# The Hyperrigidity of Tensor Algebras of C*-Correspondences

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# THE HYPERRIGIDITY OF TENSOR ALGEBRAS OF C*-CORRESPONDENCES

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Abstract. Given a C*-correspondence $X$, we give necessary and sufficient conditions for the tensor algebra $\mathcal{T}_X^+$ to be hyperrigid. In the case where $X$ is coming from a topological graph we obtain a complete characterization.

## 1. Introduction

A not necessarily unital operator algebra $\mathcal{A}$ is said to be *hyperrigid* if given any non-degenerate $*$-homomorphism

$$\tau\colon \mathrm{C}^*_{\mathrm{env}}(\mathcal{A}) \longrightarrow B(\mathcal{H})$$

then $\tau$ is the only completely positive, completely contractive extension of the restricted map $\tau_{|\mathcal{A}}$. Arveson coined the term hyperrigid in [**1**] but he was not the only one considering properties similar to this at the time, e.g. [**4**].

There are many examples of hyperrigid operator algebras such as those which are Dirichlet but the situation was not very clear in the case of tensor algebras of C*-correspondences. It was known that the tensor algebra of a row-finite graph is hyperrigid [**4**], [**5**] and Dor-On and Salmomon [**3**] showed that row-finiteness completely characterizes hyperrigidity for such graph correspondences. These approaches, while successful, did not lend themselves to a more general characterization.

The authors, in a previous work [**11**], developed a sufficient condition for hyperrigidity in tensor algebras. In particular, if Katsura's ideal acts non-degenerately on the left then the tensor algebra is hyperrigid. The motivation was to provide a large class of hyperrigid C*-correspondence examples as crossed products of operator algebras behave in a very nice manner when the operator algebra is hyperrigid. This theory was in turn leveraged to provide a positive confirmation to the Hao-Ng isomorphism problem in the case of graph correspondences and arbitrary groups. For further reading on the subject please see [**9, 10, 11**].

In this paper, we provide a necessary condition for the hyperrigidity of a tensor algebra, that a C*-correspondence cannot be $\sigma$-degenerate, and show

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that this completely characterizes the situation where the C*-correspondence is coming from a topological graph, which generalizes both the graph correspondence case and the semicrossed product arising from a multivariable dynamical system.

1.1. **Regarding hyperrigidity.** The reader familiar with the literature recognizes that in our definition of hyperrigidity, we are essentially asking that the restriction on $\mathcal{A}$ of any non-degenerate representation of $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$ possesses the *unique extension property* (abbr. UEP). According to [**3**, Proposition 2.4] a representation $\rho : \mathcal{A} \to B(\mathcal{H})$, degenerate or not, has the UEP if and only if $\rho$ is a maximal representation of $\mathcal{A}$, i.e., whenever $\pi$ is a representation of $\mathcal{A}$ dilating $\rho$, then $\pi = \rho \oplus \pi'$ for some representation $\pi'$. Our definition of hyperrigidity is in accordance with Arveson's nomenclature [**1**], our earlier work [**7, 11**] and the works of Dor-On and Salomon [**3**] and Salomon [**16**], who systematized quite nicely the non-unital theory.

An alternative definition of hyperrigidity for $\mathcal{A}$ may ask that *any* representation of $\mathrm{C}^*_{\mathrm{env}}(\mathcal{A})$, not just the non-degenerate ones, possesses the UEP when restricted on $\mathcal{A}$. It turns out that for operator algebras with a positive contractive approximate unit[1], such a definition would be equivalent to ours [**16**, Proposition 3.6 and Theorem 3.9] . However when one moves beyond operator algebras with an approximate unit, there are examples to show that the two definitions differ. One such example is the non-unital operator algebra $\mathcal{A}_V$ generated by the unilateral forward shift $V$. It is easy to see that $\mathcal{A}_V$ is hyperrigid according to our definition and yet the zero map, as a representation on $\mathcal{H} = \mathbb{C}$, does not have the UEP. (See for instance [**16**, Example 3.4].)

## 2. Main results

A C*-correspondence $(X, \mathcal{C}, \varphi_X)$ (often just $(X, \mathcal{C})$) consists of a C*-algebra $\mathcal{C}$, a Hilbert $\mathcal{C}$-module $(X, \langle\ ,\ \rangle)$ and a (non-degenerate) $*$-homomorphism $\varphi_X : \mathcal{C} \to \mathcal{L}(X)$ into the C*-algebra of adjointable operators on $X$.

An isometric (Toeplitz) representation $(\rho, t, \mathcal{H})$ of a C*-correspondence $(X, \mathcal{C})$ consists of a non-degenerate $*$-homomorphism $\rho\colon \mathcal{C} \to B(\mathcal{H})$ and a linear map $t\colon X \to B(\mathcal{H})$, such that

$$\rho(c)t(x) = t(\varphi_X(c)(x)), \quad \text{and}$$
$$t(x)^*t(x') = \rho(\langle x, x' \rangle),$$

for all $c \in \mathcal{C}$ and $x, x' \in X$. These relations imply that the C*-algebra generated by this isometric representation equals the closed linear span of

$$t(x_1)\cdots t(x_n)t(y_1)^*\cdots t(y_m)^*, \quad x_i, y_j \in X.$$

Moreover, there exists a $*$-homomorphism $\psi_t : \mathcal{K}(X) \to B$, such that

$$\psi_t(\theta_{x,y}) = t(x)t(y)^*,$$

[1]which includes all operator algebras appearing in this paper

where $\mathcal{K}(X) \subset \mathcal{L}(X)$ is the subalgebra generated by the operators $\theta_{x,y}(z) = x\langle y, z\rangle,\ x, y, x \in X$, which are called by analogy the compact operators.

The Cuntz-Pimsner-Toeplitz C*-algebra $\mathcal{T}_X$ is defined as the C*-algebra generated by the image of $(\rho_\infty, t_\infty)$, the universal isometric representation. This is universal in the sense that for any other isometric representation there is a $*$-homomorphism of $\mathcal{T}_X$ onto the C*-algebra generated by this representation in the most natural way.

The *tensor algebra* $\mathcal{T}_X^+$ of a C*-correspondence $(X, \mathcal{C})$ is the norm-closed subalgebra of $\mathcal{T}_X$ generated by $\rho_\infty(\mathcal{C})$ and $t_\infty(X)$. See [**14**] for more on these constructions.

Consider Katsura's ideal

$$\mathcal{J}_X \equiv \ker \varphi_X^\perp \cap \varphi_X^{-1}(\mathcal{K}(X)).$$

An isometric representation $(\rho, t)$ of $(X, \mathcal{C}, \varphi_X)$ is said to be covariant (or Cuntz-Pimsner) if and only if

$$\psi_t(\varphi_X(c)) = \rho(c),$$

for all $c \in \mathcal{J}_X$. The Cuntz-Pimsner algebra $\mathcal{O}_X$ is the universal C*-algebra for all isometric covariant representations of $(X, \mathcal{C})$, see [**13**] for further details. Furthermore, the first author and Kribs [**8**, Lemma 3.5] showed that $\mathcal{O}_X$ contains a completely isometric copy of $\mathcal{T}_X^+$ and $\mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_X^+) \simeq \mathcal{O}_X$.

We turn now to the hyperrigidity of tensor algebras. In [**11**] a sufficient condition for hyperrigidity was developed, Katsura's ideal acting non-degenerately on the left of $X$. To be clear, non-degeneracy here means that $\overline{\varphi_X(\mathcal{J}_X)X} = X$ which by Cohen's factorization theorem implies that we actually have $\varphi_X(\mathcal{J}_X)X = X$.

**Theorem 2.1** (Theorem 3.1, [**11**])**.** *Let $(X, \mathcal{C})$ be a C*-correspondence with $X$ countably generated as a right Hilbert $\mathcal{C}$-module. If $\varphi_X(\mathcal{J}_X)$ acts non-degenerately on $X$, then $\mathcal{T}_X^+$ is a hyperrigid operator algebra.*

The proof shows that if $\tau' \colon \mathcal{O}_X \longrightarrow B(\mathcal{H})$ is a completely contractive and completely positive map that agrees with a $*$-homomorphism of $\mathcal{O}_X$ on $\mathcal{T}_X^+$ then the multiplicative domain of $\tau'$ must be everything. This is accomplished through the multiplicative domain arguments of [**2**, Proposition 1.5.7] and the fact that by $X$ being countably generated, Kasparov's Stabilization Theorem implies the existence of a sequence $\{x_n\}_{n=1}^\infty$ in $X$ so that $\sum_{n=1}^k \theta_{x_n,x_n}$, $k = 1, 2, \ldots$, is an approximate unit for $\mathcal{K}(X)$. After quite a lot of inequality calculations one arrives at the fact that all of $\mathcal{T}_X^+$ is in the multiplicative domain and thus so is $\mathcal{O}_X$.

A C*-correspondence $(X, \mathcal{C})$ is called *regular* if and only if $\mathcal{C}$ acts faithfully on $X$ by compact operators, i.e., $\mathcal{J}_X = \mathcal{C}$. We thus obtain the following which also appeared in [**11**].

**Corollary 2.2.** *The tensor algebra of a regular, countably generated C*-correspondence is necessarily hyperrigid.*

We seek a converse to Theorem 2.1.

**Definition 2.3.** Let $(X, \mathcal{C})$ be a C*-correspondence and let $\mathcal{J}_X$ be Katsura's ideal. We say that $\varphi_X(\mathcal{J}_X)$ acts *$\sigma$-degenerately* on $X$ if there exists a representation $\sigma \colon \mathcal{C} \to B(\mathcal{H})$ so that

$$\varphi_X(\mathcal{J}_X) X \otimes_\sigma \mathcal{H} \neq X \otimes_\sigma \mathcal{H}.$$

**Remark 2.4.** In particular, if there exists $n \in \mathbb{N}$ so that

$$(\varphi_X(\mathcal{J}_X) \otimes \mathrm{id}) X^{\otimes n} \otimes_\sigma \mathcal{H} \neq X^{\otimes n} \otimes_\sigma \mathcal{H}.$$

then by considering the Hilbert space $\mathcal{K} := X^{\otimes n-1} \otimes_\sigma \mathcal{H}$, we see that

$$\varphi_X(\mathcal{J}_X) X \otimes_\sigma \mathcal{K} \neq X \otimes_\sigma \mathcal{K}.$$

and so $\varphi_X(\mathcal{J}_X)$ acts $\sigma$-degenerately on $X$.

The following gives a quick example of a $\sigma$-degenerate action. Note that this is possibly stronger than having a not non-degenerate action.

**Proposition 2.5.** *Let $(X, \mathcal{C})$ be a* C*-*correspondence. If $(\varphi_X(\mathcal{J}_X)X)^\perp \neq \{0\}$, then $\varphi_X(\mathcal{J}_X)$ acts $\sigma$-degenerately on $X$.*

**Proof.** Let $0 \neq f \in (\varphi_X(\mathcal{J}_X)X)^\perp$. Let $\sigma : \mathcal{C} \to B(\mathcal{H})$ be a $*$-representation and $h \in \mathcal{H}$ so that $\sigma\big(\langle f, f\rangle^{1/2}\big)h \neq 0$. Then,

$$\langle f \otimes_\sigma h, f \otimes_\sigma h\rangle = \langle h, \sigma((\langle f, f\rangle)h\rangle = \|\sigma\big(\langle f, f\rangle^{1/2}\big)h\| \neq 0.$$

A similar calculation shows that

$$0 \neq f \otimes_\sigma h \in (\varphi_X(\mathcal{J}_X) X \otimes_\sigma \mathcal{H})^\perp$$

and we are done. ∎

We need the following

**Lemma 2.6.** *Let $(X, \mathcal{C})$ be a* C*-*correspondence and $(\rho, t)$ an isometric representation of $(X, \mathcal{C})$ on $\mathcal{H}$.*

(i) *If $\mathcal{M} \subseteq \mathcal{H}$ is an invariant subspace for $(\rho \rtimes t)(\mathcal{T}_X^+)$, then the restriction $(\rho_{|\mathcal{M}}, t_{|\mathcal{M}})$ of $(\rho, t)$ on $\mathcal{M}$ is an isometric representation.*

(ii) *If $\rho(c)h = \psi_t(\varphi_X(c))h$, for all $c \in \mathcal{J}_X$ and $h \in [t(X)\mathcal{H}]^\perp$, then $(\rho, t)$ is a Cuntz-Pimsner representation.*

**Proof.** (i) If $p$ is the orthogonal projection on $\mathcal{M}$, then $p$ commutes with $\rho(\mathcal{C})$ and so $\rho_{|\mathcal{M}}(\cdot) = p\rho(\cdot)p$ is a $*$-representation of $\mathcal{C}$.

Furthermore, for $x, y \in X$, we have

$$\begin{aligned} t_{|\mathcal{M}}(x)^* t_{|\mathcal{M}}(y) &= pt(x)^* pt(y)p \\ &= pt(x)^* t(y)p \\ &= p\rho(\langle x, y\rangle)p = \rho_{|\mathcal{M}}(\langle x, y\rangle) \end{aligned}$$

and the conclusion follows.

(ii) It is easy to see on rank-one operators and therefore by linearity and continuity on all compact operators $K \in \mathcal{K}(X)$ that

$$t(Kx) = \psi_t(K)t(x), \quad x \in X.$$

Now if $c \in \mathcal{J}_X$, then for any $x \in X$ and $h \in \mathcal{H}$ we have

$$\rho(c)t(x)h = t(\varphi_X(c)x)h = \psi_t(\varphi_X(c))t(x)h.$$

By assumption $\rho(c)h = \psi_t(\varphi_X(c))h$, for any $h \in [t(X)\mathcal{H}]^\perp$ and the conclusion follows. ∎

**Theorem 2.7.** *Let $(X, \mathcal{C})$ be a $\mathrm{C}^*$-correspondence. If Katsura's ideal $\mathcal{J}_X$ acts $\sigma$-degenerately on $X$ then the tensor algebra $\mathcal{T}_X^+$ is not hyperrigid.*

**Proof.** Let $\sigma\colon \mathcal{C} \to B(\mathcal{H})$ so that

$$\varphi_X(\mathcal{J}_X)X \otimes_\sigma \mathcal{H} \neq X \otimes_\sigma \mathcal{H}$$

and let $\mathcal{M}_0 := (\varphi_X(\mathcal{J}_X)X \otimes_\sigma \mathcal{H})^\perp$.

We claim that

$$(\varphi_X(\mathcal{J}_X) \otimes I)\mathcal{M}_0 = \{0\}. \tag{1}$$

Indeed for any $f \in \mathcal{M}_0$ and $j \in \mathcal{J}_X$ we have

$$\big\langle (\varphi_X(j) \otimes I)f \, , \, (\varphi_X(j) \otimes I)f \big\rangle = \langle f, (\varphi_X(j^*j) \otimes I)f \rangle = 0$$

since $f \in (\varphi_X(\mathcal{J}_X)X \otimes_\sigma \mathcal{H})^\perp$. This proves the claim.

We also claim that

$$(\varphi_X(\mathcal{C}) \otimes I)\mathcal{M}_0 = \mathcal{M}_0. \tag{2}$$

Indeed this follows from the fact that

$$(\varphi_X(\mathcal{C}) \otimes I)(\varphi_X(\mathcal{J}_X)X \otimes_\sigma \mathcal{H}) = \varphi_X(\mathcal{J}_X)X \otimes_\sigma \mathcal{H},$$

which is easily verified.

Using the subspace $\mathcal{M}_0$ we produce a Cuntz-Pimsner representation $(\rho, t)$ of $(X, \mathcal{C})$ as follows. Let $(\rho_\infty, t_\infty)$ be the universal representation of $(X, \mathcal{C})$ on the Fock space $\mathcal{F}(X) = \oplus_{n=0}^\infty X^{\otimes n}$, $X^{\otimes 0} := \mathcal{C}$. Let

$$\begin{aligned} \rho_0\colon \mathcal{C} &\longrightarrow B(\mathcal{F}(X) \otimes_\sigma \mathcal{H}); c \longmapsto \rho_\infty(c) \otimes I \\ t_0\colon X &\longrightarrow B(\mathcal{F}(X) \otimes_\sigma \mathcal{H}); x \longmapsto t_\infty(x) \otimes I. \end{aligned}$$

Define

$$\begin{aligned} \mathcal{M} :&= 0 \oplus \mathcal{M}_0 \oplus (X \otimes \mathcal{M}_0) \oplus (X^{\otimes 2} \otimes \mathcal{M}_0) \oplus \dots \\ &= (\rho_0 \rtimes t_0)(\mathcal{T}_X^+)(0 \oplus \mathcal{M}_0 \oplus 0 \oplus 0 \oplus \dots) \subseteq \mathcal{F}(X) \otimes_\sigma \mathcal{H}, \end{aligned}$$

with the second equality following from (2). Clearly, $\mathcal{M}$ is an invariant subspace for $(\rho_0 \rtimes t_0)(\mathcal{T}_X^+)$.

Let $\rho := \rho_{0|_\mathcal{M}}$ and $t := t_{0|_\mathcal{M}}$. By Lemma 2.6(i), $(\rho, t)$ is a representation of $(X, \mathcal{C})$. We claim that $(\rho, t)$ is actually Cuntz-Pimsner.

Indeed by Lemma 2.6(ii) it suffices to examine whether $\psi_t(\varphi_X(j))h = \rho(j)h$, for any $h \in \mathcal{M} \ominus t(X)\mathcal{M}$. Note that since

$$t(X)\mathcal{M} = 0 \oplus 0 \oplus (X \otimes \mathcal{M}_0) \oplus (X^{\otimes 2} \otimes \mathcal{M}_0) \oplus ...,$$

we have that

$$\mathcal{M} \ominus t(X)\mathcal{M} = 0 \oplus \mathcal{M}_0 \oplus 0 \oplus 0 \oplus \dots .$$

From this it follows that for any $h \in \mathcal{M} \ominus t(X)\mathcal{M}$ we have

$$t_0(x)^* h \in (\mathcal{C} \otimes_\sigma \mathcal{H}) \oplus 0 \oplus 0 \oplus ..., \quad x \in X$$

and so in particular for any $j \in \mathcal{J}_X$ we obtain

$$\psi_t(\varphi_X(j))h \in t_{0|_\mathcal{M}}(X)(t_{0|_\mathcal{M}})(X)^* h = \{0\}.$$

On the other hand,

$$\rho(j)h \in 0 \oplus (\varphi_X(\mathcal{J}_X) \otimes I)\mathcal{M}_0 \oplus 0 \oplus 0 \oplus \dots = \{0\},$$

because of (2). Hence $(\rho, t)$ is Cuntz-Pimsner.

At this point by restricting on $\mathcal{T}_X^+$, we produce the representation $\rho \rtimes t\mid_{\mathcal{T}_X^+}$ of $\mathcal{T}_X^+$ coming from a $*$-representation of its C*-envelope $\mathcal{O}_X$, which admits a dilation, namely $\rho_0 \rtimes t_0 \mid_{\mathcal{T}_X^+}$. If we show now that $\rho_0 \rtimes t_0 \mid_{\mathcal{T}_X^+}$ is a non-trivial dilation of $\rho \rtimes t\mid_{\mathcal{T}_X^+}$, i.e. $\mathcal{M}_0$ is not reducing for $(\rho_0 \rtimes t_0)(\mathcal{T}_X^+)$, then $\rho \rtimes t\mid_{\mathcal{T}_X^+}$ is not a maximal representation of $\mathcal{T}_X^+$. Proposition 2.4 **[3]** shows $\rho \rtimes t\mid_{\mathcal{T}_X^+}$ does not have the UEP and so $\mathcal{T}_X^+$ is not hyperrigid, as desired.

Towards this end, note that

$$\mathcal{M}^\perp = \mathcal{C} \oplus (\varphi_X(\mathcal{J}_X)X \otimes_\sigma H) \oplus (X \otimes \mathcal{M}_0)^\perp \oplus \dots$$

and so

$$t_0(X)\mathcal{M}^\perp = 0 \oplus (X\mathcal{C} \otimes_\sigma \mathcal{H}) \oplus 0 \oplus 0 \oplus \dots \nsubseteq \mathcal{M}^\perp$$

Therefore $\mathcal{M}^\perp$ is not an invariant subspace for $(\rho_0 \rtimes t_0)(\mathcal{T}_X^+)$ and so $\mathcal{M}$ is not a reducing subspace for $(\rho_0 \rtimes t_0)(\mathcal{T}_X^+)$. This completes the proof. ∎

## 3. Topological graphs

A broad class of C*-correspondences arises naturally from the concept of a topological graph. For us, a topological graph $G = (G^0, G^1, r, s)$ consists of two $\sigma$-locally compact spaces $G^0$, $G^1$, a continuous proper map $r : G^1 \to G^0$ and a local homeomorphism $s : G^1 \to G^0$. The set $G^0$ is called the base (vertex) space and $G^1$ the edge space. When $G^0$ and $G^1$ are both equipped with the discrete topology, we have a discrete countable graph.

With a given topological graph $G = (G^0, G^1, r, s)$ we associate a C*-correspondence $X_G$ over $C_0(G^0)$. The right and left actions of $C_0(G^0)$ on $C_c(G^1)$ are given by

$$(fFg)(e) = f(r(e))F(e)g(s(e))$$

for $F \in C_c(G^1)$, $f, g \in C_0(G^0)$ and $e \in G^1$. The inner product is defined for $F, H \in C_c(G^1)$ by

$$\langle F \mid H \rangle (v) = \sum_{e \in s^{-1}(v)} \overline{F(e)} H(e)$$

for $v \in G^0$. Finally, $X_G$ denotes the completion of $C_c(G^1)$ with respect to the norm

$$\|F\| = \sup_{v \in G^0} \langle F \mid F \rangle (v)^{1/2}. \tag{3}$$

When $G^0$ and $G^1$ are both equipped with the discrete topology, then the tensor algebra $\mathcal{T}_G^+ \equiv \mathcal{T}_{X_G}^+$ associated with $G$ coincides with the quiver algebra of Muhly and Solel [**14**]. See [**15**] for further reading.

Given a topological graph $G = (G^0, G^1, r, s)$, we can describe the ideal $\mathcal{J}_{X_G}$ as follows. Let

$$G^0_{\text{sce}} = \{v \in G^0 \mid v \text{ has a neighborhood } V \text{ such that } r^{-1}(V) = \emptyset\}$$
$$G^0_{\text{fin}} = \{v \in G^0 \mid v \text{ has a neighborhood } V \text{ such that } r^{-1}(V) \text{ is compact}\}$$

Both sets are easily seen to be open and in [**12**, Proposition 1.24] Katsura shows that

$$\ker \varphi_{X_G} = C_0(G^0_{\text{sce}}) \quad \text{and} \quad \varphi^{-1}_{X_G}(\mathcal{K}(X_G)) = C_0(G^0_{\text{fin}}).$$

From the above it is easy to see that $\mathcal{J}_{X_G} = C_0(G^0_{\text{reg}})$, where

$$G^0_{\text{reg}} := G^0_{\text{fin}} \backslash \overline{G^0_{\text{sce}}}.$$

We need the following

**Lemma 3.1.** *Let $G = (G^0, G^1, r, s)$ be a topological graph. Then $r^{-1}\big(G^0_{\text{reg}}\big) = G^1$ if and only if $r : G^1 \to G^0$ is a proper map satisfying $r(G^1) \subseteq \big(\overline{r(G^1)}\big)^{\circ}$.*

**Proof.** Notice that

$$r^{-1}(G^0_{\text{reg}}) = r^{-1}(G^0_{\text{fin}}) \cap r^{-1}(\overline{G^0_{\text{sce}}})^c$$

and so $r^{-1}\big(G^0_{\text{reg}}\big) = G^1$ is equivalent to $r^{-1}(G^0_{\text{fin}}) = r^{-1}(\overline{G^0_{\text{sce}}}) = G^1$

First we claim that $r^{-1}(G^0_{\text{fin}}) = G^1$ if and only if $r$ is a proper map. Indeed, assume that $r^{-1}(G^0_{\text{fin}}) = G^1$ and let $K \subseteq r(G^1)$ compact in the relative topology. For every $x \in K$, let $V_x$ be a compact neighborhood of $x$ such that $r^{-1}(V_x)$ is compact and so $r^{-1}(V_x \cap K))$ is also compact. By compactness, there exist $x_1, x_2, \ldots, x_n \in K$ so that $K = \cup_{i=1}^n (V_{x_i} \cap K)$ and so

$$r^{-1}(K) = \cup_{i=1}^n r^{-1}(V_{x_i} \cap K)$$

and so $r^{-1}(K)$ is compact.

Conversely, if $r$ is proper then any compact neighborhood V of any point in $G^0$ is inverted by $r^{-1}$ to a compact set and so $r^{-1}(G^0_{\text{fin}}) = G^1$.

We now claim that $r^{-1}(\overline{G^0_{\text{sce}}}) = \emptyset$ if and only $r(G^1) \subseteq \big(\overline{r(G^1)}\big)^{\circ}$.

Indeed, $e \in r^{-1}(\overline{G^0_{\rm sce}})$ is equivalent to $r(e) \in \overline{(r(G^1)^c)^\circ}$ and so $r^{-1}(\overline{G^0_{\rm sce}}) = \emptyset$ is equivalent to

$$r(G^1) \subseteq \left(\overline{(r(G^1)^c)^\circ}\right)^c = \left(\overline{r(G^1)}\right)^\circ,$$

as desired. ■

If $G = (G^0, G^1, r, s)$ is a topological graph and $S \subseteq G^1$, then $N(S)$ denotes the collection of continuous functions $F \in X_G$ with $F_{|S} = 0$, i.e., vanishing at $S$. The following appears as Lemma 4.3(ii) in **[6]**.

**Lemma 3.2.** *Let $G = (G^0, G^1, r, s)$ be a topological graph. If $S_1 \subseteq G^0$, $S_2 \subseteq G^1$ closed, then*

$$N(r^{-1}(S_1) \cup S_2) = \overline{\rm span}\{(f \circ r)F \mid f_{|S_1} = 0, F_{|S_2} = 0\}$$

**Theorem 3.3.** *Let $G = (G^0, G^1, r, s)$ be a topological graph and let $X_G$ the* C*-*correspondence associated with $G$. Then the following are equivalent*

(i) *the tensor algebra $\mathcal{T}^+_{X_G}$ is hyperrigid*
(ii) *$\varphi(\mathcal{J}_{X_G})$ acts non-degenerately on $X_G$*
(iii) *$r : G^1 \to G^0$ is a proper map satisfying $r(G^1) \subseteq \left(\overline{r(G^1)}\right)^\circ$*

**Proof.** If $\varphi(\mathcal{J}_{X_G})$ acts non-degenerately on $X_G$, then Theorem 2.1 shows that $\mathcal{T}^+_{X_G}$ is hyperrigid. Thus (ii) implies (i).

For the converse, assume that $\varphi(\mathcal{J}_{X_G})$ acts degenerately on $X_G$. If we verify that $\varphi(\mathcal{J}_{X_G})$ acts $\sigma$-degenerately on $X_G$, then Theorem 2.7 shows that $\mathcal{T}^+_{X_G}$ is not hyperrigid and so (i) implies (ii).

Towards this end note that $\mathcal{J}_{X_G} = \mathcal{C}_0(\mathcal{U})$ for some proper open set $\mathcal{U} \subseteq G^0$. (Actually we know that $\mathcal{U} = G^0_{\rm reg}$ but this is not really needed for this part of the proof!) Hence

$$\begin{aligned} \varphi(\mathcal{J}_{X_G})X_G &= \overline{\rm span}\{(f \circ r)F \mid f_{|\mathcal{U}^c} = 0\} \\ &= N(r^{-1}(\mathcal{U})^c), \end{aligned} \tag{4}$$

according to Lemma 3.2.

Since $\varphi(\mathcal{J}_{X_G})$ acts degenerately on $X_G$, (4) shows that $r^{-1}(\mathcal{U})^c \neq \emptyset$. Let $e \in r^{-1}(\mathcal{U})^c$ and let $F \in C_c(G^1) \subseteq X_G$ with $F(e) = 1$ and $F(e') = 0$, for any other $e' \in G^1$ with $s(e') = s(e)$. Consider the one dimensional representation $\sigma : C_0(G_0) \to \mathbb{C}$ coming from evaluation at $s(e)$. We claim that

$$\varphi_{X_G}(\mathcal{J}_{X_G})X_G \otimes_\sigma \mathbb{C} \neq X_G \otimes_\sigma \mathbb{C}.$$

Indeed for any $G \in \varphi(\mathcal{J}_{X_G})X_G = N(r^{-1}(\mathcal{U})^c)$ we have

$$\begin{aligned} \langle F \otimes_\sigma 1, G \otimes_\sigma 1\rangle &= \langle 1, \sigma(\langle F, G\rangle 1) = \langle F, G\rangle s(e) \\ &= \sum_{s(e')=s(e)} \overline{F(e')}G(e') \\ &= \overline{F(e)}G(e) = 0. \end{aligned}$$

Furthermore,

$$\langle F \otimes_\sigma 1, F \otimes_\sigma 1 \rangle s(e) = |F(e)|^2 = 1$$

and so $0 \neq F \otimes_\sigma 1 \in (\varphi_{X_G}(\mathcal{J}_{X_G})X_G \otimes_\sigma \mathbb{C})^\perp$. This establishes the claim and finishes the proof of (i) implies (ii).

Finally we need to show that (ii) is equivalent to (iii). Notice that (4) implies that $\varphi(\mathcal{J}_{X_G})$ acts degenerately on $X_G$ if and only if

$$r^{-1}(\mathcal{U})^c = r^{-1}(G^0_{\text{reg}})^c = \emptyset.$$

The conclusion now follows from Lemma 3.1. ∎

The statement of the previous Theorem takes its most pleasing form when $G^0$ is a compact space. In that case $\mathcal{T}_X^+$ is hyperrigid if and only is $G^1$ is compact and $r(G^1) \subseteq G^0$ is clopen.

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