Novikov Groups are Right-Orderable

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1 Introduction

Novikov groups were introduced in [8] as examples of finitely presented groups with unsolvable conjugacy problem. It was Bokut [2] who showed that each Novikov group has a standard basis and thus a solvable word problem. Further, he showed [3] that for every recursively enumerable degree of unsolvability d there is a Novikov group whose conjugacy problem is of degree d. In the present work, we show that Novikov groups are also right-orderable, thus exhibiting the first known examples of finitely presented right-orderable groups with solvable word problem and unsolvable conjugacy problem. We further remark that this may provide a means for producing a lattice-ordered group with the same properties, which would answer one of the foremost open problems in the theory of lattice-ordered groups.

2 Novikov Groups

It is known that for any recursively enumerable degree of unsolvability there is a finitely presented semigroup of the form $S = \langle a_j \mid A_i = B_i \ 1 \leq i \leq$

$\lambda,\ 1 \leq j \leq n\rangle$ whose word problem is of that degree. It was shown by Bokut that the degree of unsolvability of the word problem for S is equal to the degree of unsolvability of the conjugacy problem for the Novikov group $A_{p_1p_2}$ defined below using the relations of S.

The group $A_{p_1p_2}$ is most easily dealt with when defined via an ascending sequence of four groups, each an H.N.N. extension of the previous,

- $G_0 = \langle \Sigma_0 \mid \Phi_0 \rangle$
- $G_1 = \langle \Sigma_0 \cup \Sigma_1 \mid \Phi_0 \cup \Phi_1 \rangle$
- $G_2 = \langle \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \mid \Phi_0 \cup \Phi_1 \cup \Phi_2 \rangle$
- $A_{p_1p_2} = G_3 = \langle \cup_{i=0}^{3} \Sigma_i \mid \cup_{i=0}^{3} \Phi_i \rangle$

Where

- $\Sigma_0 = \{q_i, t_i, q_i^+, t_i^+, 1 \leq i \leq \lambda\}$
- $\Sigma_1 = \{a_j, a_j^+, 1 \leq j \leq n\}$
- $\Sigma_2 = \{l_i, l_i^+, 1 \leq i \leq \lambda\}$
- $\Sigma_3 = \{p_1, p_2\}$
- $\Phi_0 = \emptyset$
- $\Phi_1 = \{q_i a_j = a_j q_i^2,\ t_i^2 a_j = a_j t_i,\ a_j^+ q_i^+ = (q_i^+)^2 a_j^+,\ t_i^+ a_j^+ = a_j^+ (t_i^+)^2\}$
- $\Phi_2 = \{l_i a_j = a_j l_i,\ l_i^+ a_j^+ = a_j^+ l_i^+\}$
- $\Phi_3 = \{(A_i^+)^{-1} q_i^+ l_i^+ p_1 = p_1 A_i q_i^{-1} l_i^{-1},\ (t_i^+)^{-1} p_1 = p_1 t_i,\ B_i^{-1} t_i l_i p_2 = p_2 B_i^+ (t_i^+)^{-1} (l_i^+)^{-1},\ q_i^{-1} p_2 = p_2 q_i^+\}$

with A_i's and B_i's the distinct words in $\langle a_j, 1 \leq j \leq n\rangle$ such that $A_i = B_i$ are the defining relations of the semigroup S.

It bares noting that each G_i is the free product of two anti-isomorphic groups G_i^+ and G_i^*.

3 A Standard Basis for $A_{p_1 p_2}$

A group G has a standard basis if there exists a subset L of words in the generators of G, and a bijection between L and G, such that each element of the basis is equivalent to one and only one group element. Thus if we assume the axiom of choice, then technically every group has a standard basis. However, giving an explicit finite presentation of the basis is usually not possible because of the dependence on the axiom of choice. Therefore we reserve the term standard basis for those groups that have a finite, or at least recursive, presentation.

Even in this stricter sense of the term, $A_{p_1 p_2}$ has a standard basis which is defined in terms of the ascending sequence of groups $G_0 \subset G_1 \subset G_2 \subset G_3 = A_{p_1 p_2}$ as follows:

Each of the sets C_0, C_1, C_2, and C_3 is a standard basis for G_0, G_1, G_2, and G_3 respectively.

C_0 consists of all irreducible group words of the alphabet Σ_0. A word is said to be irreducible if it does not contain subwords of the form xx^{-1} or $x^{-1}x$.

C_1 consists of all words of the from

$$(**)\ w = u_1 x_1^{\epsilon_1} u_2 x_2^{\epsilon_2} ... u_k x_k^{\epsilon_k} u_{k+1},$$

where $k \geq 0,\ x_i \in \Sigma_1,\ \epsilon_i = \pm 1,\ u_i \in C_0$ and w is irreducible and does not contain the subwords:

1. $q_i^{\epsilon}a_j,\ q_i^{-2}a_j^{-1},\ q_ia_j^{-1},\ t_i^2a_j,\ t_i^{-1}a_j,\ t_i^{\epsilon}a_j^{-1}$

2. $(q_i^+)^2a_j^+,\ (q_i^+)^{-1}a_j^+,\ (q_i^+)^{\epsilon}(a_j^+)^{-1},\ (t_i^+)^{\epsilon}a_j^+,\ t_i^+(a_j^+)^{-1},\ (t_i^+)^{-2}(a_j^+)^{-1}$

where $\epsilon = \pm 1,\ 1 \leq i \leq \lambda,\ 1 \leq j \leq n.$

C_2 consists of irreducible words of the form of $(**)$ in which $k \geq 0,\ u_i \in C_1,\ x_i \in \Sigma_2,\ \epsilon_i = \pm 1$, and which do not contain the following subwords:

3. $a_jV(q_s^2, t_s)l_i^{\epsilon},\ a_jV(q_s, t_s^2)l_i^{\epsilon}$

4. $a_j^+V(a_s^+, (t_s^+)^2)(l_i)^{\epsilon},\ (a_j^+)^{-1}V((q_s^+)^2, t_s^+)(l_i^+)^{\epsilon}$

where $\epsilon = \pm 1,\ 1 \leq j \leq n,\ 1 \leq i \leq \lambda$ and $V(x, y)$ are irreducible words of G_0 in x and y.

C_3 consists of irreducible words of the form $(**)$ in which $k \geq 0,\ u_i \in C_2,\ x_i \in \Sigma_3,\ \epsilon_i = \pm 1$, and which do not contain the following subwords:

5. $(t_i^+)^{\epsilon}p_1,\ t_i^{\epsilon}p_1^{-1},\ q_i^{\epsilon}p_2,\ (q_i^+)^{\epsilon}p_2^{-1}$

6. $l_i^+V(a_j^+)W(t_j^+)p_1,\ l_i^{-1}V(a_j)W(t_j)p_1^{-1}$

7. $l_iV(a_j)W(q_j)p_2,\ (l_i^+)^{-1}V(a_j^+)W(q_j^+)p_2^{-1},$

8. $(l_i^+)^{-1}V(a_j^+)C((q_i^+)^{-1}A_i^+)W(t_j^+)p_1,\ l_iV(a_j^{)}C(q_iA_i^{-1})W(t_j)p_1^{-1},$

9. $l_i^{-1}V(a_j)C(t_i^{-1}B_i)W(q_j)p_2,\ l_i^+V(a_j^+)C(t_i^+(B_i^+)^{-1})W(q_j^+)p_2^{-1}$

where V and W are reduced words and $C(U)$ denotes a canonical word equal to U.

We end this section with a statement of Britton's Lemma with a preliminary, explanatory excerpt from [1].

Let $\overline{G} = \langle \Sigma; \Phi \rangle$ be a group with generators Σ and relations Φ. The lemma is a useful tool when dealing with H.N.N. extensions like $A_{p_1 p_2}$.

Let $\overline{G} = \langle \Sigma; \Phi \rangle$ be a group with generators Σ and relations Φ. The group

$$G = \langle \Sigma,\ \mathcal{B};\ \Phi,\ A_i p_{m_i} = p_{n_i} B_i,\ i \in I \rangle$$

where $\Sigma \bigcap \mathcal{B}$ is empty, $p_{m_i}, p_{n_i} \in \mathcal{B}$ and A_i, B_i are Σ-words in the group with stable letters $\mathcal{B}$ and base group $\overline{G}$.

Lemma 1 (Britton's lemma [1]) *Let $\mathcal{B}$ be a regular system of stable letters of the group G, with base group $\overline{G}$ and let W be a $(\Sigma \bigcup \mathcal{B})$-word. If $W = e$ in G then either W is a Σ-word and $W = e$ in $\overline{G}$ or W contains a subword of the form $p_n^{-\epsilon} U p_m^{\epsilon}$ where U is a Σ-word and for some $U = \mathcal{U}_{p_m^{\epsilon} p_n^{\epsilon}}$.*

By a σ-word, where σ is an alphabet, we mean a group word constructed from this alphabet. A system of stable letters $\mathcal{B}$ of the group G is a subset of the defining alphabet of G such that no relation of G decreases the number of occurrences of $\mathcal{B}$-letters in any word in G, except the trivial relations where stable letters are juxtaposed with their inverses. For example p_1 and p_2 are the stable letters of $A_{p_1 p_2}$, the l_i's are the stable letters of G_2, and the a_j's are the stable letters of G_1. A system of stable letters is regular if for every relation $A_i p_{m_i} = p_{n_i} B_i$, $B_i = e$ if and only if $A_i = e$. Finally, a word $U = \mathcal{U}_{p_m^{\epsilon} p_n^{\epsilon}}$ is simply a product of A_i's and/or B_i's such that $p_n^{-\epsilon} U p_m^{\epsilon} = p_n^{-\epsilon} p_n^{\epsilon} U'$ or $p_n^{-\epsilon} U p_m^{\epsilon} = U'' p_m^{-\epsilon} p_m^{\epsilon}$, for some U' or U''.

4 Ordered Groups

We assume a basic knowledge of right-ordered and lattice-ordered group theory, and refer the reader to [5] and [6] for a more complete introduction.

We therefore state only the following previous results which are used directly in the proof of our main result.

Theorem 1 *[5] The free product $G*$ of right-ordered $\{G_\alpha \mid \alpha \in I\}$ is a right-orderable group, and for every group G_α its right-order can be extended to a right-order on the group $G*$.*

Theorem 2 (corollary to Kurosh Subgroup Theorem [7]) *Let G be a free product of A, B, C with amalgamations from the factor A, i.e., all defining relations either involve one type of generator, or have the form $U(a_\nu) = V(b_\mu)$ or $U(a_\nu) = W(c_\zeta)$. Then any subgroup H of G, whose intersection with the conjugates of A, B, and C is e, must be a free group.*

Theorem 3 *[5] A group G is fully-orderable (right-orderable) if and only if every finitely generated subgroup is fully-orderable (right-orderable).*

Theorem 4 *[6] If G is an ordered group or a right-ordered group and H is any subgroup of G then H is ordered or right-ordered respectively. If G is a lattice-ordered group then H need not be a lattice-ordered group.*

Theorem 5 *[5] (Levi) Let N be a normal subgroup of a group G, P_N be a partial right-order on the group N, and $\overline{P}$ be a partial right-order on the quotient group $\overline{G} = G/N$. Then there is a partial right-order P on the group G such that (G, P) is the lexicographic extension of (N, P_N) by $(\overline{G}, \overline{P})$. If the groups (N, P_N) and $(\overline{G}, \overline{P})$ are partially-ordered and $g^{-1}P_N g = P_n$ for any $g \in G$, then the group (G, P) is also partially-ordered if $\overline{P}$ is a partial-order on $\overline{G}$.*

5 A Right-Ordering of $A_{p_1p_2}$

In this section we prove the main result, i.e. the existence of a finitely presented group, which admits a right-ordering and has solvable word problem and unsolvable conjugacy problem. We do so by proving the following theorem.

Theorem 6 *The group $A_{p_1p_2}$ is right-orderable.*

We prove the above result by defining the normal series $A_{p_1p_2} \triangleright HK \triangleright K$ and constructing right-orders on $A_{p_1p_2}/HK$, HK/K and K separately. Theorem 12 then implies that $A_{p_1p_2}$ is right-orderable. First, however, we need the following lemma.

Lemma 2 *The subgroup G_2 of $A_{p_1p_2}$ is right-orderable.*

To show that G_2 is right-orderable, it is enough to construct a right-order on G_2^* because G_2 is the free product of anti-isomorphic subgroups G_2^* and G_2^+. We begin by labeling certain subgroups of G_2' for easier reference. Recall

$$G_2^* = \langle a_1, ..., a_n, q_1, ..., q_\lambda, t_1, ..., t_\lambda, l_1, ..., l_\lambda \rangle.$$

Let

$$A = \langle a_1, ..., a_n \rangle$$

$$Q = \langle q_1, ..., q_\lambda \rangle$$

$$L = \langle l_1, ..., l_\lambda \rangle$$

$$T = \langle t_1, ..., t_\lambda \rangle$$

Let $B = (Q * T * L)^A = \langle u^{-1}vu \mid u \in A, v \in Q * T * L \rangle$. Then by definition $A \leq N_{G_2^*}(B)$ so B is normal in G_2^* with $G_2^* = AB$. Furthermore,

$G_2^*/B = AB/B \cong A \cong F_n$ so G_2^*/B is right-orderable. Here F_n denotes a free-group of rank n which is right-orderable by theorem.......... . By theorem 12, to show G_2^* is right-orderable it is sufficient to show that B is right-orderable.

Recall the relations of $G*_2$ are

$$a_j^{-1} q_i a_j = q_i^2,\ for\ 1 \le i \le \lambda,\ 1 \le j \le n,$$

$$a_j t_i a_j^{-1} = t_i^2,\ for\ 1 \le i \le \lambda,\ 1 \le j \le n,\ and$$

$$a_j^{-1} l_i a_j = l_i\ for\ 1 \le i \le \lambda,\ 1 \le j \le n.$$

In light of these relations we can think of elements of B of the form $u^{-1}xu$ where $u \in A$ and $x \in \{q_i^j, t_i^j\}$ as k-th roots of $\langle x \rangle$ because we will show that for each $u^{-1}xu$ there exists a smallest positive integer k such that $(u^{-1}xu)^k \in \langle x \rangle$.

It is obvious, as there are no non-trivial relations which hold in $\langle Q, T, L \rangle$, that $B = (\langle q_1 \rangle * \langle q_2 \rangle * ... * \langle q_\lambda \rangle * \langle t_1 \rangle * ... * \langle t_\lambda \rangle * \langle l_1 \rangle * ... * \langle l_\lambda \rangle)^A$. We now show, using Britton's Lemma, that in fact

$$B = \langle q_1 \rangle^A * \langle q_2 \rangle^A * ... * \langle q_\lambda \rangle^A * \langle t_1 \rangle^A * ... * \langle t_\lambda \rangle^A * \langle l_1 \rangle * ... * \langle l_\lambda \rangle.$$

In the definition of $A_{p_1 p_2}$, the generators $\{l_1, ..., l_\lambda\}$ are used as the stable letters of G_2^* because $l_i^{-1} a_j l_i = a_j$ and $\{a_1, ..., a_n\}$ are used as the stable letters of G_1^* because $a_j^{-1} q_i a_j = q_i^2$ and $a_j t_i a_j^{-1} = t_i^2$. However we could also view $\{a_1, ..., a_n\}$ as the stable letters of G_2^* because $a_j^{-1} l_i a_j = l_i$, so long as we realize that the base group would then be $\langle Q, T, L \rangle$ instead of G_1^*.

With this new set of stable letters, suppose that $R = e$ is a relation that holds in B. Then R is a word in the generators (and their inverses) of B that is equal to e in B and hence in G_2^*. Therefore, by Britton's Lemma, either R is a word in the generators (and their inverses) of $\langle Q, T, L\rangle$, or there exists a pinch of the form $a_j^{-\epsilon} U a_j^{\epsilon}$ where U is a word in the generators (and their inverses) of $\langle Q, T, L\rangle$, $\epsilon = \pm 1$ and $a_j^{-\epsilon} U a_j^{\epsilon} = U' a_j^{-\epsilon} a_j^{\epsilon}$. Therefore U is generated by:

- $\{q_i^{\pm 1}, l_i^{\pm 1}, t_i^{\pm 2} \mid 1 \leq i \leq \lambda\}$ if $\epsilon = 1$
- $\{q_i^{\pm 2}, l_i^{\pm 1}, t_i^{\pm 1} \mid 1 \leq i \leq \lambda\}$ if $\epsilon = -1$.

Since $R = e$, we can perform as many pinches of the above form as necessary until we arrive at $R = R_2$ where R_2 is a word in $\{q_i^{\pm 1}, l_i^{\pm 1}, t_i^{\pm 1}\}$ and $R_2 = e$. But $\{q_i^{\pm 1}, l_i^{\pm 1}, t_i^{\pm 1}\}$ generates a free group so R_2 freely reduces to the identity, i.e., $R = v_1 v_2 ... v_\alpha$ such that for each v_j there is a fixed x_i from $\{q_i, l_i, t_i\}$ such that v_j is a word in powers of x_i, with the sum of the powers being 0. This proves that B is the free product we claimed because R must be the word $v_1' v_2' ... v_\alpha'$ where $v_j' = v_j$ in G_2' and so $v_j' \in (\langle x_i\rangle)^A$ where v_j is a word in $\{x_i^{\pm 1}\}$.

Therefore, Theorem 8 implies that, to show B is right-orderable, we need only show that $\langle q_1\rangle^A = \langle u^{-1} q_1 u \mid u \in A\rangle$ is right-orderable because B is a free product of groups isomorphic to $\langle q_1\rangle^A$. Theorem ...... further reduces the task to showing that all finitely generated subgroups of $\langle q_1\rangle^A$ are right-orderable. For ease of notation, as it does not matter which q_i we demonstrate on, let $q_1 = q$.

We begin by showing that every subgroup of $\langle q\rangle^A$ generated by two elements is right-orderable. Fix $u_1, u_2 \in A$ and consider the group $\langle u_1^{-1} q u_1, u_2^{-1} q u_2\rangle$. In actuality the most general form of a subgroup of $\langle q\rangle^A$ generated by two elements would be $\langle u_1^{-1} q^{k_1} u_1, u_2^{-1} q^{k_2} u_2\rangle$ where k_1 and k_2 are integers but $\langle u_1^{-1} q^{k_1} u_1, u_2^{-1} q^{k_2} u_2\rangle$ is a subgroup of $\langle u_1^{-1} q u_1, u_2^{-1} q u_2\rangle$, so right-orderability of the latter implies right orderability of the former.

Lemma 3 *There exist integers n_1, n_2 such that $(u_1^{-1}qu_1)^{n_1} \in \langle q\rangle$, and $(u_2^{-1}qu_2)^{n_2} \in \langle q\rangle$. We may assume that n_1, n_2 have the smallest magnitude possible.*

Proof: Since $u_1 = a_{i_1}^{\alpha_1}a_{i_2}^{\alpha_2}...a_{i_k}^{\alpha_k}$, such that α_i are integers and each $a_{i_j} \in \{a_1, ..., a_n\}$, we set $d_1 = min\{\alpha_1, \alpha_1 + \alpha_2, ..., \Sigma_{i=1}^k \alpha_i\}$.

First we show that if $d_1 \geq 0$ then $u_1^{-1}qu_1 \in \langle q\rangle$ and $n_1 = 0$. Proceeding by induction, $\alpha_1 \geq 0$ so $a_{i_1}^{-\alpha_1}qa_{i_1}^{\alpha_1} = q^{2^{\alpha_1}} \in \langle q\rangle$. By the inductive assumption $(a_{i_1}^{\alpha_1}a_{i_2}^{\alpha_2}...a_{i_j}^{\alpha_j})^{-1}qa_{i_1}^{\alpha_1}a_{i_2}^{\alpha_2}...a_{i_j}^{\alpha_j} = q^{2^{\Sigma_{i=1}^j \alpha_i}} \in \langle q\rangle$. Thus $a_{j+1}^{-\alpha_{j+1}}q^{2^{\Sigma_{i=1}^j \alpha_i}}a_{j+1}^{\alpha_{j+1}} = q^{2^{\Sigma_{i=1}^{j+1}\alpha_i}}$ which is in $\langle q\rangle$ since $\Sigma_{i=1}^{j+1}\alpha_i \geq 0$.

Otherwise if $d_1 < 0$ then let $n_1 = 2^{-d_1} > 0$. Then $(u_1^{-1}qu_1)^{n_1} = u_1^{-1}q^{2^{-d_1}}u_1 = q^{2^{-d_1+\Sigma_{i=1}^k \alpha_i}} \in Q$. We can find n_2 in the same manner so the proof is complete.

To illustrate the method we use the following example. Let

$$u_1 = a_1^3 a_2^{-5} a_3,$$

$$u_2 = a_4^{-3} a_5^{-2} a_6^{-2}.$$

Then

$$d_1 = min\{3, -2, -1\} = -2,$$

$$d_2 = min\{-3, -5, -7\} = -7$$

so $n_1 = 2^{-d_1} = 2^2$ and

$$(u_1^{-1}qu_1)^{2^2} = a_3^{-1}a_2^5a_1^{-3}q^{2^2}a_1^3a_2^{-5}a_3 = a_3^{-1}a_2^5q^{2^5}a_2^{-5}a_3 = a_3^{-1}qa_3 = q^2 \in \langle q\rangle.$$

Similarly $n_2 = 2^{-d_2} = 2^7$ and

$$(u_2^{-1}qu_2)^{2^7} = a_6^2a_5^2a_3^3q^{2^7}a_4^{-3}a_5^{-2}a_6^{-2} = a_6^2a_5^2q^{2^4}a_5^{-2}a_6^{-2} = a_6^2q^{2^2}q_6^{-2} = q \in \langle q\rangle.$$

In the proof above we do not show that n_1 and n_2 are of minimal magnitude, even though they are. Because the natural order on the positive (negative) integers is a well ordering, we are guaranteed that integers of smallest magnitude exist so we can assume that m_1 and m_2 are said integers. Clearly $(u_1^{-1}q_iu_1)^{m_1}, (u_2^{-1}q_iu_2)^{m_2} \in \langle q_i \rangle$ and $\langle q_i \rangle$ is cyclic so there exist smallest integers m_1' and m_2' such that $(u_1^{-1}q_iu_1)^{m_1'} = (u_2^{-1}q_iu_2)^{m_2'}$. This implies that every relation that holds in the group $\langle x \rangle * \langle y \rangle / \langle x^{m_1'} y^{-m_2'} \rangle^{\langle x \rangle * \langle y \rangle}$, also holds in $H = \langle u_1^{-1}qu_1, u_2^{-1}qu_2 \rangle$ via the homomorphism $x \to u_1^{-1}qu_1$, $y \to u_2^{-1}qu_2$. We now show that in fact these groups are isomorphic by showing that the relations of $\langle x \rangle * \langle y \rangle / \langle x^{m_1'} y^{-m_2'} \rangle^{\langle x \rangle * \langle y \rangle}$ are the only non-trivial relations that hold in H.

First note that the element $(u_1^{-1}qu_1)^{m_1'}$ is a power of both $u_1^{-1}qu_1$ and $u_2^{-1}qu_2$ so it generates a central subgroup of H that is identical to the one generated by $(u_2^{-1}qu_2)^{m_2'}$. For ease of notation let $x_1 = u_1^{-1}qu_1$ and $x_2 = u_2^{-1}qu_2$. If $R = e$ is a relation that holds in H, then we can express R as $R = v_1^{i_1}v_2^{i_2}...v_\beta^{i_\beta}$ where each i_j is a non-zero integer except i_1 and i_β either or both of which might be zero and such that $v_i = x_1$ if i is odd and $v_i = x_2$ if i is even. Note that if $| \ i_{2j+1} \ | \geq m_1'$ then we can rewrite R as $x_1^{\pm m_1'} v_1^{i_1}v_2^{i_2}...v_{2j+1}^{i_{2j+1} \pm m_1'}...v_\beta^{i_\beta}$ and similarly if $| \ i_{2j} \ | \geq m_2'$. Therefore we can assume that R has the form $q^\gamma v_1^{i_1}v_2^{i_2}...v_\beta^{i_\beta}$ where each $| \ i_{2j+1} \ | < m_1'$ and $| \ i_{2j} \ | < m_2'$.

We now apply Britton's Lemma to R. Either R is a power of q or we have a pinch or the form $a_j^{-\epsilon} q a_j^{\epsilon}$. We can continue to apply pinches until we have an expression equivalent to R written only in terms or powers of q, the powers of which sum to 0. But m_1' and m_2' are the smallest integral powers of $u_1^{-1}qu_1$ and $u_2^{-1}qu_2$ respectively, which lie in $\langle q \rangle$. Therefore, because each $| \ i_{2j+1} \ | < m_1'$ and $| \ i_{2j} \ | < m_2'$, we must have that they are all zeros; i.e., $R = q^\gamma$ and $\gamma = 0$. This proves that no other relations can hold in H.

Therefore,

$$H \cong \langle x \rangle * \langle y \rangle / \langle x^{m_1'} y^{-m_2'} \rangle^{\langle x \rangle * \langle y \rangle}$$

where $\langle x^{m'_1}y^{-m'_2}\rangle^{\langle x\rangle*\langle y\rangle} = \langle u^{-1}vu \mid u \in \langle x\rangle * \langle y\rangle, v \in \langle x^{m'_1}y^{-m'_2}\rangle\rangle$.

Thus H is an amalgamated free product which we show is right-orderable, by first considering the subgroup

$$I([H,H]) = \langle w \in H \mid \exists n \neq 0, w^n \in [H,H]\rangle$$

called the isolator of $[H,H]$. Naturally $H/[H,H]$ is abelian and $[H,H] \leq I([H,H])$ so $H/I([H,H])$ is an abelian group. Furthermore, $H/I([H,H])$ is torsion-free because if $wI([H,H])$ has finite order then $\exists i_1$ such that $w^{i_1} \in I([H,H])$ which implies that $\exists i_2$ such that $w^{i_1 i_2} \in [H,H]$ which means $w \in I([H,H])$. Therefore $H/I([H,H])$ is torsion-free abelian and hence right-orderable. To show that $I([H,H])$ is right-orderable, by virtue of Theorem 8, we need only show that it is a free group. Theorem 9 however, implies that $I([H,H])$ is free if

$$I([H,H])^H \bigcap \langle u_1^{-1}qu_1\rangle = I([H,H])^H \bigcap \langle u_2^{-1}qu_2\rangle = e$$

where $I([H,H])^H = \langle u^{-1}vu \mid u \in H, v \in I([H,H])\rangle$. But $I([H,H])$ is a normal subgroup of H so $I([H,H])^H = I([H,H])$. Suppose there exists integer i_1 such that $(u_1^{-1}qu_1)^{i_1} \in I([H,H])$. Then by definition, there exists integer i_2 such that $(u_1^{-1}qu_1)^{i_1 i_2} \in [H,H]$. Therefore, $u_1 - 1qu_1 \in I([H,H])$. But there exist integers m_1 and m_2 such that $(u_1^{-1}qu_1)^{m_1} = (u_2^{-1}qu_2)^{m_2}$ so $(u_2^{-1}qu_2)^{m_2} \in I([H,H])$ and thus $u_2^{-1}qu_2 \in I([H,H])$. This implies $H/I([H,H]) \cong e$ and that H has no non-trivial abelian torsion-free quotients. But $H/[H,H] \cong \langle x, y \mid [x,y] = e, x^{m'_1} = y^{m'_2}\rangle$ by virtue of the isomorphism between H and $\langle x\rangle * \langle y\rangle / \langle x^{m'_1}y^{-m'_2}\rangle^{\langle x\rangle*\langle y\rangle}$. Thus $H/I([H,H])$ has an infinite cyclic subgroup and so an infinite cyclic quotient group which is a contradiction. Therefore, the supposition that there exists integer i_1 such that $(u_1^{-1}qu_1)^{i_1} \in I([H,H])$ is false and $I([H,H])$ is free and hence right-orderable.

This proves that every subgroup $\langle x_1, x_2\rangle$ of $\langle q\rangle^A$ generated by two elements is right-orderable. We extend the proof to cover subgroups $\langle x_1, x_2, ..., x_i\rangle$, generated by i elements, by expressing H_j's iteratively as amalgamated free products of the first j generators. That is, given the subgroup

$$\langle x_1, ..., x_i \mid x_j = u_j^{-1} q u_j, u_j \in A\rangle$$

we express the subgroup generated by $\langle x_1, x_2\rangle$ as

$$H_2 = \langle x_1\rangle * \langle x_2\rangle / \langle h_2\rangle^{\langle x_1\rangle * \langle x_2\rangle} \ where \ h_2 = x_1^{m_1} x_2^{-m_2}$$

and in general we say

$$H_j = H_{j-1} * \langle x_j\rangle / \langle h_{j-1}^{m"_{j-1}} x_j^{-m_j}\rangle^{H_{j-1} * \langle x_j\rangle}.$$

Such a construction is always possible but may not yield a presentation of the intended group, unless the x_j's are first arranged in non-descending order with respect to the smallest positive integers k_i such that $x_i^{m_i} = q^{2^{k_i}}$ as the following example illustrates.

If $\langle x_1, x_2, x_3\rangle = \langle u_1^{-1} q u_1, u_2^{-1} q u_2, u_3^{-1} q u_3\rangle$ such that

$$u_1 = a_1^{-2} a_2^5,$$

$$u_2 = a_3^2 a_4^4,$$

$$u_3 = a_5^{-3} a_6^3.$$

Then finding n_1, n_2, and n_3 as before we have

$$(u_1^{-1} q u_1)^{2^2} = a_2^{-5} a_1^2 q^{2^2} a_1^{-2} a_2^5 = a_2^{-5} q a_2^5 = q^{2^5}$$

$$(u_2^{-1} q u_2)^1 = a_4^{-4} a_3^{-2} q a_3^2 a_4^4 = a_4^{-4} q^{2^2} a_4^4 = q^{2^6}$$

$$(u_3^{-1} q u_3)^{2^3} = a_6^{-3} a_5^3 q^{2^3} a_5^{-3} a_6^3 = a_6^{-3} q a_6^3 = q^{2^3}.$$

Now if we keep the order $x_1 = u_1^{-1}qu_1$, $x_2 = u_2^{-1}qu_2$, $x_3 = u_3^{-1}qu_3$ then

$$H_2 = \langle x_1 \rangle * \langle x_2 \rangle / \langle x_1^{2^3} x_2^{-1} \rangle^{\langle x_1 \rangle * \langle x_2 \rangle} \textit{ and}$$

$$H_3 = H_2 * \langle x_3 \rangle / \langle (x_1^{2^3} x_2^{-1})^1 x_3^{-2^6} \rangle^{H_2 * \langle x_3 \rangle}.$$

But note that $x_3^{2^5} \neq x_1^{2^2}$ in H_3 but $(u_1^{-1}qu_1)^{2^2} = q^{2^5}$ and $(u_3^{-1}qu_3)^{2^5} = ((u_3^{-1}qu_3)^{2^3})^{2^2} = (q^{2^3})^{2^2} = q^{2^5}$ in $\langle u_1^{-1}qu_1, u_2^{-1}qu_2, u_3^{-1}qu_3 \rangle$.

However we can remedy this problem by taking $x_1 = u_3^{-1}qu_3$, $x_2 = u_1^{-1}qu_1$, and $x_3 = u_2^{-1}qu_2$. Then

$$H_2 = \langle x_1 \rangle * \langle x_2 \rangle / \langle x_1^{2^5} x_2^{-2^2} \rangle^{\langle x_1 \rangle * \langle x_2 \rangle} \textit{ and}$$

$$H_3 = H_2 * \langle x_3 \rangle / \langle (x_1^{2^5} x_2^{-2^2})^2 x_3^{-1} \rangle^{H_2 * \langle x_3 \rangle} = \langle u_1^{-1}qu_1, u_2^{-1}qu_2, u_3^{-1}qu_3 \rangle$$

because H_3 has defining relations $x_1^{2^5} = x_2^{2^2}$ and $x_1^{2^6} = x_3$ which are precisely the defining relations of $\langle u_1^{-1}qu_1, u_2^{-1}qu_2, u_3^{-1}qu_3 \rangle$ under the above mapping.

It remains to show H_j is right-orderable. But this is done analogously to the two-generator subgroup case. $H_j/I([H_j, H_j])$ is torsion-free abelian and hence right-orderable. By Theorem 1, if $I([H_j, H_j])$ is not a free group then there exists $w \neq e$ such that $w \in H_{j-1}$ or $w \in \langle x_j \rangle$ and $w \in I([H_j, H_j])$. To see that this is not possible, recall that every element of $\langle q \rangle^A$ has a power in $\langle q \rangle$ so every element of H_{j-1} and every element or $\langle x_j \rangle$ must also have a power in $\langle q \rangle$. Therefore if $w \in I([H_j, H_j])$ then some power of q is in $I([H_j, H_j])$ and thus every power or q in H_j is in $I([H_j, H_j])$. But then every element of H_j is in $I([H_j, H_j])$ since every element or H_j has a power which is a power of q. But $H_j \neq I([H_j, H_j])$ since H_j has an infinite cyclic quotient. Therefore $I([H_j, H_j])$ is a free group and hence right-orderable and hence so is H_j.

Therefore every finitely generated subgroup of $\langle q_i \rangle^A$ is right-orderable and therefore $\langle q_i \rangle^A$ itself is right-orderable for every $i \in \{1, 2, ..., \lambda\}$. But the groups $\langle t_i \rangle^A$ are completely analogous if we replace each u with u^{-1} in the above proof so each $\langle t_i \rangle^A$ is also right-orderable. Now $\langle l_i \rangle^A = \langle l_i \rangle$ which is infinite cyclic and so definitely right-orderable. Thus the free product of these groups is right-orderable so B is right-orderable. And A is free and so right-orderable so $G_2' = BA$ is right-orderable, and hence G_2^+ is also right-orderable. Finally $G_2 = G_2' * G_2^+$ so G_2 is right-orderable.

5.1 $A_{p_1p_2}/K$ is right-orderable

Let

$$K = \langle [u, v] \mid u \in G_2, v \in P \rangle$$

where P is the free group $\langle p_1, p_2 \rangle$. Further, let H_1 be the subgroup generated by the diagonal elements, $x^{-1}x^+$ and x^+x^{-1} as x runs over the generators of G_2' and their inverses, and let H be the normal closure of H_1 in $A_{p_1p_2}$. By definition H is normal in $A_{p_1p_2}$. To see that K is also normal in $A_{p_1p_2}$ we note that $\forall g \in G_2, g^{-1}[u, v]g = [ug, v][g, v]^{-1} \in K$ and $\forall g \in P, g^{-1}[u, v]g = [u, g]^{-1}[u, vg] \in K$. Thus to right-order $A_{p_1p_2}$, we can simply right-order the groups $A_{p_1p_2}/HK$, HK/K, and K.

Note that $A_{p_1p_2}/HK$ is isomorphic to $P \times G_2'$ because $G_2' = G_2^+$ modulo H and elements of P and G_2 commute modulo K. As shown earlier, G_2 and G_2' are right-orderable, P is a free group of rank 2 and so also right-orderable, and so $A_{p_1p_2}/HK$ is right-orderable.

Now $HK/K \cong H/H \bigcap K$ is isomorphic to a subgroup of G_2 because the elements of H are conjugates of elements of G_2, which modulo K are only conjugated by elements of G_2, i.e. $\forall w \in \langle x^{-1}x^+ \rangle$, $\forall p \in \langle p_1, p_2 \rangle$, $p^{-1}wpK = wK$. Thus HK/K inherits from G_2 a right-order.

5.2 K is right-orderable

Finally, we show that K is just a free group of countable rank and thus also right-orderable.

Lemma 4 *The subgroup* $K = \langle [u, v] \mid u \in G_2, v \in P \rangle$ *of* $A_{p_1p_2}$*, is a free group of countable rank.*

The group $A_{p_1p_2}$ is finitely generated, and thus countable. $K \leq A_{p_1p_2}$ so it must be countably generated. To show K is free, we apply Britton's Lemma to the groups G_3, G_2, G_1, and G_0 in turn to show that no non-trivial relation of the form $W = e$ holds in K.

Beginning our proof by way of contradiction, assume we have

$$W = [u_1, v_1]^{n_1}[u_2, v_2]^{n_2}...[u_k, v_k]^{n_k} = e.$$

Applying Britton's Lemma in $K < G_3$ (recall $G_3 = A_{p_1p_2}$), it is not possible for the above presentation of W to be a $\Sigma_0 \bigcup \Sigma_1 \bigcup \Sigma_2$-word since each v_i is a Σ_3-word. Thus W contains a subword $p_j^{\epsilon} U p_j^{-\epsilon}$ where U is a word generated by

- $\{(A_i^+)^{-1} q_i^+ l_i^+,\ (t_i^+)^{-1}; 1 \leq i \leq \lambda\}$ if $p_j^{\epsilon} = p_1^{-1}$
- $\{A_i q_i^{-1} l_i^{-1},\ t_i; 1 \leq i \leq \lambda\}$ if $p_j^{\epsilon} = p_1$
- $\{B_i^{-1} t_i l_i,\ q_i^{-1}; 1 \leq i \leq \lambda\}$ if $p_j^{\epsilon} = p_2^{-1}$
- $\{B_i^+ (t_i^+)^{-1} (l_i^+)^{-1},\ q_i^+; 1 \leq i \leq \lambda\}$ if $p_j^{\epsilon} = p_2$

Let us consider how such a subword $p_j^{\epsilon} U p_j^{-\epsilon}$, also called a pinch, can occur in W.

One possibility is that a pinch could be completely contained in a single commutator $[u_i, v_i]$. In this case v_i must equal p_1, p_2, p_1^{-1} or p_2^{-1} because if one of the subwords $p_i^{\epsilon_1}U$ can be replaced, using the group relations, with a subword of the form $\overline{U}p_i^{\epsilon_1}$ then $p_j^{\epsilon_2}\overline{U}$ can not be replaced using the group relations in a similar way, unless $p_i^{\epsilon_1}p_j^{\epsilon_2} = e$, which is trivial. A similar argument applies to subwords of the form $Up_i^{\epsilon_1}$. For example $p_1 t_i$ can be replaced by $(t_i^+)^{-1}p_i$ but $p_1(t_i^+)^{-1}$ and $p_2^{\pm 1}(t_i^+)^{-1}$ can not be replaced using the group relations.

Therefore we have

$$[u_i, v_i] = u_i^{-1}v_i^{-1}u_iv_i = u_i^{-1}\gamma(u_i)v_i^{-1}v_i = u_i^{-1}\gamma(u_i)$$

where γ is the map defined by

$$\gamma : x^{-1} \mapsto x^+, \ x \in G_2'$$

$$\gamma : x^+ \mapsto x^{-1}, \ x \in G_2^+$$

The other possibility is that $v_i = v_{i+1}$ for some i and $n_in_{i+1} < 0$, and the pinch occurs between $[u_i, v_i]$ and $[u_{i+1}, v_{i+1}]$. Assume, without loss of generality, that $n_i < 0$. We have

$$[u_i, v_i][u_{i+1}, v_{i+1}]^{-1} = u_i^{-1}v_i^{-1}u_iv_iv_{i+1}^{-1}u_{i+1}^{-1}v_{i+1}u_{i+1} = u_i^{-1}v_i^{-1}u_iu_{i+1}^{-1}v_{i+1}u_{i+1}$$

in which case $v_i \in \{p_1, p_2, p_1^{-1}, p_2^{-1}\}$ and

$$[u_i, v_i][u_{i+1}, v_{i+1}]^{-1} = u_i\gamma(u_iu_{i+1}^{-1})u_{i+1}.$$

Thus, all of the pinches from G_3 yield subwords of the form

1. $u_i^{-1}\gamma(u_i)$

2. $u_i\gamma(u_iu_{i+1}^{-1})u_{i+1}$

where each u_i in 1 and $u_i u_{i+1}^{-1}$ in 2, is generated by one of the bulleted sets above. Note that each pinch can use one and only one bulleted set in this way, but any and all of the bulleted sets may be used in different pinches throughout the expression W. We can continue applying Britton's Lemma until we produce a $\Sigma_0 \bigcup \Sigma_1 \bigcup \Sigma_2$-word equivalent to W. We label this equivalent expression W_2. Note that W_2 will be the product of subwords of the form of 1 and 2. Furthermore note that each such subword can be created in one and only one way, i.e. given the subword we can tell exactly what the pinch was and recover the original commutator or pair of commutators.

We again apply Britton's Lemma, this time to W_2 in G_2. In G_2, $\{l_i, l_i^+ \mid 1 \leq i \leq \lambda\}$ is the set of stable letters so either W_2 is a $\Sigma_0 \bigcup \Sigma_1$-word or W_2 has a pinch of the form

$$l_i^{-\epsilon} U l_i^{\epsilon}$$

or

$$(l_i^+)^{-\epsilon} U (l_i^+)^{\epsilon}$$

where U is a $\{a_j \mid 1 \leq j \leq n\}$-word for l_i and U is a $\{a_j^+ \mid 1 \leq j \leq n\}$-word for l_i^+. However there is no way to produce such a pinch using a product of words of the form of 1 and 2.

It is very easy to see that l_i^{ϵ} and $l_i^{+\epsilon}$ in the same u_i or $\gamma(u_i)$ or $\gamma(u_i u_{i+1}^{-1})$ can not form a pinch because they are necessarily separated by a word of the form $(A_i^+)^{-1} q_i^+$ or $A_i q_i^{-1}$ or $B_i^{-1} t_i$ or $B_i^+ (l_i^+)^{-1}$ and in order for $l_i^{-\epsilon} U l_i^{\epsilon}$ or $(l_i^+)^{-\epsilon} U (l_i^+)^{\epsilon}$ to be a pinch, we must have $U \in \langle a_i \rangle$ or $U \in \langle a_i^+ \rangle$ respectively.

Now consider the leftmost subword of the form 1. or 2. If the left most subword is of type 1, then any l_i^{ϵ} or $l_i^{+\epsilon}$ in u_i^{-1} is not part of a pinch and therefore not removable, since the pinches involving $\gamma(u_i)$ will not completely remove $\gamma(u_i)$ and any pinch involving letters of u_i^{-1} can not involve letters of $\gamma(u_i)$. The case for type 2 is the same for the l_i^ϵ or $l_i^{+\epsilon}$ in $\gamma(u_i u_{i+1}^{-1})$.

Therefore W_2 must already be a $\Sigma_0 \bigcup \Sigma_1$-word. But if the subwords of the form of 1 and 2, do not contain l_i^ϵ and $(l_i^+)^\epsilon$, then they can not contain a_j's and a_j^+'s either, which are the stable letters of G_1.

Therefore, W_2 must be a Σ_0-word. But G_0 is a free group so W_2 must already be the identity. This means that the subwords of the form 1 and 2 in W_2 must cancel one another out, so one the subwords must be adjacent to its inverse. Since each of these subwords is created in a unique way, one of the commutators of W must be adjacent to its inverse, yielding a contradiction. We have shown that K is free and, therefore, right-orderable.

6 The Lattice-Ordered Group $L(A_{p_1p_2})$

We end this chapter by showing how to construct a lattice-ordered group $L(A_{p_1p_2})$ that may be a candidate to answer the question by A.M.W. Glass affirmatively. The author plans to investigate this possibility in the future. The method of embedding a right-ordered group into a lattice-ordered one is not new. We have shown that $A_{p_1p_2}$ is right-orderable, so taking the right regular representation of $A_{p_1p_2}$ yields a faithful homomorphism of $A_{p_1p_2}$ into the group of order preserving permutations of the totally ordered set $A_{p_1p_2}$. To avoid confusion between the group $A_{p_1p_2}$ and the ordered set $A_{p_1p_2}$, we denote the latter Ω. Then for $g, h \in A_{p_1p_2}$, and $x \in \Omega$, we set $(g \vee h)(x) = max\{xg, xh\}$ and $(g \wedge h)(x) = min\{xg, xh\}$. This gives a lattice-ordered group generated by the generators of $A_{p_1p_2}$, and such that $A_{p_1p_2}$ is a subgroup. We denote this group $L(A_{p_1p_2})$.

It is important to note here that under the logical signature $\{e, \cdot, \wedge, \vee\}$

$L(A_{p_1p_2})$ is a finitely presented lattice-ordered group; the generators and defining relations of $L(A_{p_1p_2})$ are just those of $A_{p_1p_2}$. However, when viewed strictly as a group $L(A_{p_1p_2})$ is not even finitely generated. Thus, it could possibly be used to prove the existence of a group which is finitely presented in the class of lattice-ordered groups which have solvable word problem and unsolvable conjugacy problem. It would not be a group which is finitely presented as a group.

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