathbb{N}\} \cup \{n + a : n \in \mathbb{N}\}$$

We can now prove that the a-defect of $\mathbb{Z}$ has the following properties.

Proposition A.2. (a) *For each $a \in (0,1)$ and each van Hove sequence $\mathcal{A}$, we have $\delta_{\Lambda_a} \in \mathcal{M}\mathsf{ap}_{\mathcal{A}}(G)$.*
(b) *For each $a \in (0,1)$ and $A_n = [-n, n]$, we have $\delta_{\Lambda_a} \notin \mathcal{B}\mathsf{ap}_{\mathcal{A}}(G)$. In particular, for all $1 \leq p < \infty$, we have $\delta_{\Lambda_a} \notin \mathcal{B}\mathsf{ap}^p_{\mathcal{A}}(G)$ and $\delta_{\Lambda_a} \notin \mathcal{W}\mathsf{ap}(G)$.*
(c) *For each $a \in (0,1), 1 \leq p < \infty$ and $A_n = [-n, n^2]$, we have $\delta_{\Lambda_a} \in \mathcal{B}\mathsf{ap}^p_{\mathcal{A}}(G)$.*
(d) *For each $a \in (0,1)$ and each van Hove sequence $\mathcal{A}$, the autocorrelation γ of Λ_a exists with respect to $\mathcal{A}$ and $\gamma = \delta_{\mathbb{Z}}$.*
(e) *For each $a \in (0,1)$ and each $b > 0$, the Fourier–Bohr coefficients of Λ_a exists with respect to $A_n = [-n, bn]$ and satisfy*

$$a_\lambda(\Lambda_a) = \begin{cases} \frac{1+be^{2\pi \mathrm{i} \lambda a}}{b+1}, & \text{if } \lambda \in \mathbb{Z}\,, \\ 0, & \text{if } \lambda \notin \mathbb{Z}\,. \end{cases}$$

(f) *If $a \in (0,1)\backslash\mathbb{Q}$, then for all $\lambda \in \mathbb{Z}\backslash\{0\}$ the Fourier–Bohr coefficients of δ_{Λ_a} don't exist with respect to the van Hove sequence $A_n = [-n, (2+(-1)^n)n]$.*

(g) *If $a \in (0,1)\backslash\mathbb{Q}$ and $A_n = [-n,n]$, then for all $\lambda \in \mathbb{Z}\backslash\{0\}$, the Fourier–Bohr coefficients of δ_{Λ_a} exist with respect to $\mathcal{A}$ and*

$$\widehat{\gamma}(\{\lambda\}) \neq \left|a_\lambda^{\mathcal{A}}(\delta_{\Lambda_a})\right|^2 .$$

(h) *$\mathbb{X}(\Lambda_a)$ is uniquely ergodic.*

(i) *If $a \in (0,1)\backslash\mathbb{Q}$, then the dynamical spectrum of $(\mathbb{X}(\Lambda_a), G)$ is $\mathbb{Z}$, while the topological dynamical spectrum is trivial.*

Proof. (a) If $n \in \mathbb{Z}$, then $\tau_n\delta_{\Lambda_a} - \delta_{\Lambda_a}$ is a measure with compact support. It follows that, for all $\varphi \in C_{\mathsf{c}}(G)$, we have $\tau_n\delta_{\Lambda_a} * \varphi - \delta_{\Lambda_a} * \varphi \in C_{\mathsf{c}}(G) \subseteq WAP_0(G)$. Therefore, for all van Hove sequences $\mathcal{A}$, we have

$$\overline{M}_{\mathcal{A}}(|\tau_n\delta_{\Lambda_a} * \varphi - \delta_{\Lambda_a} * \varphi|) = 0 .$$

The claim follows.

(b) Fix $0 < b < \min\{a, 1-a\}$. Pick some $\varphi \in C_{\mathsf{c}}(\mathbb{R})$ such that $\varphi \geq 1_{[-\frac{b}{8},\frac{b}{8}]}$ and $\operatorname{supp}(\varphi) \subseteq (-\frac{b}{4},\frac{b}{4})$. Let f be any Bohr almost periodic function.

Since $\delta_{\mathbb{Z}} * \varphi - f$ is Bohr almost periodic, using the independence of the mean with respect to van Hove sequences we get

$$\lim_{n\to\infty} \frac{1}{n}\int_{-n}^{0} |\delta_{\mathbb{Z}} * \varphi - f| \,\mathrm{d}t = \lim_{n\to\infty} \frac{1}{n}\int_{0}^{n} |(\delta_{a+\mathbb{Z}} * \varphi)(t) - f(t)| \,\mathrm{d}t = M(|\delta_{a+\mathbb{Z}} * \varphi - f|) .$$

Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{2n}&\left(\int_{0}^{n} |(\delta_{a+\mathbb{Z}} * \varphi)(t) - f(t)| \,\mathrm{d}t + \int_{-n}^{0} |(\delta_{\mathbb{Z}} * \varphi)(t) - f(t)| \,\mathrm{d}t\right) \\
&= \frac{1}{2}\left(M(|\delta_{\mathbb{Z}} * \varphi - f|) + M(|\delta_{a+\mathbb{Z}} * \varphi - f|)\right) \\
&\geq \frac{1}{2} M(|\delta_{\mathbb{Z}} * \varphi - \delta_{a+\mathbb{Z}} * \varphi|) .
\end{aligned}$$

Now, the choice of φ implies that

$$(\delta_{\Lambda_a} * \varphi)(x) = \begin{cases} (\delta_{a+\mathbb{Z}} * \varphi)(x), & \text{for all } x > 1 , \\ (\delta_{\mathbb{Z}} * \varphi)(x), & \text{for all } x < 1 . \end{cases}$$

This yields

$$\begin{aligned}
\overline{M}(|\delta_{\Lambda_a} * \varphi - f|) &= \lim_{n\to\infty} \frac{1}{2n}\left(\int_{0}^{n} |(\delta_{a+\mathbb{Z}} * \varphi)(t) - f(t)| \,\mathrm{d}t + \int_{-n}^{0} |(\delta_{\mathbb{Z}} * \varphi)(t) - f(t)| \,\mathrm{d}t\right) \\
&\geq \frac{1}{2} M(|\delta_{\mathbb{Z}} * \varphi - \delta_{a+\mathbb{Z}} * \varphi|) .
\end{aligned}$$

Finally, the choice of the support of φ implies that, for each $x \in \mathbb{R}$, at most one of $(\delta_{a+\mathbb{Z}} * \varphi)(x)$ and $(\delta_{\mathbb{Z}} * \varphi)(x)$ can be non-zero. Therefore,

$$|\delta_{\mathbb{Z}} * \varphi - \delta_{a+\mathbb{Z}} * \varphi| = |\delta_{\mathbb{Z}} * \varphi| + |\delta_{a+\mathbb{Z}} * \varphi| = \delta_{\mathbb{Z}} * \varphi + \delta_{a+\mathbb{Z}} * \varphi .$$

We thus get

$$\overline{M}(|\delta_{\Lambda_a} * \varphi - f|) \geq \frac{1}{2}\big(M(\delta_{\mathbb{Z}} * \varphi) + M(\delta_{a+\mathbb{Z}} * \varphi)\big) = \int_{\mathbb{R}} \varphi(t) \,\mathrm{d}t \geq \frac{b}{4} .$$

This shows that for all $f \in SAP(\mathbb{R})$, and in particular for all trigonometric polynomials, one has

$$\overline{M}(|\delta_{\Lambda_a} * \varphi - f|) \geq \frac{b}{4} .$$

It follows that $\delta_{\Lambda_a} * \varphi \notin Bap_{\mathcal{A}}(\mathbb{R})$ and hence $\delta_{\Lambda_a} \notin \mathcal{B}\mathsf{ap}_{\mathcal{A}}(\mathbb{R})$. Since $\mathcal{W}\mathsf{ap}^p(\mathbb{R}) \subset \mathcal{B}\mathsf{ap}^p_{\mathcal{A}}(\mathbb{R}) \subset \mathcal{B}\mathsf{ap}_{\mathcal{A}}(\mathbb{R})$, the claim follows.

(c) Let $\varphi \in C_{\mathsf{c}}(G)$, and let A be such that $\operatorname{supp}(\varphi) \subseteq [-A, A]$. Then, for all $x > A$, we have $\delta_{\Lambda_a} * \varphi = \delta_{a+\mathbb{Z}} * \varphi$. Therefore,

$$\begin{aligned}
\|\delta_{\Lambda_a} * \varphi - \delta_{a+\mathbb{Z}} * \varphi\|_{b,p,\mathcal{A}} &= \left(\limsup_{n\to\infty} \frac{1}{n^2+n} \int_{-n}^{n^2} |(\delta_{\Lambda_a} * \varphi)(t) - (\delta_{a+\mathbb{Z}} * \varphi)(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}} \\
&\leq \Bigg(\limsup_{n\to\infty} \frac{1}{n^2+n} \int_{-n}^{A} |(\delta_{\Lambda_a} * \varphi)(t) - (\delta_{a+\mathbb{Z}} * \varphi)(t)|^p \, \mathrm{d}t \\
&\qquad + \limsup_{n\to\infty} \frac{1}{n^2+n} \int_{A}^{n^2} |(\delta_{\Lambda_a} * \varphi)(t) - (\delta_{a+\mathbb{Z}} * \varphi)(t)|^p \, \mathrm{d}t \Bigg)^{\frac{1}{p}} \\
&= \left(\limsup_{n\to\infty} \frac{1}{n^2+n} \int_{-n}^{A} |(\delta_{\Lambda_a} * \varphi)(t) - (\delta_{a+\mathbb{Z}} * \varphi)(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}} \\
&\leq \left(\limsup_{n\to\infty} \frac{A+n}{n^2+n} \|(\delta_{\Lambda_a} - \delta_{a+\mathbb{Z}}) * \varphi\|_\infty^p \right)^{\frac{1}{p}} = 0 .
\end{aligned}$$

Since $\delta_{a+\mathbb{Z}} * \varphi \in SAP(G)$, the claim follows.

(d) First, we show that, for each van Hove sequence, we have

$$\lim_{n\to\infty} \frac{1}{|A_n|} \sharp(\Lambda_a \cap A_n) = 1 .$$

To do this, fix some function $\varphi \in C_{\mathsf{c}}(G)$ such that $\operatorname{supp}(\varphi) \in [0, 2]$ and $(\delta_{\mathbb{Z}} * \varphi)(x) = 1$ for all $x \in \mathbb{R}$. Then, a trivial computation shows that $(\delta_{\Lambda_a} * \varphi)(x) = 1$ for all $x \notin [-2, 3]$. Therefore, as $C_{\mathsf{c}}(G) \subset WAP_0(G)$, we have $1 - (\delta_{\Lambda_a} * \varphi)(x) \in WAP_0(G)$ and hence, we have

$$\lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} (\delta_{\Lambda_a} * \varphi)(x) \, \mathrm{d}x = 1 .$$

Now, by a standard Fubini and van Hove computation, we get

$$\lim_{n\to\infty} \frac{1}{|A_n|} \left(\int_{A_n} (\delta_{\Lambda_a} * \varphi)(x) \, \mathrm{d}x - \int_{\mathbb{R}} \varphi(t) \, \mathrm{d}t \, \sharp(\Lambda_a \cap A_n) \right) = 0 .$$

As

$$1 = M(1) = M(\delta_{\mathbb{Z}} * \varphi) = \operatorname{dens}(\mathbb{Z}) \int_{\mathbb{R}} \varphi(t) \, \mathrm{d}t$$

the claim follows.

Next, since Λ_a is a Meyer set, a standard argument (compare [7]) shows that its autocorrelation exists with respect to $\mathcal{A}$ if and only if the limit

$$\eta(z) = \lim_{n} \frac{1}{|A_n|} \sharp(\Lambda_a \cap (z + \Lambda_a) \cap A_n)$$

exists for all $z \in \mathbb{Z}$. Moreover, in this case we have $\gamma = \sum_{z\in\mathbb{R}} \eta(z)\delta_z$ (compare [7]).

Now, it is easy to see that for all $z \in \mathbb{Z}$ the sets Λ_a and $z+\Lambda_a$ agree outside the compact set $[-|z|, |z|+1]$. Therefore,

$$\eta(z) = \lim_{n\to\infty} \frac{1}{|A_n|} \sharp(\Lambda_a \cap (z+\Lambda_a) \cap A_n) = \lim_{n\to\infty} \frac{1}{|A_n|} \sharp(\Lambda_a \cap A_n) = 1$$

for all $z \in \mathbb{Z}$. Also, for all $z \notin \mathbb{Z}$, it is easy to see that $\Lambda_a \cap (z+\Lambda_a)$ is a finite set and hence

$$\eta(z) = \lim_{n\to\infty} \frac{1}{|A_n|} \sharp(\Lambda_a \cap (z+\Lambda_a) \cap A_n) = 0$$

for all $z \notin \mathbb{Z}$. The claim now follows.

(e) We compute

$$\begin{aligned}\frac{1}{bn+n} \int_{-n}^{bn} e^{-2\pi \mathrm{i} \lambda t} \, \mathrm{d}\delta_{\Lambda_a}(t) &= \frac{1}{b+1} \frac{1}{n} \int_{-n}^{0} e^{-2\pi \mathrm{i} \lambda t} \, \mathrm{d}\delta_{\Lambda_a}(t) + \frac{b}{b+1} \frac{1}{bn} \int_{0}^{bn} e^{-2\pi \mathrm{i} \lambda t} \, \mathrm{d}\delta_{\Lambda_a}(t) \\ &= \frac{1}{b+1} \frac{1}{n} \int_{-n}^{0} e^{-2\pi \mathrm{i} \lambda t} \, \mathrm{d}\delta_{\mathbb{Z}}(t) + \frac{b}{b+1} \frac{1}{bn} \int_{0}^{bn} e^{-2\pi \mathrm{i} \lambda t} \, \mathrm{d}\delta_{a+\delta_{\mathbb{Z}}}(t) \,.\end{aligned}$$

Now, since $\delta_{\mathbb{Z}}, \delta_{a+\mathbb{Z}}$ are weakly almost periodic measures their Fourier–Bohr coefficients exists with respect to any van Hove sequence, and they are independent of the choice of the van Hove sequence [35]. Therefore,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{bn+n} \int_{-n}^{bn} e^{-2\pi \mathrm{i} \lambda t} \, \mathrm{d}\delta_{\Lambda_a}(t) &= \frac{1}{b+1} a_\lambda(\delta_{\mathbb{Z}}) + \frac{b}{b+1} a_\lambda(\delta_{a+\mathbb{Z}}) \\ &= \left(\frac{1}{b+1} + e^{2\pi \mathrm{i} \lambda a} \frac{b}{b+1} \right) a_\lambda(\delta_{\mathbb{Z}}) \,.\end{aligned}$$

Since $\widehat{\delta_{\mathbb{Z}}} = \delta_{\mathbb{Z}}$, the claim follows.

(f) By (d), we have

$$\lim_{n\to\infty} \frac{1}{|A_{2n}|} \int_{A_{2n}} e^{-2\pi \mathrm{i} \lambda t} \, \mathrm{d}\delta_{\Lambda_a}(t) = \frac{1+3e^{2\pi \mathrm{i} \lambda a}}{4}$$

and

$$\lim_{n\to\infty} \frac{1}{|A_{2n+1}|} \int_{A_{2n+1}} e^{-2\pi \mathrm{i} \lambda t} \, \mathrm{d}\delta_{\Lambda_a}(t) = \frac{1+e^{2\pi \mathrm{i} \lambda a}}{2} \,.$$

Now,

$$\frac{1+3e^{2\pi \mathrm{i} \lambda a}}{4} = \frac{1+e^{2\pi \mathrm{i} \lambda a}}{2} \iff e^{2\pi \mathrm{i} \lambda a} = 1 \iff \lambda a \in \mathbb{Z} \,.$$

Since $a \notin \mathbb{Q}$, we have $\frac{1+3e^{2\pi \mathrm{i} \lambda a}}{4} \neq \frac{1+e^{2\pi i \lambda a}}{2}$ for all $\lambda \in \mathbb{Z}$ and hence

$$\lim_{n\to\infty} \frac{1}{|A_{2n}|} \int_{A_{2n}} e^{-2\pi \mathrm{i} \lambda t} \, \mathrm{d}\delta_{\Lambda_a}(t) \neq \lim_{n\to\infty} \frac{1}{|A_{2n+1}|} \int_{A_{2n+1}} e^{-2\pi \mathrm{i} \lambda t} \, \mathrm{d}\delta_{\Lambda_a}(t) \,,$$

showing that $\left(\frac{1}{|A_n|} \int_{A_n} e^{-2\pi \mathrm{i} \lambda t} \, \mathrm{d}\delta_{\Lambda_a}(t) \right)$ is not convergent.

(g) By (d), we have for all $\lambda \in \mathbb{Z}$,

$$\lim_{n\to\infty} \frac{1}{|A_n|} \int_{A_n} e^{-2\pi \mathrm{i} \lambda t} \, \mathrm{d}\delta_{\Lambda_a}(t) = \frac{1+e^{2\pi \mathrm{i} \lambda a}}{2} \,.$$

Hence, the Fourier–Bohr coefficients exist. Also, we have $|\frac{1+e^{2\pi i\lambda a}}{2}|^2 = 1$ if and only if $e^{2\pi i\lambda a} = 1$, which again by the irrationality of a implies that

$$\widehat{\gamma}(\{\lambda\}) \neq \left|a_\lambda^{\mathcal{A}}(\delta_{\Lambda_a})\right|^2 \qquad \text{for all } \lambda \in \mathbb{Z}\backslash\{0\}\,.$$

(h) Note that $\mathbb{X}(\mu) = \{\tau_t\mu : t \in \mathbb{R}\} \sqcup \{\delta_{t+\mathbb{Z}} : t \in \mathbb{R}/\mathbb{Z}\}$. Set $\mathbb{T} := \{\delta_{t+\mathbb{Z}} : t \in \mathbb{R}/\mathbb{Z}\}$. This is a compact Abelian group, and the action of $\mathbb{R}$ is simply addition modulo 1: $\tau_s\delta_{t+\mathbb{Z}} = \delta_{t+s+\mathbb{Z}}$. Also, set $\Omega := \{\tau_t\mu : t \in \mathbb{R}\}$.

We show that any ergodic measure is equal to the probability Haar measure on $\mathbb{T}$. This proves unique ergodicity.

Let m be a $\mathbb{R}$-invariant ergodic measure on $\mathbb{X}(\mu)$. Next, define $\varphi : C_{\mathsf{c}}(\mathbb{R}) \to C(\mathbb{X}(\mu))$ via $\varphi(f)(\tau_t\mu) = f(t)$ and $\varphi(f)(\delta_{t+\mathbb{Z}}) = 0$ for all $t \in \mathbb{R}$. It is trivial to see that $\varphi(f)$ is indeed continuous. Define,

$$\eta(f) = m(\varphi(f)) \qquad \text{for all } f \in C_{\mathsf{c}}(\mathbb{R})\,.$$

It is easy to see that η is linear, and for all $f \in C_{\mathsf{c}}(\mathbb{R})$ we have

$$|\eta(f)| = |m(\varphi(f))| \leq \|\varphi(f)\|_\infty = \|f\|_\infty\,.$$

Therefore, by Riesz' representation theorem, η is a finite measure on $\mathbb{R}$. Also, it is easy to see from the definition that for all $t \in \mathbb{R}$ and all $f \in C_{\mathsf{c}}(\mathbb{R})$ we have $\varphi(\tau_t f) = \tau_t\varphi(f)$. Therefore, since m is $\mathbb{R}$-invariant, so is η.

This implies that η is a finite Haar measure on $\mathbb{R}$ and hence $\eta = 0$.

Next, for each $n \in \mathbb{N}$ pick some $f_n \in C_{\mathsf{c}}(\mathbb{R})$ such that $1_{[-n,n]} \leq f_n \leq 1_{[-n-1,n+1]}$, and let $\psi_n := \varphi(f_n)$. Then, (ψ_n) is an increasing sequence of functions in $C(\mathbb{X}(\mu))$ which converges pointwise to the characteristic function of Ω.

Let $g \in C(\mathbb{X})$ and define $h_n : \mathbb{R} \to \mathbb{C}$ via $h_n(t) = \psi_n(\tau_t\mu)g(\tau_t\mu)$. Then, $h_n \in C_{\mathsf{c}}(\mathbb{R})$ and $\varphi(h_n) = \psi_n g$. Now, the monotone convergence theorem implies

$$\begin{aligned}
\int_{\mathbb{X}(\mu)} g(\omega)\,\mathrm{d}m(\omega) &= \int_{\mathbb{T}} g(\omega)\,\mathrm{d}m(\omega) + \int_{\Omega} g(\omega)\,\mathrm{d}m(\omega) \\
&= \int_{\mathbb{T}} g(\omega)\,\mathrm{d}m(\omega) + \lim_{n\to\infty}\int_{\Omega} \psi_n g(\omega)\,\mathrm{d}m(\omega) \\
&= \int_{\mathbb{T}} g(\omega)\,\mathrm{d}m(\omega) + \lim_{n\to\infty}\int_{\Omega} \varphi(h_n)(\omega)\,\mathrm{d}m(\omega) \\
&= \int_{\mathbb{T}} g(\omega)\,\mathrm{d}m(\omega) + \lim_{n\to\infty} m(\varphi(h_n)) \\
&= \int_{\mathbb{T}} g(\omega)\,\mathrm{d}m(\omega) + \lim_{n\to\infty} \eta(h_n) = \int_{\mathbb{T}} g(\omega)\,\mathrm{d}m(\omega)\,.
\end{aligned}$$

This implies that m is supported on $\mathbb{T}$, and hence it is an $\mathbb{R}$-invariant probability measure on $\mathbb{T}$. Thus, m is the probability Haar measure on $\mathbb{T}$.

(i) The first part follows from (c).

Now, we show that if $f_\lambda \not\equiv 0$ is a continuous eigenfunction, then $\lambda = 0$. We know that the measurable spectrum is $\mathbb{Z}$, so $\lambda \in \mathbb{Z}$. Let f_λ be a continuous eigenfunction, and let

$c = f_\lambda(\delta_{\Lambda_a})$. Now, for $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \tau_n \delta_{\Lambda_a} = \delta_{a+\mathbb{Z}}$$

in the local topology, and hence, since f_λ is continuous,

$$e^{2\pi \mathrm{i} \lambda a} f_\lambda(\delta_{\mathbb{Z}}) = f_\lambda(\delta_{a+\mathbb{Z}}) = \lim_{n\to\infty} f_\lambda(\tau_n \delta_{\Lambda_a}) = \lim_{n\to\infty} e^{2\pi \mathrm{i} \lambda n} f_\lambda(\delta_{\Lambda_a}) = c\,.$$

In the same way, for $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \tau_{-n} \delta_{\Lambda_a} = \delta_{\mathbb{Z}}\,,$$

and hence, since f_λ is continuous,

$$f_\lambda(\delta_{\mathbb{Z}}) = \lim_{n\to\infty} f_\lambda(\tau_{-n} \delta_{\Lambda_a}) = c\,.$$

This yields

$$c = e^{2\pi \mathrm{i} \lambda a} f_\lambda(\delta_{\mathbb{Z}}) = e^{2\pi \mathrm{i} \lambda a} c\,,$$

and hence $c = 0$ or $e^{2\pi \mathrm{i} \lambda a} = 1$. In the first case, we get $f_\lambda \equiv 0$, which is not possible, while the second case gives $\lambda = 0$. $\square$

Remark A.3. (a) In Proposition A.2, (d) also follows from (h).
(b) In Proposition A.2, (h) can alternately be proved by Corollary 6.6.

We next discuss an example of a mean almost periodic measure with respect to some van Hove sequence $\mathcal{A}$ such that the autocorrelation does not exists with respect to $\mathcal{A}$. Since the computations are straightforward and similar to the ones done for the proof of Proposition A.2, we skip them.

Example A.4. Let $\Lambda := \{n, -2n : n \in \mathbb{N}\}$. Then Λ is mean almost periodic with respect to $A_n = [-n, (2 + (-1)^n)n]$, but its autocorrelation does not exist with respect to this van Hove sequence.

Example A.5. Let $\mu = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \delta_{2^n+k}$. Then, μ is Besicovitch almost periodic with respect to $A_n = [-n, n]$, but not Weyl almost periodic. Indeed, let $\varphi \in C_{\mathsf{c}}(\mathbb{R})$ be arbitrary. We show that $\overline{M}(|\mu * \varphi|) = 0$. This yields Besicovitch almost periodicity. Let A be such that $\operatorname{supp}(\varphi) \subseteq [-A, A]$. Since $\mu([0, 2^m]) = 1 + 2 + \ldots + m - 1 = \frac{m(m-1)}{2}$, a simple computation yields

$$\frac{1}{2n} \int_{[-n,n]} |(\mu * \varphi)(t)| \, \mathrm{d}t \le \|\varphi\|_1 \frac{(\log_2(n + A))^2}{2n}\,.$$

From here, Besicovitch almost periodicity follows. One can show via a similar approximation that μ is not Weyl almost periodic. This can also be seen via Theorem 6.15: since $\mathbb{X}(\mu)$ is not uniquely ergodic, μ cannot be Weyl almost periodic.

Appendix B. Cut and project schemes

In this Appendix, we give a brief review of cut and project schemes (CPS). For a detailed review of this, we recommend [40, 3, 54, 32, 46, 47].

A triple $(G, H, \mathcal{L})$ is called a cut and project scheme (CPS) if G and H are LCA groups and $\mathcal{L}$ is a **lattice** in $G \times H$ (i.e., a cocompact discrete subgroup) such that the restriction of canonical projection $\pi^G : G \times H \longrightarrow G$ to $\mathcal{L}$ is one to one, and $\pi^H(\mathcal{L})$ is dense in H.

Let $L := \pi^G(\mathcal{L})$. We can then define the star mapping $(\cdot)^\star \colon L \longrightarrow H$ as follows: If $x \in L$, then $x^\star$ is a unique $y \in H$ such that $(x, y) \in \mathcal{L}$. Under this mapping we have

$$\mathcal{L} = \{(x, x^\star) : x \in L\} .$$

$$\begin{tikzcd}
G & G \times H \arrow[l, "\pi^G"'] \arrow[r, "\pi^H"] & H \\
L \arrow[u, hook] \arrow[rr, bend right, "\star"'] & \mathcal{L} \arrow[u, hook] \arrow[l, "1-1"'] \arrow[r, "\pi^H"] & L^\star \arrow[u, hook, "\text{dense}"']
\end{tikzcd}$$

Given a cut and project scheme, we can associate to any $W \subset H$, called the *window*, the set

$$\curlywedge(W) := \{x \in L : x^\star \in W\} .$$

If W is relatively compact, then $\curlywedge(W)$ is called a **weak model set**. Any weak model set is uniformly discrete. If in addition $W^\circ \neq \varnothing$, then $\curlywedge(W)$ is called a **model set**. Any weak model set is uniformly discrete and any model set is a Delone set.

If, in addition, the model set $\curlywedge(W)$ satisfies $|\partial W| = 0$, it is called a **regular model set**.

Given a CPS $(G, H, \mathcal{L})$, for each function $h : H \to \mathbb{C}$, we can define a formal sum via

$$\omega_h := \sum_{x \in L} h(x^\star) \delta_x .$$

If h is compactly supported and bounded, then ω_h is a measure. The same holds under various decaying conditions of h [46, 32, 47, 54, 55]. Also, if $h = 1_W$ is the characteristic function of a window, we have $\omega_h = \delta_{\curlywedge(W)}$.

Given a CPS $(G, H, \mathcal{L})$, we can define a new CPS $(\widehat{G}, \widehat{H}, \mathcal{L}^0)$, called the **dual lattice** where $\mathcal{L}^0$ is the annihilator, or the dual lattice, of $\mathcal{L}$ in $\widehat{G} \times \widehat{H} \simeq \widehat{G \times H}$. For details that this is a CPS see [40, 41].

We now list some of the essential properties of such combs [46, 32, 47, 3, 4, 54, 55].

Theorem B.1. [47] *Let $(G, H, \mathcal{L})$ be a CPS and $h \in C_{\mathsf{c}}(H)$. Then,*

(a) *$\omega_h \in \mathcal{SAP}(G)$ and $M(\omega_h) = \mathrm{dens}(\mathcal{L}) \int_H h(t)\, dt$.*

(b) *ω_h is Fourier transformable if and only if $\widehat{h} \in L^1(\widehat{H})$. Moreover, in this case $\omega_{\check{h}}$ is a measure in the dual CPS and*

$$\widehat{\omega_h} = \mathrm{dens}(\mathcal{L}) \omega_{\check{h}} .$$

Next, we introduce the concept of weak model sets of maximal density. First, let us recall the following result.

Proposition B.2. [23, 54] *Let $(G, H, \mathcal{L})$ be a CPS, and $W \subset H$ be a pre-compact set. Then, for each van Hove sequence $\mathcal{A}$, we have*

$$\operatorname{dens}(\mathcal{L})\,|W^\circ| \;\leq\; \liminf_{m\to\infty} \frac{\delta_{\curlywedge(W)}(A_m)}{|A_m|} \;\leq\; \limsup_{m\to\infty} \frac{\delta_{\curlywedge(W)}(A_m)}{|A_m|} \;\leq\; \operatorname{dens}(\mathcal{L})\,|\overline{W}|\,.$$

We can now introduce the following definition.

Definition B.3. [5, 26] Given a CPS $(G, H, \mathcal{L})$, a van Hove sequence $\mathcal{A}$ and a compact set $W \subset H$, we say that the weak model set $\curlywedge(W)$ has **maximal density with respect to $\mathcal{A}$** if

$$\lim_{m\to\infty} \frac{\delta_{\curlywedge(W)}(A_m)}{|A_m|} \;=\; \operatorname{dens}(\mathcal{L})\,|W|\,.$$

Appendix C. Semi-measures and their Fourier transform

In this section, we collect the basic results we need about semi-measures (see Def 1.27). Let us stat with the following consequence of the definition.

Lemma C.1. *Let ϑ be a Fourier transformable semi-measure. Then,*

(a) *For all $\psi \in K_2(G)$, we have $\check{\psi} \in L^1(|\widehat{\vartheta}|)$ and*

$$\vartheta(\psi) = \widehat{\vartheta}(\check{\psi})\,.$$

(b) *For all $\psi \in K_2(G)$, we have*

$$(\vartheta * \psi)(t) = \int_{\widehat{G}} \chi(t)\,\widehat{\psi}(\chi)\,d\widehat{\vartheta}(\chi) = \widecheck{\widehat{\psi}\widehat{\vartheta}}(t)\,.$$

Proof. (a) By the polarisation identity [43, p. 244], we get the claim for $\psi = \varphi * \phi$ with $\varphi, \phi \in C_{\mathsf{c}}(G)$. (a) follows now by linearity.

(b) Since $K_2(G)$ is closed under reflection and translation, we get

$$(\vartheta * \psi)(t) = \vartheta(\tau_t\varphi^\dagger) = \widehat{\vartheta}(\widecheck{\tau_t\varphi^\dagger}) = \widecheck{\widehat{\psi}\widehat{\vartheta}}(t)\,.$$

□

Next, let us recall the following definition [56].

Definition C.2. A measure μ is called **weakly admissible**, if for all $\varphi \in K_2(\widehat{G})$, we have $\widehat{\varphi} \in L^1(|\mu|)$.

We start with the following result, which emulates the standard proof that positive definite measures are Fourier transformable [10, Thm. 4.5], [43, Thm. 4.11.5].

Lemma C.3. *Let $\{\sigma_\varphi\}_{\varphi\in C_{\mathsf{c}}(G)}$ be a family of finite measures on $\widehat{G}$ which satisfy the compatibility condition*

$$|\widehat{\varphi}|^2\,\sigma_\psi = \big|\widehat{\psi}\big|^2\sigma_\varphi \qquad \text{for all } \varphi, \psi \in C_{\mathsf{c}}(G)\,. \tag{10}$$

Then, there exists a weakly admissible measure σ on $\widehat{G}$ such that, for all $\varphi \in C_{\mathsf{c}}(G)$, we have

$$\sigma_\varphi = |\widehat{\varphi}|^2\,\sigma\,.$$

Proof. We follow closely the proof of [10, Thm. 4.5]. For each $f \in C_{\mathsf{c}}(\widehat{G})$, we define

$$\sigma(f) = \sigma_{\varphi}\Big(\frac{f}{|\widehat{\varphi}|^2}\Big)\,,$$

where $\varphi \in C_{\mathsf{c}}(G)$ is any function such that $\widehat{\varphi}$ is not vanishing on $\operatorname{supp}(f)$. Such a function always exists by [10, Prop. 2.4], [43, Cor. 4.9.12]. The compatibly condition Eq. (10) ensures that our definition doesn't depend on the choice of φ. It is easy to see that $\sigma : C_{\mathsf{c}}(G) \to \mathbb{C}$ is linear.

We show next that σ is continuous with respect to the inductive topology. To do this, fix some compact set K. Fix some $\varphi \in C_{\mathsf{c}}(G)$ such that $\widehat{\varphi} \geq 1_K$. Such a function exists again by [10, Prop. 2.4], [43, Cor. 4.9.12]. Then, for all $f \in C_{\mathsf{c}}(\widehat{G})$ with $\operatorname{supp}(f) \subset K$, we have $\left|\frac{f}{|\widehat{\varphi}|^2}\right| \leq \|f\|_\infty 1_K$ and hence

$$|\sigma(f)| \leq |\sigma_{\varphi}|\,(K) \cdot \|f\,\|_\infty\,.$$

Since σ_φ is a (finite) measure, the claim follows.

Next, we show that $\sigma_\varphi = |\widehat{\varphi}|^2\,\sigma$ for all $\varphi \in C_{\mathsf{c}}(G)$.

Let $\varphi \in C_{\mathsf{c}}(G)$ be arbitrary. Pick some $f \in C_{\mathsf{c}}(\widehat{G})$, and choose some $\psi \in C_{\mathsf{c}}(G)$, such that $\widehat{\psi}$ is not vanishing on $\operatorname{supp}(f)$. Then,

$$(|\widehat{\varphi}|^2\,\sigma)(f) = \sigma(|\widehat{\varphi}|^2\,f) = \sigma_\psi\left(\frac{|\widehat{\varphi}|^2\,f}{|\widehat{\psi}|^2}\right) = \big(|\widehat{\varphi}|^2\,\sigma_\psi\big)\left(\frac{f}{|\widehat{\psi}|^2}\right) = \big(|\widehat{\psi}|^2\sigma_\varphi\big)\left(\frac{f}{|\widehat{\psi}|^2}\right) = \sigma_\varphi(f)\,.$$

This shows that

$$\sigma_\varphi = |\widehat{\varphi}|^2\,\sigma\,.$$

Finally, since σ_φ is finite, so is $|\widehat{\varphi}|^2\,\sigma$, which gives the weak admissibility of σ. □

We can now prove the following simple result.

Proposition C.4. *Let μ be a measure on $\widehat{G}$. Then, there exists a semi-measure ϑ on G such that $\widehat{\vartheta} = \mu$ if and only if μ is weakly admissible.*

Proof. $\Longrightarrow$ follows from the definition of the Fourier transformability.

$\Longleftarrow$ Since μ is weakly admissible, we have $|\check{\psi}| \in L^1(|\mu|)$ for all $\psi \in K_2(G)$.

Then, we can define a semi-measure ϑ via

$$\vartheta(\psi) := \mu(\check{\psi}) \qquad \text{for all } \psi \in K_2(G)\,.$$

□

We can now give an example of a semi-measure which is not a measure.

Example C.5. On $G = \mathbb{R}$, the Lebesgue measure is weakly admissible, and hence, so is its restriction to $[0, \infty)$ [56, Lem. 3.2(ii)]. Therefore, by Proposition C.4,

$$\vartheta(f) := \int_0^\infty \check{f}(s)\,\mathrm{d}s\,, \quad f \in K_2(G)\,, \tag{11}$$

is well defined and a semi-measure on $\mathbb{R}$.

However, ϑ is not a measure. Assume by contradiction that it is. Then, by Eq. (11), ϑ is Fourier transformable as a measure and its Fourier transform as a measure is $\nu := \lambda|_{[0,\infty)]}$.

Then, by [17, Thm. 11.1], $\nu \in \mathcal{WAP}(\mathbb{R})$ and hence ν has a mean which does not depend on the choice of the van Hove sequence, which is a contradiction.

We introduce the concept of positive definiteness for a semi-measure, similar to a measure. We then show that positive definiteness implies Fourier transformability.

Definition C.6. A semi-measure ϑ is called **positive definite**, if for all $\varphi \in C_{\mathsf{c}}(G)$, we have $\vartheta(\varphi * \tilde{\varphi}) \geq 0$.

Remark C.7. Similarly to [10, Thm. 4.5], [43, Thm. 4.11.5] one can prove that a semi-measure ϑ is Fourier transformable with positive Fourier transform if and only if ϑ is positive definite and, for all $\varphi \in K_2(G)$, the function $\vartheta * \varphi$ is continuous at $t = 0$.

We complete Appendix C by discussing when a semi-measure is a measure.

Lemma C.8. *Let ϑ be a semi-measure. Then, ϑ is a measure if and only if, for all $K \subset G$, there exists a constant $C_K > 0$ such that, for all $\psi \in K_2(G)$ with supp$(\psi) \subseteq K$, we have*

$$|\vartheta(\psi)| \leq C_K \, \|\psi\|_\infty \, .$$

Proof. $\Longrightarrow$ follows from the definition of measures.

$\Longleftarrow$ Fix some $K \subseteq G$. Then, the set $C(G:K) := \{\varphi \in C_{\mathsf{c}}(G) : \operatorname{supp}(\varphi) \subset K\}$ is a Banach space with respect to $\| \cdot \|_\infty$. Now, the given relation says that ϑ is bounded on the dense subspace $C(G:K) \cap K_2(G)$ and hence, has a unique extension to a continuous mapping $\mu_K : C(G:K) \to \mathbb{C}$.

Now, if K', K'' are arbitrary compacts with non-empty intersection $K = K' \cap K''$, it is easy to see that $\mu_{K'}|_{C(G:K)} = \mu_{K''}|_{C(G:K)}$. Therefore, we can define $\mu : C_{\mathsf{c}}(G) \to \mathbb{C}$ via

$$\mu(\varphi) = \mu_K(\varphi) \, ,$$

where K is any compact set containing the $\operatorname{supp}(\varphi)$. It is easy to see that μ is a measure. $\square$

Appendix D. Averaging along arbitrary van Hove sequences

We denote the open ball around $z \in \mathbb{C}$ with radius $r > 0$ by $U_r(z)$.

Proposition D.1. *Let $h : G \longrightarrow \mathbb{C}$ be a bounded measurable function. Let A be an open relatively compact subset of G and assume that there exist $r > 0$ and $z \in \mathbb{C}$ with*

$$\frac{1}{|A|} \int_{A+s} h(t) \, dt \in U_r(z)$$

for all $s \in G$. Then, for any van Hove sequence $\mathcal{B}$ and any $R > r$, there exists a natural number N with

$$\frac{1}{|B_n|} \int_{B_n+v} h(t) \, dt \in U_R(z)$$

for all $v \in G$ and $n \geq N$.

Proof. A short computation shows

$$\left|\int_{B_n+v} h(t+u)\mathrm{d}t - \int_{B_n+v} h(t)\,\mathrm{d}t\right| \leq \|h\|_\infty\, |\partial^{A\cup(-A)} B_n|$$

for all $u \in A$ and $v \in G$. In particular, we have

$$\left|\frac{1}{|A|}\int_A \left(\int_{B_n+v} h(t+u)\,\mathrm{d}t\right)\mathrm{d}u - \int_{B_n+v} h(t)\,\mathrm{d}t\right| \leq \|h\|_\infty\, |\partial^{A\cup(-A)} B_n|$$

for all $v \in G$. Now, from the assumption we find

$$\frac{1}{|A|}\int_A \left(\frac{1}{|B_n|}\int_{B_n+v} h(t+u)\,\mathrm{d}t\right)\mathrm{d}u = \frac{1}{|B_n|}\int_{B_n+v}\left(\frac{1}{|A|}\int_A h(t+u)\,\mathrm{d}u\right)\mathrm{d}t \in U_r(z)$$

for all $v \in G$ and $n \in \mathbb{N}$. Taking these statements together we infer that

$$\frac{1}{|B_n|}\int_{B_n+v} h(t)\,\mathrm{d}t \in U_{r+\delta(n)}(z)$$

with

$$\delta(n) = \frac{1}{|B_n|}\, \|h\|_\infty\, |\partial^{A\cup(-A)} B_n|$$

for all $v \in G$ and $n \in \mathbb{N}$. This easily gives the statement. $\square$

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