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L^p SPACES OF OPERATOR-VALUED FUNCTIONS

CHRISTOPHER RAMSEY AND ADAM REEVES

ABSTRACT. We define a p -norm in the context of quantum random variables, measurable operator-valued functions with respect to a positive operator-valued measure. This norm leads to a operator-valued L^p space that is shown to be complete. Various other norm candidates are considered as well as generalizations of Hölder's inequality to this new context.

1. INTRODUCTION

In recent works [14, 15], the first author and Sarah Plosker have defined the spaces of L^∞ and L^1 quantum random variables with respect to an operator-valued measure, meaning measurable operator-valued functions on a nice set. These Banach spaces are suitably well-behaved to look into the possibility of L^p spaces in this context. This paper will discuss these new spaces and the difficulties of trying to establish versions of Hölder's and triangle (Minkowski's) inequalities in this context. One of the main difficulties is the (mis)behaviour of the Loewner (positive) order. Sections 2 and 3 introduce the p -norms and their decomposable versions as well as other possibilities for norms in these contexts. Section 4 proves some Hölder inequality analogues but all with extra assumptions.

First we remind the reader of the main objects of study. Suppose that \mathcal{H} is a finite dimensional or separable Hilbert space, X is a locally compact Hausdorff space and $\mathcal{O}(X)$ is the σ -algebra of Borel sets on X .

An *operator-valued measure (OVM)* $\nu : \mathcal{O}(X) \rightarrow \mathcal{B}(\mathcal{H})$ is an ultraweakly countably additive function. That is, for every countable collection $\{E_k\}_{k \in \mathbb{N}} \subset \mathcal{O}(X)$ of disjoint Borel sets one has

$$\nu\left(\bigcup_{k \in \mathbb{N}} E_k\right) = \sum_{k \in \mathbb{N}} \nu(E_k),$$

with the sum converging in the ultraweak topology. To define this topology one first needs to define the set of *states* or *density operators* $\mathcal{S}(\mathcal{H})$, that is the set of all positive, trace-one operators in $\mathcal{B}(\mathcal{H})$. Thus, to every $s \in \mathcal{S}(\mathcal{H})$ one has that $\text{Tr}(s \cdot)$ is a state on $\mathcal{B}(\mathcal{H})$ in the other sense, a unital positive linear functional. Recall, that there are many more states in this latter sense that do not arise from $\mathcal{S}(\mathcal{H})$. However, the density operators do separate points in $\mathcal{B}(\mathcal{H})$ and norm any positive operator. With all of this in mind, $A_n \rightarrow A$ ultraweakly in $\mathcal{B}(\mathcal{H})$ if and only if

$$\text{Tr}(sA_n) \rightarrow \text{Tr}(sA), \quad \forall s \in \mathcal{S}(\mathcal{H}).$$

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An OVM $\nu : \mathcal{O}(X) \rightarrow \mathcal{B}(\mathcal{H})$ is called a positive, operator-valued measure (POVM) if $\nu(E) \geq 0$ for every $E \in \mathcal{O}(X)$. Notice that this implies that ν is a finite measure since $\nu(E) \leq \nu(X) \in \mathcal{B}(\mathcal{H})$, $E \in \mathcal{O}(X)$. Such POVMs arise as one of the main objects of study in quantum physics and quantum information theory in particular. See [2] for a canonical presentation of the probabilistic structure of quantum mechanics. In the operator theoretic context these objects have been studied in [5–8, 10, 12, 14] with a lot of development in the past decade due to the growing connections between pure mathematics and quantum theory.

For any full-rank density operator $\rho \in \mathcal{S}(\mathcal{H})$, that is, ρ is a positive, trace-one operator with no null space, define $\nu_\rho(E) := \text{Tr}(\rho\nu(E))$, $E \in \mathcal{O}(X)$. Then ν_ρ is a finite, positive measure into \mathbb{C} that is mutually absolutely continuous with ν . Note that the definition of absolute continuity of measures extends easily to all OVMs as it does not depend on the dimension of the codomain.

Suppose now that \mathcal{H} has an orthonormal basis $\{e_i\}$, finite or countable. Define $\nu_{ij}(E) := \langle \nu(E)e_j, e_i \rangle$, $E \in \mathcal{O}(X)$ which is a finite measure such that $\nu_{ij} \ll_{ac} \nu \sim_{ac} \nu_\rho$. Hence, by the Radon-Nikodým Theorem there exists a unique $\frac{d\nu_{ij}}{d\nu_\rho} \in L^1(X, \nu_\rho)$ such that $\nu_{ij}(E) = \int_E \frac{d\nu_{ij}}{d\nu_\rho} d\nu_\rho$ for all $E \in \mathcal{O}(X)$. Finally, define the *Radon-Nikodým derivative of ν* as

$$\frac{d\nu}{d\nu_\rho} := \left[\frac{d\nu_{ij}}{d\nu_\rho} \right]_{i,j}.$$

We are only interested in POVMs whose Radon-Nikodým derivative gives a bounded operator a.e., which happens automatically in finite-dimensions. It was shown in [12] that this is the same under any choice of full-rank ρ or orthonormal basis. Moreover, if ν is a POVM then $\frac{d\nu}{d\nu_\rho}$ maps into the positive operators.

A *quantum random variable* $f : X \rightarrow \mathcal{B}(\mathcal{H})$ is a Borel measurable function, that is, for each $s \in \mathcal{S}(\mathcal{H})$ the function $\text{Tr}(sf(x))$ is Borel measurable. Now for each state $s \in \mathcal{S}(\mathcal{H})$ define

$$f_s(x) := \text{Tr} \left(s \left(\frac{d\nu}{d\nu_\rho}(x) \right)^{1/2} f(x) \left(\frac{d\nu}{d\nu_\rho}(x) \right)^{1/2} \right).$$

A quantum random variable $f : X \rightarrow \mathcal{B}(\mathcal{H})$ is ν -*integrable* if and only if f_s is ν_ρ -integrable for every $s \in \mathcal{S}(\mathcal{H})$. Finally, the integral of f with respect to ν is implicitly defined by

$$\text{Tr} \left(s \int_X f d\nu \right) = \int_X f_s d\nu_\rho, \quad s \in \mathcal{S}(\mathcal{H}).$$

It is helpful to realize that if μ is a finite, positive measure on X and $\nu = \mu I_{\mathcal{H}}$ then $\frac{d\nu}{d\nu_\rho} = I_{\mathcal{H}}$ and integration is just entrywise integration. Refer to [12, 14, 15] for further reading on quantum random variable integration.

2. THE L^p NORM

Here and in the remainder of the paper we will be assuming that $\nu : \mathcal{O}(X) \rightarrow \mathcal{B}(\mathcal{H})$ is a POVM where the Radon-Nikodým derivative $\frac{d\nu}{d\nu_\rho}$ is a quantum random variable, maps into bounded operators.

As mentioned in the introduction, the only L^p spaces of quantum random variables defined previously are those for $p = 1$ and ∞ . First, in [14] the von Neumann

algebra of all essentially bounded quantum random variables,

$$L_{\mathcal{H}}^{\infty}(X, \nu) = L^{\infty}(X, \nu_{\rho}) \bar{\otimes} \mathcal{B}(\mathcal{H})$$

was introduced. Here, the ∞ -norm is defined to be the least upper essential bound, $\|f\|_{\infty} \geq \|f(x)\|$ almost everywhere with respect to ν (or equivalently ν_{ρ}). Second, in [15] an L^1 norm and Banach space were defined which will be given below in the more general p -norm definition. For

$$\mathcal{L}_{\mathcal{H}}(X, \nu) = \text{span}\{\text{positive quantum random variables}\}$$

we define:

Definition 2.1. *If $f \in \mathcal{L}_{\mathcal{H}}(X, \nu)$ consider the set of 4-tuples of positive quantum random variables that combine to give f :*

$$Pos_f := \left\{ (f_1, f_2, f_3, f_4) \in \mathcal{L}_{\mathcal{H}}(X, \nu)^4 : f = f_1 - f_2 + if_3 - if_4, f_i \geq 0, 1 \leq i \leq 4 \right\}$$

Now define

$$\begin{aligned} \|f\|_p &= \inf_{Pos_f} \sup_{s \in \mathcal{S}(\mathcal{H})} \left(\int_X (f_1 + f_2 + f_3 + f_4)_s^p d\nu_{\rho} \right)^{1/p} \\ &= \inf_{Pos_f} \sup_{s \in \mathcal{S}(\mathcal{H})} \|(f_1 + f_2 + f_3 + f_4)_s\|_p \end{aligned}$$

where the latter p -norm is that of $L^p(X, \nu_{\rho})$.

When $p = 1$ this is indeed the 1-norm defined in [15] since

$$\begin{aligned} \inf_{Pos_f} \sup_{s \in \mathcal{S}(\mathcal{H})} \left(\int_X \text{Tr}(s(f_1 + f_2 + f_3 + f_4)) d\nu_{\rho} \right) \\ &= \inf_{Pos_f} \sup_{s \in \mathcal{S}(\mathcal{H})} \text{Tr} \left(s \int_X f_1 + f_2 + f_3 + f_4 d\nu \right) \\ &= \inf_{Pos_f} \left\| \int_X f_1 + f_2 + f_3 + f_4 d\nu \right\| \\ &= \|f\|_1. \end{aligned}$$

Proposition 2.2. $\|\cdot\|_p$ is a seminorm on $\mathcal{L}_{\mathcal{H}}(X, \nu)$ into the extended reals $[0, \infty]$.

Proof. Positive semidefinite is automatic. For homogeneity, if $\lambda = \lambda_1 - \lambda_2 + i\lambda_3 - i\lambda_4 \in \mathbb{C}$ with $\lambda_i \geq 0$ and $(f_1, f_2, f_3, f_4) \in Pos_f$ then we have that the sum of the positive parts of $\lambda f = (\lambda_1 - \lambda_2 + i\lambda_3 - i\lambda_4)(f_1 - f_2 + if_3 - if_4)$ is

$$\begin{aligned} (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)(f_1 + f_2 + f_3 + f_4) &= (|\text{Re}\lambda| + |\text{Im}\lambda|)(f_1 + f_2 + f_3 + f_4) \\ &= |\lambda|_1(f_1 + f_2 + f_3 + f_4). \end{aligned}$$

Thus, $\|\lambda f\|_p \leq |\lambda|_1 \|f\|_p$. But this is an equality since if $(g_1, g_2, g_3, g_4) \in Pos_{\lambda f}$ then $f = \lambda^{-1}(g_1 - g_2 + ig_3 - ig_4)$ which can be recombined as the linear combination of positive functions and so all possibilities in the infimum are realized. Hence, $\|\lambda f\|_p = |\lambda|_1 \|f\|_p$ and $\|\cdot\|_p$ is homogeneous with respect to the 1-norm on \mathbb{C} .

Now for the triangle inequality, let $f, g \in \mathcal{L}_{\mathcal{H}}(X, \nu)$ with $(f_1, f_2, f_3, f_4) \in Pos_f$ and $(g_1, g_2, g_3, g_4) \in Pos_g$. By the triangle inequality on $L^p(X, \nu_{\rho})$ we have

$$\|f + g\|_p \leq \sup_{s \in \mathcal{S}(\mathcal{H})} \left\| \left(\sum_{i=1}^4 f_i + g_i \right)_s \right\|_p$$

$$\begin{aligned}
&\leq \sup_{s \in \mathcal{S}(\mathcal{H})} \left\| \left(\sum_{i=1}^4 f_i \right)_s \right\|_p + \left\| \left(\sum_{i=1}^4 g_i \right)_s \right\|_p \\
&\leq \sup_{s \in \mathcal{S}(\mathcal{H})} \left\| \left(\sum_{i=1}^4 f_i \right)_s \right\|_p + \sup_{s \in \mathcal{S}(\mathcal{H})} \left\| \left(\sum_{i=1}^4 g_i \right)_s \right\|_p.
\end{aligned}$$

Taking the infimum over the sets Pos_f and Pos_g results in

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

□

Let $\mathcal{L}_{\mathcal{H}}^p(X, \nu) = \{f \in \mathcal{L}_{\mathcal{H}}(X, \nu) : \|f\|_p < \infty\}$ and $I_p = \{f \in \mathcal{L}_{\mathcal{H}}^p(X, \nu) : \|f\|_p = 0\}$. Naturally then, define

$$L_{\mathcal{H}}^p(X, \nu) := \mathcal{L}_{\mathcal{H}}^p(X, \nu) / I_p.$$

The very nice thing about this p -norm is that it allows us a completeness argument.

Theorem 2.3 (cf. Theorem 3.12 [15]). *$L_{\mathcal{H}}^p(X, \nu)$ is a Banach space with respect to the $\|\cdot\|_p$ norm.*

Proof. The proof is exactly that in [15] with minor changes to this new wider context. Let $\{f_n\}$ be a Cauchy sequence in $L_{\mathcal{H}}^p(X, \nu)$. This implies that there are numbers $\{k_n\}_{n \in \mathbb{N}}$ such that

$$\|f_l - f_m\|_p < \frac{1}{2^{n+1}}, \quad \forall l, m \geq k_n.$$

Since $f_{k_1} \in L_{\mathcal{H}}^p(X, \nu)$ there exist $(g_{0,1}, g_{0,2}, g_{0,3}, g_{0,4}) \in Pos_{f_{k_1}}$ such that

$$\|g_{0,1} + g_{0,2} + g_{0,3} + g_{0,4}\|_p < \|f_{k_1}\|_p + 1.$$

Similarly, $f_{k_{n+1}} - f_{k_n} \in L_{\mathcal{H}}^p(X, \nu)$ there exists $(g_{n,1}, g_{n,2}, g_{n,3}, g_{n,4}) \in Pos_{f_{k_{n+1}} - f_{k_n}}$ such that

$$\|g_{n,1} + g_{n,2} + g_{n,3} + g_{n,4}\|_p < \|f_{k_{n+1}} - f_{k_n}\|_p + \frac{1}{2^{n+1}} < \frac{1}{2^n}.$$

Now for each $s \in \mathcal{S}(\mathcal{H})$ the sequence $(\sum_{n=0}^m g_{n,1} + g_{n,2} + g_{n,3} + g_{n,4})_s$ is increasing almost everywhere on X and is bounded above

$$\begin{aligned}
\left(\int_X \left(\sum_{n=0}^m g_{n,1} + g_{n,2} + g_{n,3} + g_{n,4} \right)_s^p d\nu_\rho \right)^{1/p} &\leq \sum_{n=0}^m \|g_{n,1} + g_{n,2} + g_{n,3} + g_{n,4}\|_p \\
&< \|f_{k_1}\|_p + \sum_{n=0}^m \frac{1}{2^n} \\
&< \|f_{k_1}\|_p + 2.
\end{aligned}$$

Thus by the Monotone Convergence Theorem, $\sum_{n=0}^{\infty} g_{n,1} + g_{n,2} + g_{n,3} + g_{n,4} \in L_{\mathcal{H}}^p(X, \nu)$ with

$$\left\| \sum_{n=0}^{\infty} g_{n,1} + g_{n,2} + g_{n,3} + g_{n,4} \right\|_p < \|f_{k_1}\|_p + 2.$$

An identical argument gives that

$$\left\| \sum_{n=0}^{\infty} g_{n,i} \right\|_p \leq \left\| \sum_{n=0}^{\infty} g_{n,1} + g_{n,2} + g_{n,3} + g_{n,4} \right\|_p$$

and so $g_i := \sum_{n=0}^{\infty} g_{n,i} \geq 0$ is in $L^p_{\mathcal{H}}(X, \nu)$. Thus, $g := g_1 - g_2 + ig_3 - ig_4 \in L^p_{\mathcal{H}}(X, \nu)$. Consider now that for each $m \geq 1$, by telescoping, we have that

$$\begin{aligned} \|g - f_{k_m}\|_p &= \left\| f_{k_1} + \sum_{n=1}^{\infty} (f_{k_{n+1}} - f_{k_n}) - f_{k_m} \right\|_p \\ &= \left\| f_{k_m} + \sum_{n=m}^{\infty} (f_{k_{n+1}} - f_{k_n}) - f_{k_m} \right\|_p \\ &= \left\| \sum_{n=m}^{\infty} (f_{k_{n+1}} - f_{k_n}) \right\|_p \\ &< \sum_{n=m}^{\infty} \frac{1}{2^n} \\ &= \frac{1}{2^{m-1}}. \end{aligned}$$

Therefore, $f_n \rightarrow g$ with respect to $\|\cdot\|_p$ and the conclusion follows. \square

There are certainly other possibilities for defining a p -norm. One would be to ignore the operator structure of f and look at

$$\left\| \int_X \|f(x)\|^p I_{\mathcal{H}} d\nu \right\|^{1/p}.$$

As described in [15], this type of norm is not very useful in an infinite setting. For instance, $f(x) = \sum_{i=n}^{\infty} 2^n E_{n,n} \chi_{(1/2^n, 1/2^{n-1}]}$ on $X = [0, 1]$ with $\nu = \mu I_{\ell^2(\mathbb{N})}$, where μ is Lebesgue measure, is infinite in the above norm.

Another possibility would be to consider

$$\begin{aligned} \inf_{Pos_f} \left\| \left(\sum_{i=1}^4 f_i \right)^p \right\|_1^{1/p} &= \inf_{Pos_f} \left\| \int_X \left(\sum_{i=1}^4 f_i \right)^p d\nu \right\|^{1/p} \\ &= \inf_{Pos_f} \sup_{s \in \mathcal{S}(\mathcal{H})} \left(\int_X \text{Tr} \left(s \frac{d\nu}{d\nu_\rho}^{1/2} \left(\sum_{i=1}^4 f_i \right)^p \frac{d\nu}{d\nu_\rho}^{1/2} \right) d\nu_\rho \right)^{1/p} \end{aligned}$$

as a p -norm candidate. This is natural to consider as this is the 1-norm when $p = 1$. Unfortunately, this probably fails the triangle inequality outside of the cases $p = 1$ and 2. We say “probably” since this doesn’t seem to be known.

2.1. Schatten-type norms. A final class of potential norms to consider are those arising from a combination of Schatten norms and classical integral norms.

Recall, that in $\mathcal{B}(\mathcal{H})$, $\|A\|_{S^p} = \text{Tr}(|A|^p)^{1/p}$ is called the Schatten p -norm. When $p = 1$ this is called the trace norm and when $p = 2$ it is called the Hilbert-Schmidt norm. The operators which have a finite p -norm are called the Schatten class for p and are non-closed (in the operator norm) ideals of the compact operators. These are norms because there are Young’s, Hölder’s and Minkowski’s inequalities

in this context. See [3] for further background on these norms and their classes of operators.

Consider now this new family of seminorms:

$$\begin{aligned} \|f\|_{S^p, L^q} &= \sup_{s \in \mathcal{S}(\mathcal{H})} \left(\int_X \text{Tr} \left(\left| s^{1/2} \frac{d\nu}{d\nu_\rho}^{1/2} f \frac{d\nu}{d\nu_\rho}^{1/2} s^{1/2} \right|^p \right)^{q/p} d\nu_\rho \right)^{1/q} \\ &= \sup_{s \in \mathcal{S}(\mathcal{H})} \left\| \left\| s^{1/2} \frac{d\nu}{d\nu_\rho}^{1/2} f \frac{d\nu}{d\nu_\rho}^{1/2} s^{1/2} \right\|_{S^p} \right\|_{L^q}. \end{aligned}$$

Lemma 2.4. $\|\cdot\|_{S^p, L^q}$ is a seminorm on $\mathcal{L}_{\mathcal{H}}(X, \nu)$ with $\|f\|_{S^p, L^q} \leq \|f\|_{S^r, L^q}$, for $p \geq r$. Furthermore, $\|f\|_{S^1, L^q} \leq \|f\|_q$.

Proof. It is immediate that for any fixed $s \in \mathcal{S}(\mathcal{H})$ that

$$f \mapsto \left\| \left\| s^{1/2} \left(\frac{d\nu}{d\nu_\rho}(x) \right)^{1/2} f(x) \left(\frac{d\nu}{d\nu_\rho}(x) \right)^{1/2} s^{1/2} \right\|_{S^p} \right\|_{L^q}$$

is a seminorm. Thus, $\|\cdot\|_{S^p, L^q}$ is the supremum of seminorms and so is a seminorm itself.

Monotonicity follows since the Schatten norms have this monotonic property.

For the last inequality, suppose $f \in \mathcal{L}_{\mathcal{H}}(X, \nu)$. For any $(f_1, f_2, f_3, f_4) \in \text{Pos}_f$ we have

$$\begin{aligned} \|f\|_{S^1, L^q} &= \sup_{s \in \mathcal{S}(\mathcal{H})} \left\| \left\| s^{1/2} \frac{d\nu}{d\nu_\rho}^{1/2} (f_1 - f_2 + if_3 - if_4) \frac{d\nu}{d\nu_\rho}^{1/2} s^{1/2} \right\|_{S^1} \right\|_{L^q} \\ &\leq \sup_{s \in \mathcal{S}(\mathcal{H})} \left\| \sum_{i=1}^4 \left\| s^{1/2} \frac{d\nu}{d\nu_\rho}^{1/2} f_i \frac{d\nu}{d\nu_\rho}^{1/2} s^{1/2} \right\|_{S^1} \right\|_{L^q} \\ &= \sup_{s \in \mathcal{S}(\mathcal{H})} \left\| \text{Tr} \left(s^{1/2} \frac{d\nu}{d\nu_\rho}^{1/2} (f_1 + f_2 + f_3 + f_4) \frac{d\nu}{d\nu_\rho}^{1/2} s^{1/2} \right) \right\|_{L^q} \\ &= \|f_1 + f_2 + f_3 + f_4\|_q. \end{aligned}$$

Taking the infimum over all such positive decompositions yields the desired result. \square

One of the main difficulties with this seminorm is that there is no clear path to showing that it would lead to a Banach space (at least to the authors). A possibility would be to show that $\|\cdot\|_p$ and $\|\cdot\|_{S^1, L^p}$ are comparable, though this may prove to not be true.

3. THE DECOMPOSABLE p -NORM

In 1985 Haagerup [9] introduced the decomposition (or decomposable) norm for the completely bounded maps that are in the span of the completely positive maps. In particular, for such a map $u : \mathcal{A} \rightarrow \mathcal{B}$ he defined

$$\|u\|_{dec} = \inf_{S_1, S_2} \{\max\{\|S_1\|, \|S_2\|\}\}$$

where S_1 and S_2 are completely positive maps such that

$$a \mapsto \begin{bmatrix} S_1(a) & u(a) \\ u(a^*)^* & S_2(a) \end{bmatrix}$$

is a completely positive map. For further reference see Chapter 6 of Pisier's book [13]. Junge and Ruan [11] used this to define the decomposable norm of their non-commutative L^p space, $L^p(\mathcal{M})$ for a von Neumann algebra \mathcal{M} . The original norm that this decomposable norm is formed from is the Schatten p -norm.

In the same way, we can use these ideas to create a decomposable norm in our context.

Definition 3.1. *If $f \in \mathcal{L}_{\mathcal{H}}(X, \nu)$ then define*

$$\|f\|_{p,dec} = \inf \left\{ \max\{\|S_1\|_p, \|S_2\|_p\} : S_1, S_2 \in \mathcal{L}_{\mathcal{H}}(X, \nu), S_1, S_2 \geq 0, \right. \\ \left. \begin{bmatrix} S_1(x) & f(x) \\ f(x)^* & S_2(x) \end{bmatrix} \geq 0 \text{ a.e.} \right\}$$

Proving that this is a seminorm doesn't depend on the norm one starts with. For completeness we provide the proof but one can equally find something similar in [9] or [13].

Proposition 3.2. $\|\cdot\|_{p,dec}$ is a seminorm on $\mathcal{L}_{\mathcal{H}}(X, \nu)$.

Proof. Positive semidefinite is automatic. For $\lambda \in \mathbb{C}, \lambda \neq 0$ let $\lambda^{1/2}$ be the principle square root and then

$$\text{diag}(\lambda^{1/2}, \overline{\lambda^{1/2}}) \begin{bmatrix} S_1(x) & f(x) \\ f(x)^* & S_2(x) \end{bmatrix} \text{diag}(\overline{\lambda^{1/2}}, \lambda^{1/2}) = \begin{bmatrix} |\lambda|S_1(x) & \lambda f(x) \\ (\lambda f(x))^* & |\lambda|S_2(x) \end{bmatrix}.$$

Thus, it is immediate that $\|\lambda f\|_{p,dec} = |\lambda| \|f\|_{p,dec}$. This gives that the dec-norm is homogeneous with respect to the usual Euclidean norm instead of the 1-norm on \mathbb{C} , certainly an improvement on $\|\cdot\|_p$.

Finally, if $f, g \in \mathcal{L}_{\mathcal{H}}(X, \nu)$ and $S_1, S_2, T_1, T_2 \in \mathcal{L}_{\mathcal{H}}(X, \nu)$ are positive such that

$$\begin{bmatrix} S_1 & f \\ f^* & S_2 \end{bmatrix}, \begin{bmatrix} T_1 & g \\ g^* & T_2 \end{bmatrix} \geq 0$$

then

$$\begin{bmatrix} S_1 + T_1 & f + g \\ (f + g)^* & S_2 + T_2 \end{bmatrix} \geq 0$$

and

$$\max\{\|S_1 + T_1\|_p, \|S_2 + T_2\|_p\} \leq \max\{\|S_1\|_p, \|S_2\|_p\} + \max\{\|T_1\|_p, \|T_2\|_p\}.$$

Taking infimums we get that $\|f + g\|_{p,dec} \leq \|f\|_{p,dec} + \|g\|_{p,dec}$. \square

We now establish that the "nice" properties of the $\|\cdot\|_p$ norm are inherited by the decomposable p -norm by way of comparability.

Proposition 3.3. *If $f \in \mathcal{L}_{\mathcal{H}}(X, \nu)$ then $\|f^*\|_{p,dec} = \|f\|_{p,dec}$ and if $f = f^*$ then $\|f\|_{p,dec} = \|f\|_p$.*

Proof. These arguments follow very similarly to [13, Chapter 6]. The first statement follows from the fact that

$$\begin{bmatrix} S_2 & f^* \\ f & S_1 \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} S_1 & f \\ f^* & S_2 \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Now suppose that $f = f^*$. For any combination $f = f_1 - f_2$ such that $f_1, f_2 \in \mathcal{L}_{\mathcal{H}}(X, \nu)$ are positive we have

$$\begin{bmatrix} f_1 + f_2 & f \\ f & f_1 + f_2 \end{bmatrix} = \begin{bmatrix} f_1 & f_1 \\ f_1 & f_1 \end{bmatrix} + \begin{bmatrix} f_2 & -f_2 \\ -f_2 & f_2 \end{bmatrix}.$$

Thus, $\|f\|_{p,dec} \leq \|f_1 + f_2\|_p$ and taking infimums gives one direction.

For the other direction assume that $\begin{bmatrix} S_2(x) & f(x) \\ f(x) & S_1(x) \end{bmatrix} \geq 0$ in $M_2(\mathcal{B}(\mathcal{H}))$ a.e. By [13, Lemma 1.37]

$$|\langle f\xi, \xi \rangle| \leq \frac{1}{2}\langle S_1\xi, \xi \rangle + \frac{1}{2}\langle S_2\xi, \xi \rangle, \quad \forall \xi \in \mathcal{H}$$

which implies that

$$\left\langle \left(\frac{1}{2}(S_1 + S_2) \pm f \right) \xi, \xi \right\rangle \geq 0, \quad \forall \xi \in \mathcal{H}.$$

Thus,

$$\begin{aligned} \|f\|_p &= \left\| \frac{1}{2} \left(\frac{1}{2}(S_1 + S_2) + f \right) - \frac{1}{2} \left(\frac{1}{2}(S_1 + S_2) - f \right) \right\|_p \\ &\leq \left\| \frac{1}{2} \left(\frac{1}{2}(S_1 + S_2) + f \right) + \frac{1}{2} \left(\frac{1}{2}(S_1 + S_2) - f \right) \right\|_p \\ &= \left\| \frac{1}{2}(S_1 + S_2) \right\|_p \\ &\leq \max\{\|S_1\|_p, \|S_2\|_p\} \end{aligned}$$

Therefore, $\|f\|_p \leq \|f\|_{p,dec}$. \square

Proposition 3.4. *For $f \in \mathcal{L}_{\mathcal{H}}(X, \nu)$, $\|Re f\|_p, \|Im f\|_p \leq \|f\|_p$ and $\|Re f\|_{p,dec}, \|Im f\|_{p,dec} \leq \|f\|_{p,dec}$. Moreover,*

$$\frac{1}{2}\|f\|_p \leq \|f\|_{p,dec} \leq 2\|f\|_p.$$

Proof. Suppose $(f_1, f_2, f_3, f_4) \in Pos_f$. Then

$$\|Re f\|_p \leq \|f_1 + f_2\|_p \leq \|f_1 + f_2 + f_3 + f_4\|_p$$

since $(f_1 + f_2)_s \leq (f_1 + f_2 + f_3 + f_4)_s$ for all $s \in \mathcal{S}(\mathcal{H})$. Taking the infimum over Pos_f we get $\|Re f\|_p \leq \|f\|_p$. The imaginary part follows similarly.

For the dec-norm suppose that $S_1, S_2 \in \mathcal{L}_{\mathcal{H}}(X, \nu), S_1, S_2 \geq 0$ with $\begin{bmatrix} S_1 & f \\ f^* & S_2 \end{bmatrix} \geq 0$.

This implies

$$0 \leq \frac{1}{2} \begin{bmatrix} S_1 & f \\ f^* & S_2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} S_2 & f^* \\ f & S_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(S_1 + S_2) & Re f \\ Re f & \frac{1}{2}(S_1 + S_2) \end{bmatrix}$$

and so

$$\|Re f\|_{p,dec} \leq \frac{1}{2}\|S_1 + S_2\|_p \leq \max\{\|S_1\|_p, \|S_2\|_p\}.$$

Taking the infimum gives $\|Re f\|_{p,dec} \leq \|f\|_{p,dec}$. Again, the imaginary part follows similarly.

Lastly, the comparability of the norms follows immediately by the triangle inequality, the agreement of the two norms on self-adjoint terms, and the arguments above. \square

Corollary 3.5. $L_{\mathcal{H}}^p(X, \nu)$ is a Banach space under the $\|\cdot\|_{p,dec}$ norm.

Proof. This is a direct consequence of the comparability of the norms and Theorem 2.3. \square

Finally, one can estimate the decomposable p -norm by the operator absolute value of the function and its adjoint.

Lemma 3.6. Suppose $f \in \mathcal{L}_{\mathcal{H}}(X, \nu)$ has polar decompositions $f = u|f| = |f^*|u$ with partial isometry $u \in \mathcal{L}_{\mathcal{H}}(X, \nu)$. Then

$$\|f\|_{p,dec} \leq \max \left\{ \| |f| \|_p, \| |f^*| \|_p \right\}.$$

Proof. Observe

$$\begin{bmatrix} |f^*| & f \\ f^* & |f| \end{bmatrix} = \begin{bmatrix} u & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} |f| & |f| \\ |f| & |f| \end{bmatrix} \begin{bmatrix} u^* & 0 \\ 0 & I \end{bmatrix} \geq 0.$$

Hence, $\|f\|_{p,dec} \leq \max\{\| |f| \|_p, \| |f^*| \|_p\}$. \square

4. HÖLDER'S INEQUALITY

In the classical L^p context one proves Young's inequality to prove Hölder's inequality to prove the triangle inequality (or Minkowski's inequality). Here we have already arrived at the triangle inequality in the previous sections. This is by design of course as Hölder's inequality does not hold in the generality one would wish. However, much can be said and this section explores the possibilities.

Theorem 4.1. Suppose $\mathcal{H} = \mathbb{C}^n$ and $\frac{1}{q} + \frac{1}{p} = 1$. If $f, g \in \mathcal{L}_{\mathcal{H}}(X, \nu)$ such that $f^p, g^q \in L_{\mathcal{H}}^1(X, \nu)$ $f, g \geq 0$ and $fg = gf$, then $fg \in L_{\mathcal{H}}^1(X, \nu)$ and

$$\|fg\|_1 \leq \|f^p\|_1^{1/p} \|g^q\|_1^{1/q}.$$

Proof. Since f and g commute then $fg \geq 0$. By Young's inequality for commuting matrices developed by Ando [1] we get:

$$\frac{f(x)g(x)}{AB} \leq \frac{f(x)^p}{A^p p} + \frac{g(x)^q}{B^q q}$$

Then by the comparison theorem

$$\begin{aligned} \left\| \int_X \frac{f(x)g(x)}{AB} d\nu \right\| &\leq \left\| \int_X \frac{f(x)^p}{A^p p} + \frac{g(x)^q}{B^q q} d\nu \right\| \\ &\leq \left\| \int_X \frac{f(x)^p}{A^p p} d\nu \right\| + \left\| \int_X \frac{g(x)^q}{B^q q} d\nu \right\| \end{aligned}$$

Hence, letting $A = \|f^p\|_1^{1/p}$ and $B = \|g^q\|_1^{1/q}$ leads to

$$\begin{aligned} \left\| \int_X \frac{f(x)g(x)}{AB} d\nu \right\| &\leq \frac{1}{A^p p} \left\| \int_X f(x)^p d\nu \right\| + \frac{1}{B^q q} \left\| \int_X g(x)^q d\nu \right\| \\ &= \frac{1}{p} + \frac{1}{q} \\ &= 1 \end{aligned}$$

Therefore,

$$\|fg\|_1 = \left\| \int_X fg d\nu \right\|$$

$$\begin{aligned} &\leq AB \\ &= \|f^p\|_1^{1/p} \|g^q\|_1^{1/q} \end{aligned}$$

□

Corollary 4.2. *Suppose $\mathcal{H} = \mathbb{C}^n$ and $\frac{1}{q} + \frac{1}{p} = 1$. If $f, g \in \mathcal{L}_{\mathcal{H}}(X, \nu)$ such that $\|f(x)\|^p I_n, g^q \in L_{\mathcal{H}}^1(X, \nu)$ and $f, g \geq 0$ then $g^{\frac{1}{2}} f g^{\frac{1}{2}} \in L_{\mathcal{H}}^1(X, \nu)$ and*

$$\|g^{\frac{1}{2}} f g^{\frac{1}{2}}\|_1 \leq \| \|f(x)\|^p I_n \|_1^{1/p} \|g^q\|_1^{1/q}$$

Proof. By positivity

$$f(x) \leq \|f(x)\| I_n$$

and so

$$g(x)^{\frac{1}{2}} f(x) g(x)^{\frac{1}{2}} \leq \|f(x)\| g(x)$$

Now since $g(x)$ and $\|f(x)\| I_n$ commute, we can apply the previous theorem to get

$$\|g^{\frac{1}{2}} f g^{\frac{1}{2}}\|_1 \leq \| \|f(x)\| g(x) \|_1 \leq \| \|f(x)\|^p I_n \|_1^{1/p} \|g^q\|_1^{1/q}.$$

□

We can use the previous theorem to establish a limited triangle inequality.

Theorem 4.3. *Suppose $\mathcal{H} = \mathbb{C}^n$. If $f, g \in \mathcal{L}_{\mathcal{H}}(X, \nu)$ such that $f^p, g^p \in L_{\mathcal{H}}^1(X, \nu)$, $f, g \geq 0$ and $fg = gf$ then $(f + g)^p \in L_{\mathcal{H}}^1(X, \nu)$ and*

$$\|(f + g)^p\|_1^{1/p} \leq \|f^p\|_1^{1/p} + \|g^p\|_1^{1/p}.$$

Proof. The classic proof of the triangle (or Minkowski) inequality still works in this case. Namely, because f and g are commuting positive operators then by the previous theorem and assuming $\frac{1}{p} + \frac{1}{q} = 1$

$$\begin{aligned} \|(f + g)^p\|_1 &= \|(f + g)(f + g)^{p-1}\|_1 \\ &\leq \|f(f + g)^{p-1}\|_1 + \|g(f + g)^{p-1}\|_1 \\ &\leq \|f^p\|_1^{1/p} \|(f + g)^{q(p-1)}\|_1^{1/q} + \|g^p\|_1^{1/p} \|(f + g)^{q(p-1)}\|_1^{1/q} \\ &= (\|f^p\|_1^{1/p} + \|g^p\|_1^{1/p}) \|(f + g)^p\|_1^{(p-1)/p}. \end{aligned}$$

The conclusion then follows. □

While one can prove the following result using the variation of Hölder's inequality for not necessarily commuting functions, it is simpler to get it as a direct result of the previous theorem.

Corollary 4.4. *Suppose $\mathcal{H} = \mathbb{C}^n$ and $p \geq 1$. If $f, g \in \mathcal{L}_{\mathcal{H}}(X, \nu)$ such that $\|f(x)\|^p I_n, \|g(x)\|^p I_n \in L_{\mathcal{H}}^1(X, \nu)$ and $f, g \geq 0$ then $(f + g)^p \in L_{\mathcal{H}}^1(X, \nu)$ and*

$$\|(f + g)^p\|_1^{1/p} \leq \| \|f(x)\|^p I_n \|_1^{1/p} + \| \|g(x)\|^p I_n \|_1^{1/p}$$

Proof. Since $f + g \geq 0$ then by functional calculus

$$\begin{aligned} (f(x) + g(x))^p &\leq \| (f(x) + g(x))^p \| I_n \\ &= \|f(x) + g(x)\|^p I_n \\ &\leq (\|f(x)\| + \|g(x)\|)^p I_n \\ &= (\|f(x)\| I_n + \|g(x)\| I_n)^p. \end{aligned}$$

Thus, by comparison and the previous theorem, because multiples of the identity commute,

$$\begin{aligned} \|(f+g)^p\|_1^{1/p} &\leq \|(\|f(x)\|_{I_n} + \|g(x)\|_{I_n})^p\|_1^{1/p} \\ &\leq \| \|f(x)\|_{I_n}^p \|_1^{1/p} + \| \|g(x)\|_{I_n}^p \|_1^{1/p} \end{aligned}$$

□

These do not generalize well into infinite dimensions since Ando's matrix Young's inequality only really makes sense for compact operators [4]. As well, the above two results do not extend from positive to arbitrary functions. The crux of the problem is that multiplication does not work well with positive operators and the Loewner order (for P, Q self-adjoint, $P \leq Q$ if and only if $Q - P$ is a positive operator).

A possibility to fix this is to consider tensor products, forcing commutativity. Suppose μ is a finite, positive measure on X and ν_1 and ν_2 are POVMs into $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$, respectively, such that $\nu_1, \nu_2 \ll_{ac} \mu$. One can define a POVM

$$\nu_1 \otimes_{\mu} \nu_2(E) = \int_E \left(\frac{d\nu_1}{d\mu} \otimes \frac{d\nu_2}{d\mu} \right) d\mu I_{\mathcal{H}_1 \otimes \mathcal{H}_2}$$

provided that $\nu_1 \otimes_{\mu} \nu_2(X) \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

The choice of μ makes a difference: if γ is a finite, positive measure such that $\mu \ll_{ac} \gamma$ then

$$\begin{aligned} \nu_1 \otimes_{\gamma} \nu_2(E) &= \int_E \left(\frac{d\nu_1}{d\gamma} \otimes \frac{d\nu_2}{d\gamma} \right) d\gamma I_{\mathcal{H}_1 \otimes \mathcal{H}_2} \\ &= \int_E \left(\frac{d\nu_1}{d\mu} \frac{d\mu}{d\gamma} I_{\mathcal{H}_1} \otimes \frac{d\nu_2}{d\mu} \frac{d\mu}{d\gamma} I_{\mathcal{H}_2} \right) d\gamma I_{\mathcal{H}_1 \otimes \mathcal{H}_2} \\ &= \int_E \frac{d\mu}{d\gamma} \left(\frac{d\nu_1}{d\mu} \otimes \frac{d\nu_2}{d\mu} \right) d\gamma I_{\mathcal{H}_1 \otimes \mathcal{H}_2} \end{aligned}$$

which in general is not equal to $\nu_1 \otimes_{\mu} \nu_2(E)$.

An alternative definition would be to take the square root of each Radon-Nikodým derivative. While this would give a tensor product of measures that is the same no matter the choice of μ , the downside is that one loses the structure discussed below.

Recall that if $s_1 \in \mathcal{S}(\mathcal{H}_1)$ and $s_2 \in \mathcal{S}(\mathcal{H}_2)$ then $s_1 \otimes s_2 \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, called a *product state*. In this way one can see that

$$\mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)_{sep} := \overline{\text{conv}\{s_1 \otimes s_2 : s_1 \in \mathcal{S}(\mathcal{H}_1), s_2 \in \mathcal{S}(\mathcal{H}_2)\}} \subset \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2).$$

These are called the *separable states* and it is a well known fact that not all states are separable, calling such a non-separable state an *entangled state*.

Define a new seminorm using only the separable states

$$\|f \otimes g\|_{p, sep, \nu_1 \otimes_{\mu} \nu_2} := \sup\{\|(f \otimes g)_s\|_{p, \mu} : s \in \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)_{sep}\}$$

and in the same way define a decomposable version

$$\begin{aligned} \|f \otimes g\|_{p, sep, dec} &:= \inf \left\{ \max\{\|S_1\|_{p, sep, \nu_1 \otimes_{\mu} \nu_2}, \|S_2\|_{p, sep, \nu_1 \otimes_{\mu} \nu_2}\} : \right. \\ &\quad \left. S_1, S_2 \in L_{\mathcal{H}_1 \otimes \mathcal{H}_2}^p(X, \nu_1 \otimes_{\mu} \nu_2), S_1, S_2 \geq 0, \begin{bmatrix} S_1(x) & f(x) \\ f(x)^* & S_2(x) \end{bmatrix} \geq 0 \text{ a.e.} \right\}. \end{aligned}$$

Using these we can establish a version of Hölder's inequality.

Theorem 4.5. *If $f \in L^p_{\mathcal{H}}(X, \nu)$ and $g \in L^q_{\mathcal{H}}(X, \nu)$ for $\frac{1}{p} + \frac{1}{q} = 1$ then*

$$\|f \otimes g\|_{1, sep, \nu \otimes_{\nu} \nu} \leq \|f\|_p \|g\|_q$$

and

$$\|f \otimes g\|_{1, sep, dec} \leq \|f\|_{p, dec} \|g\|_{q, dec}.$$

Proof. First, if f and g are positive and $s_1, s_2 \in \mathcal{S}(\mathcal{H})$ then

$$\begin{aligned} \|(f \otimes g)_{s_1 \otimes s_2}\|_1 &= \int_X \text{Tr} \left(\left(s_1 \frac{d\nu}{d\nu_\rho}^{1/2} f \frac{d\nu}{d\nu_\rho}^{1/2} \right) \otimes \left(s_2 \frac{d\nu}{d\nu_\rho}^{1/2} g \frac{d\nu}{d\nu_\rho}^{1/2} \right) \right) d\nu_\rho \\ &= \int_X \text{Tr} \left(s_1 \frac{d\nu}{d\nu_\rho}^{1/2} f \frac{d\nu}{d\nu_\rho}^{1/2} \right) \text{Tr} \left(s_2 \frac{d\nu}{d\nu_\rho}^{1/2} g \frac{d\nu}{d\nu_\rho}^{1/2} \right) d\nu_\rho \\ &= \|f_{s_1} g_{s_2}\|_1 \\ &\leq \|f_{s_1}\|_p \|g_{s_2}\|_q \\ &\leq \|f\|_p \|g\|_q, \end{aligned}$$

using the classic Hölder's inequality. Now suppose that $s_{i,j} \in \mathcal{S}(\mathcal{H}), i = 1, 2, 1 \leq j \leq n$ and $\lambda_j \geq 0$ such that $\sum_{j=1}^n \lambda_j = 1$. If $s = \sum_{j=1}^n \lambda_j (s_{1,j} \otimes s_{2,j})$ then $s \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H})_{sep}$ and

$$\begin{aligned} \|(f \otimes g)_s\|_1 &= \int_X (f \otimes g)_s d\nu_\rho \\ &= \sum_{j=1}^n \lambda_j \|(f \otimes g)_{s_{1,j} \otimes s_{2,j}}\|_1 \\ &\leq \sum_{j=1}^n \lambda_j \|f\|_p \|g\|_q \\ &= \|f\|_p \|g\|_q. \end{aligned}$$

It is straightforward that if $s_k \rightarrow s$ in $\mathcal{S}(\mathcal{H})$ then $f_{s_k} \rightarrow f_s$ in $L^1(X, \nu_\rho)$. Hence, for all $s \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H})_{sep}$

$$\|(f \otimes g)_s\|_1 \leq \|f\|_p \|g\|_q.$$

Secondly, suppose $(f_1, f_2, f_3, f_4) \in Pos_f$ and $(g_1, g_2, g_3, g_4) \in Pos_g$. As elsewhere, we have

$$\begin{aligned} \|f \otimes g\|_{1, sep, \nu \otimes_{\nu} \nu} &\leq \|(f_1 + f_2 + f_3 + f_4) \otimes (g_1 + g_2 + g_3 + g_4)\|_{1, sep, \nu \otimes_{\nu} \nu} \\ &= \sup_{s \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H})_{sep}} \|((f_1 + f_2 + f_3 + f_4) \otimes (g_1 + g_2 + g_3 + g_4))_s\|_1 \\ &\leq \|f_1 + f_2 + f_3 + f_4\|_p \|g_1 + g_2 + g_3 + g_4\|_q. \end{aligned}$$

Taking the infimum over all possible combinations gives the first desired result.

Lastly, for general f and g suppose that $S_1, S_2 \geq 0$ in $L^p_{\mathcal{H}}(X, \nu)$, $T_1, T_2 \geq 0$ in $L^q_{\mathcal{H}}(X, \nu)$ such that

$$\begin{bmatrix} S_1 & f \\ f^* & S_2 \end{bmatrix} \geq 0 \quad \text{and} \quad \begin{bmatrix} T_1 & g \\ g^* & T_2 \end{bmatrix} \geq 0.$$

This implies that

$$\begin{bmatrix} S_1 \otimes T_1 & f \otimes g \\ (f \otimes g)^* & S_2 \otimes T_2 \end{bmatrix} \geq 0.$$

Thus,

$$\begin{aligned} \|f \otimes g\|_{1,sep,dec} &\leq \max\{\|S_1 \otimes T_1\|_{1,sep,\nu \otimes_{\nu_\rho} \nu}, \|S_2 \otimes T_2\|_{1,sep,\nu \otimes_{\nu_\rho} \nu}\} \\ &\leq \max\{\|S_1\|_p \|T_1\|_q, \|S_2\|_p \|T_2\|_q\} \\ &\leq \max\{\|S_1\|_p, \|S_2\|_p\} \max\{\|T_1\|_q, \|T_2\|_q\} \end{aligned}$$

and taking infimums over all such S_1, S_2, T_1, T_2 leads to the second desired inequality. \square

In [15] it was shown that $L_{\mathcal{H}}^\infty(X, \nu)$ functions are not bounded multipliers of $L_{\mathcal{H}}^1(X, \nu)$ in general. Using the tensor product gets around this under invertibility and boundedness conditions on the Radon-Nikodým derivative.

Theorem 4.6. *Suppose $\frac{d\nu}{d\nu_\rho} \in L_{\mathcal{H}}^\infty(X, \nu)$ and $\frac{d\nu}{d\nu_\rho}(x)$ is invertible a.e. If $f \in L_{\mathcal{H}}^1(X, \nu)$ and $g \in L_{\mathcal{H}}^\infty(X, \nu)$ then $f \otimes g \in L_{\mathcal{H}}^1(X, \nu \otimes_{\nu_\rho} \nu)$ such that*

$$\|f \otimes g\|_1 \leq 2 \left\| \frac{d\nu}{d\nu_\rho} \right\|_\infty \|f\|_1 \|g\|_\infty$$

and

$$\|f \otimes g\|_{1,dec} \leq \left\| \frac{d\nu}{d\nu_\rho} \right\|_\infty \|f\|_{1,dec} \|g\|_\infty.$$

Proof. This uses calculations very similar to [15, Proposition 3.13]. Let $(f_1, f_2, f_3, f_4) \in Pos_f$. If g is self-adjoint then its positive and negative parts are also essentially bounded. Moreover,

$$\begin{aligned} \|f \otimes g\|_1 &= \|(f_1 - f_2 + i(f_3 - f_4)) \otimes (g_+ - g_-)\|_1 \\ &= \left\| f_1 \otimes g_+ + f_2 \otimes g_- - f_1 \otimes g_- - f_2 \otimes g_+ \right. \\ &\quad \left. + i(f_3 \otimes g_+ + f_4 \otimes g_- - f_3 \otimes g_- - f_4 \otimes g_+) \right\|_1 \\ &\leq \left\| \int_X f_1 \otimes g_+ + f_2 \otimes g_- + f_1 \otimes g_- + f_2 \otimes g_+ \right. \\ &\quad \left. + f_3 \otimes g_+ + f_4 \otimes g_- + f_3 \otimes g_- + f_4 \otimes g_+ d\nu \otimes_{\nu_\rho} \nu \right\| \\ &= \left\| \int_X (f_1 + f_2 + f_3 + f_4) \otimes (g_+ + g_-) d\nu \otimes_{\nu_\rho} \nu \right\| \\ &= \left\| \int_X (f_1 + f_2 + f_3 + f_4) \otimes |g| d\nu \otimes_{\nu_\rho} \nu \right\|. \end{aligned}$$

By [15, Lemma 3.8] this last equation is equal to

$$\begin{aligned} &\left\| \int_X \frac{d\nu}{d\nu_\rho}^{1/2} (f_1 + f_2 + f_3 + f_4) \frac{d\nu}{d\nu_\rho}^{1/2} \otimes \frac{d\nu}{d\nu_\rho}^{1/2} |g| \frac{d\nu}{d\nu_\rho}^{1/2} d\nu_\rho I_{\mathcal{H} \otimes \mathcal{H}} \right\| \\ &\leq \left\| \int_X \frac{d\nu}{d\nu_\rho}^{1/2} (f_1 + f_2 + f_3 + f_4) \frac{d\nu}{d\nu_\rho}^{1/2} \otimes \left\| \frac{d\nu}{d\nu_\rho} \right\|_\infty \|g\|_\infty I_{\mathcal{H}} d\nu_\rho I_{\mathcal{H} \otimes \mathcal{H}} \right\| \\ &= \left\| \int_X \frac{d\nu}{d\nu_\rho}^{1/2} (f_1 + f_2 + f_3 + f_4) \frac{d\nu}{d\nu_\rho}^{1/2} d\nu_\rho I_{\mathcal{H}} \right\| \left\| \frac{d\nu}{d\nu_\rho} \right\|_\infty \|g\|_\infty \end{aligned}$$

$$= \left\| \int_X f_1 + f_2 + f_3 + f_4 d\nu \right\| \left\| \frac{d\nu}{d\nu_\rho} \right\|_\infty \|g\|_\infty$$

By taking the infimum over Pos_f we get that $\|f \otimes g\|_1 \leq \left\| \frac{d\nu}{d\nu_\rho} \right\|_\infty \|f\|_1 \|g\|_\infty$.

For arbitrary g the triangle inequality gives

$$\begin{aligned} \|f \otimes g\|_1 &\leq \|f \otimes Re g\|_1 + \|f \otimes Im g\|_1 \\ &\leq \left\| \frac{d\nu}{d\nu_\rho} \right\|_\infty \|f\|_1 \|Re g\|_\infty + \left\| \frac{d\nu}{d\nu_\rho} \right\|_\infty \|f\|_1 \|Im g\|_\infty \\ &\leq 2 \left\| \frac{d\nu}{d\nu_\rho} \right\|_\infty \|f\|_1 \|g\|_\infty. \end{aligned}$$

For the decomposable norm suppose $S_1, S_2 \in L^1_{\mathcal{H}}(X, \nu)$ with $S_1, S_2 \geq 0$ and $\begin{bmatrix} S_1 & f \\ f^* & S_2 \end{bmatrix} \geq 0$. Copying the ideas of Lemma 3.6 we get that

$$\begin{bmatrix} S_1 \otimes |g^*| & f \otimes g \\ f^* \otimes g^* & S_2 \otimes |g| \end{bmatrix} \geq 0.$$

This implies that by using the previous arguments

$$\begin{aligned} \|f \otimes g\|_{1, dec} &\leq \max\{\|S_1 \otimes |g^*|\|_1, \|S_2 \otimes |g|\|_1\} \\ &\leq \max\left\{\|S_1\|_1 \left\| \frac{d\nu}{d\nu_\rho} \right\|_\infty \| |g^*| \|_\infty, \|S_2\|_1 \left\| \frac{d\nu}{d\nu_\rho} \right\|_\infty \| |g| \|_\infty\right\} \\ &= \max\{\|S_1\|_1, \|S_2\|_1\} \left\| \frac{d\nu}{d\nu_\rho} \right\|_\infty \|g\|_\infty. \end{aligned}$$

Taking the infimum over all possible S_1, S_2 yields the desired result. \square

For comparison, we end this section with a variation of Hölder's inequality with the Schatten type norm. However, it is a tad unsatisfying as one cannot make the left-hand side behave itself.

Lemma 4.7. *If $f, g \in \mathcal{L}_{\mathcal{H}}(X, \nu)$ and $\frac{1}{p} + \frac{1}{q} = 1$ then*

$$\sup_{s \in \mathcal{S}(\mathcal{H})} \left\| \left\| s^{1/2} \frac{d\nu}{d\nu_\rho}^{1/2} f \frac{d\nu}{d\nu_\rho}^{1/2} s \frac{d\nu}{d\nu_\rho}^{1/2} g \frac{d\nu}{d\nu_\rho}^{1/2} s^{1/2} \right\| \right\|_{S^1} \leq \|f\|_{S^p, L^p} \|g\|_{S^q, L^q}$$

Proof. First label $f' = s^{1/2} \frac{d\nu}{d\nu_\rho}^{1/2} f \frac{d\nu}{d\nu_\rho}^{1/2} s^{1/2}$ and $g' = s^{1/2} \frac{d\nu}{d\nu_\rho}^{1/2} g \frac{d\nu}{d\nu_\rho}^{1/2} s^{1/2}$. By Young's inequality for the trace norm, where $A, B > 0$,

$$\frac{\text{Tr}(|f'(x)g'(x)|)}{AB} \leq \frac{\text{Tr}(|f'(x)|^p)}{A^p p} + \frac{\text{Tr}(|g'(x)|^q)}{B^q q},$$

for every $x \in X$. Hence, for $A = (\int_X \text{Tr}(|f'|^p) d\nu_\rho)^{1/p}$ and $B = (\int_X \text{Tr}(|g'|^q) d\nu_\rho)^{1/q}$ one has

$$\begin{aligned} \int_X \frac{\text{Tr}(|f'g'|)}{AB} d\nu_\rho &\leq \int_X \frac{\text{Tr}(|f'|^p)}{A^p p} + \frac{\text{Tr}(|g'|^q)}{B^q q} d\nu_\rho \\ &= \frac{1}{p} + \frac{1}{q} = 1. \end{aligned}$$

Therefore, the conclusion follows. \square

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