Operator Algebras and Symbolic Dynamical Systems Associated to Symbolic Substitutions

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Abstract

In this thesis, we shall look at symbolic dynamical systems and operator algebras that are associated with these systems. We shall focus on minimal shift dynamical systems generated by symbolic substitutions. By first characterizing the shift spaces associated to primitive substitutions, we shall see that all minimal dynamical systems generated by symbolic substitutions are conjugate to proper primitive substitutions. In doing this, we will look at ordered Bratteli diagrams and strongly maximal TAF-algebras and we will see how we can associate these to a type of dynamical system called a Cantor minimal system, of which infinite minimal shift spaces are an example. We also shall see how we can associate a semi-crossed product algebra to a topological dynamical system and how isomorphism of two semi-crossed product algebras is equivalent to conjugacy of their associated dynamical systems. The semi-crossed product algebra is more general in that it can be associated to any dynamical system, whereas TAF-algebras can only be associated to Cantor minimal systems.

Contents

1. Introduction

The study of dynamical systems is fundamentally the study of how things change over time. Dynamical systems emerged as a sub-discipline of physics in the 1600s when Newton used differential equations to describe the motion of the planets. It has since become a sizeable mathematical discipline with applications in almost all areas of science. Applications include, but are not limited to, modelling how physical systems change over time, modelling chemical reactions, modelling changes in population, pricing of financial instruments, fractals and data storage in computers, to name a few. Dynamical systems unsurprisingly are mathematically vibrant and have many connections to other areas of mathematics. This includes, but again is not limited to, topology, measure theory, functional analysis, differential geometry and operator algebras. Dynamical systems fit into two main categories. Systems where we assume that time is continuous (such as differential equations) and systems where we assume time is discrete (iterated maps). Note that when we say time, this is a result of the dynamical systems originating as a way to model physical systems. In an abstract dynamical system, "time" may be measured by other objects, such as the complex numbers, causing time to lose its typical physical meaning. Additionally, in generalizations of dynamical systems, such as multivariable dynamical systems, a notion of time may not make sense.

This thesis will explore a type of discrete system called a symbolic dynamical system and its relationship to operator algebras. In particular, we shall focus on minimal symbolic dynamical systems. For us, a dynamical system consists of a pair (X, ϕ) where X is a locally compact Hausdorff space and ϕ is a proper continuous map. A symbolic dynamical system is a space consisting of either infinite or bi-infinite sequences of letters from a finite alphabet and a shift map that "shifts" elements of the sequence to the left by one position. Symbolic dynamical systems were introduced in the late 1800s to study continuous dynamical systems, with the first successful application being credited to Hadamard in 1898 [16]. To obtain a symbolic system from a continuous system, one would partition the space into a finite number of pieces and then associate to each piece a symbol. One would then get a "symbolic trajectory" by looking at how a point moves between these different pieces over time [33]. For a simple example of this, we can look at the motion of a pendulum swinging back and forth. We could assign the symbols L, C, and R, where L denotes when the pendulum swung to the left, C for when it is in the center and R for when it is swung to the right. If we assume no loss of energy, then a potential symbolic trajectory could look like $...LCRCL.CRCLCR...$ or $...LLCRRC.CLLCRR...$ where the "."

keeps track of the symbol at position zero. The shift map would then describe how the pendulum's position changes over discrete jumps in time. The first paper studying symbolic systems for their own sake was in [18] by Morse and Hedlund. Here they also coined the name. Since then, they have been studied extensively, finding many applications. Some of these applications outside of dynamical systems include computer science (think sequences of 0's and 1's), linear algebra, graph theory, and operator algebras [22].

A symbolic substitution is a function from a finite set of letters into the set of all finite words made from these letters. This function can naturally be extended to sequences of letters by applying the function to the individual letters and then concatenating the resulting words together to create a new sequence. Symbolic substitutions appear naturally in various areas of mathematics. Some examples include the study of formal languages (here, substitutions are typically called iterated morphisms. The name comes from the fact that we can think of substitutions as morphisms on a free monoid generated by a finite number of elements), automata theory, aperiodic order and fractal geometry [11]. To these substitutions, we can associate a symbolic dynamical system based on the words that appear by applying the substitution repeatedly to the letters in the alphabet. These systems tell us information about the substitution and vice-versa. The focus of this paper will be on systems generated from what are called primitive substitutions. Primitive substitutions are by far the most well-studied type of substitution. This, in part, is because the primitive substitutions have nice properties, allowing us to better characterize their shift spaces and providing extra tools for determining conjugacy between two shift spaces. Additionally, primitive substitutions are the easiest way to generate minimal shift spaces.

Operator algebras have been used extensively to study dynamical systems. To a dynamical system, there are various ways that one can associate an operator algebra, then using this algebra, one can determine properties of the dynamical system and vice-versa. Typically, one is interested in associating an algebra such that two algebras are isomorphic if and only if the associated dynamical systems are equivalent up to some notion of conjugacy (isomorphism for dynamical systems). The study of associating operator algebras to dynamical systems began with work done by F. J. Murray and J. von Neumann in [24]. Since then, it has grown substantially, with a large amount of research being devoted to studying C^*-algebras associated with dynamical systems. In more recent decades, the study of associating non-selfadjoint operator algebras such as tensor algebras has also seen much research. Non-selfadjoint operator algebras, tend to be able to encode more information about the dynamical system, as we shall see in an upcoming section. This thesis will focus on two types of algebras: semi-crossed product algebras and the strongly maximal TAF-algebras. J. Peters introduced the semi-crossed product algebra in [27], and he was able to show that the algebras associated to $(X, \phi), (Y, \psi)$ were isomorphic (as algebras) if and only if the dynamical systems were conjugate assuming the spaces were compact and there were no fixed points. This result was extended in [9] to all topological dynamical systems $(X, \phi), (Y, \psi)$. Strongly maximal TAF-algebras is a particular type of AF-subalgebra. Bratteli introduced AF-algebras in [4] and are defined as the inductive limit of finite-dimensional C^*-algebras. A strongly maximal TAF-algebra can only be associated with systems conjugate to Cantor minimal systems (minimal dynamical systems on a Cantor set) where the system's map is invertible. However for these dynamical systems, they can determine conjugacy. As we shall see, all minimal symbolic shifts that are not finite end up being conjugate to Cantor minimal systems.

This thesis has three main aims. First, we will introduce shift dynamical systems and symbolic substitutions. We will see how we can associate to a symbolic substitution a shift system. The focus here will be on primitive substitutions and characterizing their shift spaces. Second, we will see how we can associate to a dynamical system an operator algebra that can tell apart conjugacy. This will be done for both a general dynamical system and Cantor minimal systems, shift spaces of primitive substitutions being an example of the latter. Third, we shall show that all minimal shift spaces defined by a substitution are conjugate to the shift spaces defined by proper primitive substitutions (a special type of primitive substitution).

2. Shift Space

In this section, we shall briefly introduce topological dynamical systems. We shall then go on to introduce the concept of a shift space and shift dynamical systems. We will see how we can characterize shift spaces by looking at the list of all finite words which do occur or which do not occur inside the shift space. We will then look at the basic structure of conjugacies between shift spaces as well as some conjugacy invariants. We will end this section by looking at a few generalizations of shift spaces. Namely, we shall look at higher dimensional shift spaces and shift spaces defined over a countable alphabet, and we shall see some of the difficulties that occur in these situations.

2.1. **Dynamical Systems and Shift Spaces.**

Definition 2.1. *A one-variable **(topological) dynamical system** (X, ϕ) is a locally compact Hausdorff space X with a proper continuous map ϕ :*

$X \to X$, where proper means that the inverse image of a compact set is compact.

Definition 2.2. *Two one-variable dynamical systems $(X_1, \phi_1), (X_2, \phi_2)$ are said to be **conjugate** if there exists a homeomorphism $\tau : X_2 \to X_1$ such that $\tau \circ \phi_2 = \phi_1 \circ \tau$*

This is just the dynamical system version of isomorphic. Unsurprisingly, there is no general decision algorithm to determine if two dynamical systems are conjugate or not. To determine if two systems are conjugate, one often does this by finding an explicit conjugacy. If they are not conjugate, one is often forced to instead look at invariants of the system, where an invariant is any property that is preserved under conjugacy. One then concludes that the systems are not conjugate if they can find an invariant that differs. Later we shall briefly mention some of the common invariants that one uses for shift spaces, but for us, the invariants of focus will be the operator algebras mentioned in the previous section.

The primary type of dynamical system that we shall be concerned with in this thesis are minimal dynamical systems.

Definition 2.3. *A topological dynamical system (X, ϕ) is said to be **minimal** if for all closed subsets A of X, $\phi(A) \subseteq A$ implies that $A = X$ or $A = \emptyset$.*

The definition of minimality is quite intuitive. A dynamical system (X, ϕ) is minimal precisely when it has no proper subset that is also a dynamical system with respect to ϕ. Note that (X, ϕ) being minimal does not imply that (X, ϕ^n) is minimal for all $n \in \mathbb{N}$. If this second condition is satisfied, then (X, ϕ) is called totally minimal. A simple example of a minimal system that is not totally minimal is the pair $(\mathbb{Z}_2, \phi)$, where $\phi(a) = a + 1 \pmod 2$. Clearly ϕ^n is minimal for all odd n, but not minimal for all even n

Also, note that for minimal dynamical systems where ϕ is a homeomorphism, it does not follow that ϕ and ϕ^{-1} are conjugate. For instance, it was shown in [17] that if we let $K = \{z \in \mathbb{C} : |z| = 1\}$, $X = K^3$, and ϕ be the map $\phi(z_1, z_2, z_3) = a \cdot (z_1, z_1^3 z_2, z_1 z_2^3 z_3)$, were $a \in K^3$ and the first component of a is not a root of unity, then (X, ϕ) defines a minimal dynamical system and $(X, \phi), (X, \phi^{-1})$ are not conjugate. In this paper, the authors came up with necessary and sufficient conditions for a totally minimal homeomorphism of a compact abelian group with quasi-discrete spectrum to define a dynamical system conjugate to its inverse. Using this, they concluded that no conjugacy can exist between maps of the type described above.

Definition 2.4. *Let $\mathcal{A}$ be a set consisting of n letters $\{a_1, \ldots, a_n\}$. The set $\mathcal{A}^*$ consists of all finite words made from the alphabet $\mathcal{A}$. By convention,* *we call $\epsilon \in \mathcal{A}^*$ the empty word and say that it has length zero. $\mathcal{A}^+$ denotes the set of finite words minus the empty word ϵ. The set $\mathcal{A}^{\mathbb{Z}}$, consists of all bi-infinite sequences of letters from $\mathcal{A}$.*

For a finite word $u \in \mathcal{A}^+$ of length n, u_i denotes the i'th letter of u, where $0 \le i \le n-1$.

One can similarly define the space $\mathcal{A}^{\mathbb{N}}$. This space will have many analogous properties to $\mathcal{A}^{\mathbb{Z}}$, but for us, we shall only be interested in $\mathcal{A}^{\mathbb{Z}}$.

Definition 2.5 ([1]). *Let $w = \dots w_{-2}w_{-1}w_0w_1w_2\dots \in \mathcal{A}^{\mathbb{Z}}$ and let $w_{[k,l]}$ be the finite subword of w from position k to position l were $k, l \in \mathbb{Z}$, $l \le k$. Denote the number of letters in $w_{[k,l]}$ by $|w_{[k,l]}| = m$. A* ***cylinder set*** *of a finite word u of length m is defined to by $Z_k(u) = \{w \in \mathcal{A}^{\mathbb{Z}} \mid w_{[k,k+m-1]} = u\}$.*

Similarly, for $w \in \mathcal{A}^{\mathbb{Z}}$, $w_{[k,l)}$ denotes the finite subword of w from position k to position $l-1$ were $k < l$ and $w_{[k,\infty)}$ denotes the infinite subword of w from position k on wards.

The collection of all cylinder sets is a basis for $\mathcal{A}^{\mathbb{Z}}$. Moreover, the topology defined by this basis coincides with the product topology on $\mathcal{A}^{\mathbb{Z}}$, were $\mathcal{A}$ is given the discrete topology. As a result, by Tychonov's theorem, $\mathcal{A}^{\mathbb{Z}}$ is compact. There are two standard metrics that one can associate with this topology:

Theorem 2.6. *Let $\mathcal{T}$ be the topology defined above. Then the topologies defined by the following two metrics coincide with $\mathcal{T}$.*

1. $d_F(u,v) := \sum_{m \in \mathbb{N}} \frac{d(u[-m,m], v[-m,m])}{2^m}$ *where* $d(u[-m,m], v[-m,m]) = |\{i : u_i \ne v_i\}|$
2. $d(u,v) := 2^{-min\{|n| \in \mathbb{Z} : u_n \ne v_n\}}$

$\mathcal{T}$ and the metric in (1.) coinciding is explained in [1]. (1.) and (2.) coinciding follows almost immediately from their definitions. Intuitively, one can see these metrics coincide because two sequences are "close" to each other in these metrics if and only if they agree with each-other in a large region around the zero index. These metrics also provide a simple direct way to show that $\mathcal{A}^{\mathbb{Z}}$ is compact. $\mathcal{A}^{\mathbb{Z}}$ being complete with respect to either metric is essentially a proof by definition. Choosing the second metric, a sequence $\{x_m\}_{m \in \mathbb{N}}$ being Cauchy in $\mathcal{A}^{\mathbb{Z}}$ means that for any $N \in \mathbb{N}$, the tails of the sequence will agree for the first N letters around the zero index for sufficiently large m. Using this, we can construct our limit point. For compactness, note that the collection of all the cylinder sets $Z_{-n}(u)$ for all the finite words u of length $2n+1$ form a finite partition of $\mathcal{A}^{\mathbb{Z}}$. By letting $\epsilon > 0$ and choosing n so that $2^{-n} < \epsilon$, we get a covering of $\mathcal{A}^{\mathbb{Z}}$ by finitely many ϵ balls. Since $\mathcal{A}^{\mathbb{Z}}$ is complete and totally bounded, it is compact.

Definition 2.7 ([1])**.** *The operator* $S : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$*,* $S(w) = u$ *were* $w_{i+1} = u_i$ *is called the* ***shift operator****.*

The shift operator is plainly continuous and proper in our topology and it is the mapping that will be used to generate our dynamical system. The dynamical systems that we will be interested in are subsets $\mathcal{A}^{\mathbb{Z}}$ which are also closed with respect to the shift action.

Definition 2.8 ([1])**.** *Let* $\mathbb{X} \subset \mathcal{A}^{\mathbb{Z}}$*. We say that* $\mathbb{X}$ *is a* ***shift space*** *if it is a closed subset of* $\mathcal{A}^{\mathbb{Z}}$ *and it is invariant under the shift operator. Shift spaces are also called subshifts.*

A set that is invariant under the shift operator need not be closed in general. For instance, taking $\mathcal{A} = \{a, b\}$, if we let $\mathbb{X} = \{x^i \in \mathcal{A}^{\mathbb{Z}} : x^i_{[i,i]} = a$, and $x^i_{[j,j]} = b, \forall j \neq i \in \mathbb{Z}\}$, this is a shift invariant set, but it is not closed as the sequence *...bbb.bbb...* $\notin \mathbb{X}$, but *...bbb.bbb...* is clearly in its closure.

2.2. **Characterization of Shift Spaces and Shift Space Conjugacies.**

Definition 2.9 ([22])**.** *Let* $\mathcal{F}$ *be a collection of words over* $\mathcal{A}$*. Define* $\mathbb{X}_{\mathcal{F}}$ *to be the subset of* $\mathcal{A}^{\mathbb{Z}}$ *whose elements contains no finite subwords from* $\mathcal{F}$*.* $\mathcal{F}$ *is called the* ***forbidden words list*** *of* $\mathbb{X}_{\mathcal{F}}$*.*

$\mathbb{X}_{\mathcal{F}}$ is a shift space. It is clearly shift invariant and it can be seen to be closed since any sequence $x \in \mathcal{A}^{\mathbb{Z}}$ which contains a forbidden word, say $x_{[i,j]} = u \in \mathcal{F}$ will be a distance of at least $2^{-max\{|i|,|j|\}}$ from any element in X. These forbidden words lists can be at most countably infinite. Note that these lists are not unique. For instance, if $A = \{0, 1\}$ was our alphabet, and $\mathcal{F}_1 = \{1\}$, $\mathcal{F}_2 = \{01, 10, 11\}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ would both define the one element shift space $X = \{...000.000...\}$. Alternatively, we can define a shift space based on the allowed words instead.

Definition 2.10 ([22])**.** *Let* $\mathbb{X} \subseteq \mathcal{A}^{\mathbb{Z}}$ *be any subset. Let* $B_n(\mathbb{X})$ *denote the set of all words of length* n *inside* $\mathbb{X}$*. The* ***language*** *of* $\mathbb{X}$*,* $\mathcal{L}(\mathbb{X})$*, is defined to be* $\bigcup_{i=1}^{\infty} B_n(\mathbb{X})$*.* $\mathcal{L}^c(\mathbb{X})$ *is defined to be the set of all words which do not appear in* $\mathcal{L}(\mathbb{X})$*.*

If $\mathbb{X}$ was already a shift space, then the shift space defined by $\mathcal{L}(\mathbb{X})$, namely $\mathbb{X}_{\mathcal{L}^c(\mathbb{X})}$ is again $\mathbb{X}$. Not all languages define a shift space, but the following proposition characterizes the relationship between the two.

Proposition 2.11 ([22])**.** *Let* $\mathcal{A}$ *be an alphabet of n-letters and* $\mathbb{X}, \mathbb{Y} \subseteq \mathcal{A}^{\mathbb{Z}}$*. Then the following are true:*

1. *If* $\mathbb{X}$ *is a shift space and* $w \in \mathcal{L}(\mathbb{X})$ *then:*
   a. *Every subword of* w *also belongs to* $\mathcal{L}(\mathbb{X})$*.*

b. *There are nonempty words $u, v \in \mathcal{L}(\mathbb{X})$ such that $uwv \in \mathcal{L}(\mathbb{X})$.*

2. *If $\mathcal{L}(\mathbb{Y})$ is a language that satisfies conditions (a) and (b) in (1), then $\mathbb{Y}$ is a shift space.*
3. $\mathbb{X} = \mathbb{X}_{\mathcal{L}^c(\mathbb{X})}$.
4. *$\mathcal{L}^c(\mathbb{X})$ is the largest forbidden words list that determines $\mathbb{X}$.*

This proposition allows us to define shift spaces directly from languages $\mathcal{L}$ that satisfy (1). Here elements of the associated shift space are all the sequences such that the finite words that make up those sequences all belong to $\mathcal{L}$. Any shift space can be characterized by its language or its forbidden words list. There are other ways to characterize shift spaces that are less universal but more useful when available. We shall see an example of this using a fixed point criteria in an upcoming section.

We end this subsection by discussing the basic structure of conjugacy maps between shift spaces.

Definition 2.12 ([22])**.** *Let $x = ...x_{-1}x_0x_1...$ be an element in a shift space $\mathbb{X}$ over a finite alphabet $\mathcal{A}$ and let $y = ...y_{-1}y_0y_1$ be another sequence over a finite alphabet $\mathcal{B}$. Let $\mathfrak{B}_{m+n+1}(\mathbb{X})$ denote the set of all words from $\mathbb{X}$ of the form $x_{[i-m,i+n]}$ where $x \in \mathbb{X}$. A function $\Phi : \mathfrak{B}_{m+n+1}(\mathbb{X}) \to \mathcal{B}$ defined by $\Phi(x_{[i-m,i+n]}) = y_i$ where $y \in \mathcal{B}$ is called a **fixed block map** (or just block map).*

These fixed block maps will be the building blocks for conjugacy maps between shift spaces. We can extend block maps to maps into $\mathcal{B}^{\mathbb{Z}}$ with the following definition.

Definition 2.13 ([22])**.** *Let $\mathbb{X}$ be a shift space over a finite alphabet $\mathcal{A}$ and $\Phi : \mathfrak{B}_{m+n+1}(\mathbb{X}) \to \mathcal{B}$ as before. Then $\phi : \mathbb{X} \to \mathcal{B}^{\mathbb{Z}}$ defined by $y = \phi(x)$ where ϕ is induced by Φ is called a **sliding block code** with **memory** m and **anticipation** n.*

If $m = n = 0$, then we call it a 1-block code. Note that the terminology block code, memory, anticipation elude to the application of shift space to computer science and data storage.

Proposition 2.14 ([22])**.** *Let $\mathbb{X}, \mathbb{Y}$ be shift spaces and let S_x, S_y denote their respective shift maps. If $\phi : \mathbb{X} \to \mathbb{Y}$ is a sliding block code, then $\phi \circ S_x = S_y \circ \phi$. i.e., the following diagram commutes:*

$$\begin{tikzcd} \mathbb{X} \arrow[r, "S_x"] \arrow[d, "\phi"'] & \mathbb{X} \arrow[d, "\phi"] \\ \mathbb{Y} \arrow[r, "S_y"] & \mathbb{Y} \end{tikzcd}$$

The proof of the above proposition is immediate by doing a direct computation. Sliding block codes are easily seen to be continuous maps. Moreover, they are closed maps in the case of finite alphabets, and if $\mathbb{X}$ is a shift space, the image of a sliding block code is again a shift space [22].

When sliding block codes are invertible, their inverses are also sliding block codes. As a result invertible sliding block codes between shift spaces are conjugacies. In fact, all conjugacies between shift spaces are invertible sliding block codes.

Proposition 2.15 ([12])**.** *Let $\mathbb{X}$ and $\mathbb{Y}$ be shift spaces. A map $\phi : \mathbb{X} \to \mathbb{Y}$ is a conjugacy if and only if it is an invertible sliding block code.*

While helpful, the characterization of conjugacy maps by sliding block codes does not make the problem of determining conjugacy easy. While there are methods for determining conjugacy for special types of shift spaces, there is no general method. However, there are many conjugacy invariants that one can compute to determine if two spaces are not conjugate. We will list a few examples or some of the common invariants for shift spaces here but will not go into great detail as these particular invariants are not the focus of the thesis.

Definition 2.16. *Let $\mathbb{X}$ be a shift space. The **entropy** of $\mathbb{X}$ is defined as follows:*

$$\lim_{n\to\infty} \frac{1}{n} \log |B_n(\mathbb{X})|$$

where $|B_n(\mathbb{X})|$ denotes the number of unique words of length n that occur inside the shift space $\mathbb{X}$.

Entropy can be thought of as a measure of how complex the shift space is and is a conjugacy invariant. $|B_n(\mathbb{X})|$ is often denoted as $p_\phi(n) = |\mathcal{L}_\phi^n|$, where $p_\phi(n)$ is called the complexity function. Entropy can be defined for a general topological dynamical system (X, ϕ), assuming that X is compact. We will not explain it here. Just note that the general definition of entropy and our above definition coincide for shift spaces and that entropy is a conjugacy invariant in general. In the case of minimal shift spaces, entropy is always equal to zero, so this invariant is not particularly useful for the spaces we will be looking at. Despite this, the complexity function itself, or more specifically its asymptotic rate of growth can still give information about minimal shift spaces. For more information about this see [2].

Definition 2.17. *Let (X, ϕ) be a dynamical system. For $n \geq 1$, define $q_n(\phi) = |\{x \in X : \phi^n(x) = x\}|$. If $q_n(\phi) < \infty$ for all $n \in \mathbb{N}$, we define the **zeta function**:*

$$\zeta_\phi(t) = exp\left(\sum_{i=1}^{\infty} \frac{q_n(\phi)}{n} t^n\right)$$

where $exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

One can also define what is called the periodic point generating function $g_\phi(t)$ as:

$$g_\phi(t) = \sum_{n=1}^{\infty} q_n(\phi) t^n$$

Both functions contain the same information about the periodic points of the dynamical system, however, the zeta function tends to be used more frequently as for certain types of shift spaces, the zeta function ends up having nicer properties (for instance, with a class of shift spaces called Sofic shifts, which show up naturally in automata theory, the zeta function ends up being a rational function [22]). Like the first invariant, this one isn't overly useful for the shift spaces that we will be looking at as infinite minimal shift spaces do not contain fixed points with respect to the shift map.

2.3. **Generalizations of Shift Spaces.** First, we shall look at the full shift over a countable alphabet $\mathcal{A}$, i.e $\mathcal{A}^{\mathbb{Z}} = \{x = (x_i) : x_i \in \mathcal{A}, \forall i \in \mathbb{Z}\}$ where $|\mathcal{A}| = |\mathbb{N}|$. In this case, if we choose the same metric as before, our space is no longer compact [22]. Moreover, the space is not even locally compact [25]. Additionally, sliding block codes are no longer closed maps in general [22]. We also lose some of the conjugacy invariants that we had in the finite alphabet case. For instance, because our space is no longer compact, we no longer have a notion of entropy. While there are ways to define entropy for non-compact dynamical systems, these notions come with extra issues that the compact case does not have. As an example, sliding block codes preserve entropy for finite alphabets, but there are multiple notions of entropy in the countable case that are not preserved by sliding block codes [22].

One of the ways to get around some of the issues described above is to change the topology to something else that is still compatible with the questions that you are interested in. For instance, in [25], the authors, looking at one-sided shift spaces over a countable alphabet, endowed $\mathcal{A}$ with the discrete topology and took the one-point compactification $\mathcal{A}_\infty = \mathcal{A} \bigcup \{\infty\}$. They then defined an equivalence relation $\sim$ on $\mathcal{A}_\infty^{\mathbb{N}}$ by identifying sequences which contained ∞ with finite sequences based on the first occurrence of ∞. By letting $\sum_{\mathcal{A}}$ denote the set of all finite and infinite sequences, and by identifying elements of $\sum_{\mathcal{A}}$ with elements of $X_{\mathcal{A}}/\sim$, based on the first occurrence of ∞ in the sequence, the authors got a topology on $\sum_{\mathcal{A}}$ that was compact, Hausdorff and that made the shift map on $\sum_{\mathcal{A}}$ continuous on all sequences except for the empty sequence. These ideas were further extended to two-sided shift spaces in [15].

The next generalization that we shall discuss is higher-dimensional shift spaces. Here, we define the full d-dimensional shift to be $\mathcal{A}^{\mathbb{Z}^d}$. Here, to each point in $\mathbb{Z}^d$ we associate to it a symbol from $\mathcal{A}$. More information about higher-dimensional shift spaces and further resources can be found in [22]. We define the metric, in this case, to be $\rho(x,y) = 2^{-k}$ where $k \in \mathbb{Z}$ is the largest integer such that $x_{[-k,k]^d} = y_{[-k,k]^d}$ if $x \neq y$ and zero otherwise. In higher dimensions, there are multiple shift maps that one can choose. By choosing any vector in $\mathbb{Z}^d$, we can define a shift map based on that vector. As an example, if we let $d = 2$ and choose the vector $v = (2,3)$, then the symbol associated with the point (i,j) in the integer lattice $\mathbb{Z}^2$, would be shifted to the point $(i+2, j+3)$. Similar to the one-dimensional shift spaces, higher-dimensional shift spaces are defined by a forbidden patterns list where patterns are the high-dimensional version of words. As an example, the forbidden patterns list $\mathcal{F} = \{11, {}^1_1\}$ over the alphabet $A = \{0,1\}$ generates what is called the 2-Dimensional Golden Mean Shift. The set of all patterns inside a shift space creates a higher-dimensional analogue of the language of a shift that we saw before.

Higher-dimensional shift spaces are significantly more difficult to deal with. For instance, in the one dimensional case, given a finite list $\mathcal{L}$, it is always possible to know if $\mathbb{X}_{\mathcal{L}^c}$ is non-empty and if a word occurs inside $\mathbb{X}_{\mathcal{L}^c}$. For dimensions two and greater, these problems end up being undecidable i.e. there is no algorithm that works in all cases [5, 31]. A second issue is that many of the tools that one can use in the one-dimensional case either don't have higher-dimensional analogues or their higher-dimensional versions are far more difficult to compute. As an example, given a d-dimensional shift space $\mathbb{X}$, the entropy $h(\mathbb{X})$ is defined to be:

$$h(\mathbb{X}) \lim_{n\to\infty} \frac{1}{n^d} \log |\mathbb{X}_{[0,n-1]^d}|$$

where $|\mathbb{X}_{[0,n-1]^d}|$ denotes the the number of patterns on $[0, n-1]^d$ that occur in $\mathbb{X}$. Like before, this can be thought of as a measure of how complex the shift space is and is a conjugacy invariant. Unlike in the one-dimensional case however, the entropy of very few shift spaces is exactly known [22]. Many other invariants in the one-dimensional case can be computed using basic linear algebra (for example, in shifts with a finite forbidden words list, one can associate a positive definite matrix to it and calculate what is called its PF-eigenvalue. This PF-eigenvalue ends up being a conjugacy invariant), but in higher-dimensional spaces, there are not analogous results.

3. Symbolic Substitutions and Their Shift Spaces

In this section, we shall introduce symbolic substitutions and see how one can associate a shift space to a substitution using a legal word characterization. We shall then focus on the case of primitive substitutions and see how we can characterize their shift spaces using a fixed point criteria. Note that this fixed point will be with respect to the substitution map and not the shift map.

3.1. **Shift Spaces of Symbolic Substitutions Using Legal Words Characterization.**

Definition 3.1 ([1])**.** *Let $\mathcal{A}$ be a set consisting of n-letters $a_1, a_2, \dots, a_n$. A* ***substitution rule*** *ϕ on $\mathcal{A}$ is a function $\sigma : \mathcal{A} \to \mathcal{A}^+$, where $\mathcal{A}^+$ is the set of all finite words generated by $\mathcal{A}$. σ can be extended to a function on $\mathcal{A}^+$ or $\mathcal{A}^{\mathbb{Z}}$ by applying it to the individual letters of a word and then concatenating the resulting words together.*

As an example, consider the substitution σ on two letters $\mathcal{A} = \{a, b\}$ given by:

$$\sigma(a) = ab$$
$$\sigma(b) = ba$$

and the infinite word $...abab.abab... \in \mathcal{A}^{\mathbb{Z}}$, then by applying σ to this word, we get:

$$...\sigma(a)\sigma(b)\sigma(a)\sigma(b).\sigma(a)\sigma(b)\sigma(a)\sigma(b)... = ...abbaabba.abbaabba... \in \mathcal{A}^{\mathbb{Z}}$$

More generally, one can define a substitution as an endomorphism from the free group generated by $a_1, a_2, \dots, a_n$ into itself [1]. One would then have formal inverses of each letter a_i^{-1} and the empty word ϵ corresponding to the identity. One would then get a symbolic substitution by requiring the image of $\sigma(a_i)$ to contain no negative powers of any of the generators. Alternatively we could define our symbolic substitutions as an endomorphism from the free monoid generated by $a_1, a_2, \dots, a_n$ [13]. These more general substitutions defined from a free group will not be of interest to us however. In our topology, symbolic substitutions are continuous. To see this, let $\epsilon > 0$ and let $w, w' \in \mathcal{A}^{\mathbb{Z}}$ be two words such that $d(w, w') < \epsilon$. Then there exists $n \in \mathbb{N}$ and a finite word u of length $2n+1$ such that $w_{[-n,n]} = w'_{[-n,n]} = u$ and $2^{-n} < \epsilon$. Then $\sigma(w_{[-n,n]}) = \sigma(w'_{[-n,n]}) = \sigma(u)$. Since $\sigma(u)$ must have a length of at least $2n + 1$, $d(\sigma(w), \sigma(w')) \leq 2^{-n} < \epsilon$. Therefore σ is continuous.

Definition 3.2 ([1])**.** *Let σ be a substitution on n letters $a_1, ..., a_n$. The* ***substitution matrix*** *M_σ associated to σ is an $n \times n$ matrix whose (i, j) entry equals the number of copies of a_i that appear in $\sigma(a_j)$.*

It is easy to check that if M_σ,M_ϱ are the substitution matrices associated with the substitution σ, ϱ, then the substitution matrix associated to $\varrho \circ \sigma$ is $M_{\varrho\circ\sigma} = M_\varrho M_\sigma$. Substitution matrices are one of the most basic tools used in studying symbolic substitutions as one is able to determine various properties of the substitution based on the associated matrix.

Definition 3.3 ([1])**.** *A non-negative substitution rule σ on a finite alphabet $\mathcal{A}$ is called **irreducible** when for each index pair (i,j), there exists some $n \in \mathbb{N}$ such that a_j is a subword of $\sigma^n(a_i)$. σ is called **primitive** when some $k \in \mathbb{N}$ exists such that a_j is a subword of $\sigma^k(a_i)$ for all index pairs (i,j).*

Definition 3.4 ([1])**.** *A matrix M with non-negative entries is called **irreducible** if for each index pair (i,j), there exists $n \in \mathbb{N}$ such that $(M^n)_{i,j} > 0$. A matrix M with non-negative entries is called **primitive** if there exists $k \in \mathbb{N}$ such that $(M^k)_{i,j} > 0$ for every index pair (i,j).*

A substitution is irreducible (respectively primitive) if and only if its substitution matrix is irreducible (primitive). A substitution being irreducible prevents two things from occurring. First, it means that a substitution cannot be broken up into two or more smaller substitutions. For example, looking at the substitution on 3 letters $\{a,b,c\}$ given by:

$$\sigma(a) = bb$$
$$\sigma(b) = aa$$
$$\sigma(c) = cc$$

This would not be irreducible as a and b are never a subword of $\sigma^n(c)$ and vice versa. Second, it disallows substitutions where one of the letters does not appear in the image of any $\sigma(a_i)$. For instance the substitution given on two letters by $\{a,b\}$ given by:

$$\sigma(a) = aa$$
$$\sigma(b) = a$$

is also not irreducible.

A simple example of a irreducible, but not primitive substitution would be:

$$\sigma_i(a) = bb$$
$$\sigma_i(b) = aa$$

The most well known example of a primitive substitution is the Fibonacci substitution, which is a substitution on two letters $\{a,b\}$ given by:

$$\sigma(a) = ab$$
$$\sigma(b) = a$$

It is clear from the definition that a substitution is primitive if and only if there exists $N \in \mathbb{N}$ such that every a_i is in the image of every $\sigma^N(a_j)$ for all $1 \le i, j \le n$. We will primarily be interested in substitutions that are primitive. As stated before, the reason is that we can nicely characterize the shift space of primitive substitutions using a fixed point criteria. After introducing strongly maximal TAF-algebras in an upcoming section, we shall look at substitutions that are not primitive, but which lead to minimal shift spaces.

The general procedure for generating a shift space from a symbolic substitution comes from the following definition.

Definition 3.5 ([2]). *Let σ be a substitution. The **language** of σ, $\mathcal{L}(\sigma)$, is defined to be the set of all words occurring in $\sigma^n(a)$, for some $a \in \mathcal{A}$. The set $\mathbb{X}(\sigma)$ denotes the set of all sequences $y \in \mathcal{A}^{\mathbb{Z}}$ such that all the subwords of y belong to $\mathcal{L}(\sigma)$*

$\mathcal{L}(\sigma)$ is readily seen to be a language which satisfies the conditions of Proposition 2.10 and $\mathbb{X}(\sigma)$ is readily seen to be the shift space associated to $\mathcal{L}(\sigma)$. Note that $\mathcal{L}(\sigma^n) \subseteq \mathcal{L}(\mathbb{X}(\sigma)) \subseteq \mathcal{L}(\sigma)$ and $\mathbb{X}(\sigma^n) \subseteq \mathbb{X}(\sigma)$ for $n \ge 1$, but equality does not hold in general.

3.2. **Tiling Spaces.** Shift spaces defined by symbolic substitutions have new invariants that are not available for more general shift spaces. One of these invariants is the tiling space associated with a given substitution.

Definition 3.6 ([1]). *A **tiling** of $\mathbb{R}$ is a partition of $\mathbb{R}$ into sets called tiles such that these tiles can only intersect on their boundary. A tiling of $\mathbb{R}$ is called **simple** if there are a finite number of tiles up to translation and each tile is a closed interval.*

A tiling and simple tiling of $\mathbb{R}^d$ is defined analogously, except we replace closed interval by a polytope. Famous examples of tilings in the plane include the Penrose tiling and pinwheel tiling. Note that these tilings can be generated by a geometric analogue of our symbolic substitution called an inflation rule. More details about this can be found in [1].

Definition 3.7 ([2]). *Let σ be a substitution of the alphabet $\mathcal{A}$ with associated shift space $\mathbb{X}(\sigma)$. The **tiling space** associated to σ, denoted Ω_σ, is defined to be:*

$$\Omega_\sigma = (\mathbb{X}(\sigma) \times [0,1])/\sim$$

where $\sim$ is the relation $(w, 0) \sim (\sigma(w), 1)$.

Points in the tiling space can be thought of as partitions $\mathbb{R}$ into unit intervals, where each unit interval is associated with or "labelled" by a letter in the word w. Adopting this second view, we can describe Ω_σ as follows: For each letter in $\mathcal{A}$, associate to a_i a tile which is a unit interval in $\mathbb{R}$. Then

Ω_σ is the set of all tilings of $\mathbb{R}$ such that every patch is can be found inside the patch corresponding to $\sigma(a_i)^m$ for some $1 \leq i \leq n$ and $m \in \mathbb{N}$.

Definition 3.8 ([2]). *Two tilings T, T' are said to be ϵ-close if after a translation of at most ϵ, the tilings agree on a ball of radius $\frac{1}{\epsilon}$ around the origin.*

This concept of closeness defines what is called the local topology for our tiling space. From the definition of our tilings spaces, we get the following proposition.

Proposition 3.9 ([2]). *Let ϕ, η be two symbolic substitutions such that they define conjugate shift spaces. Then Ω_ϕ and Ω_η are homeomorphic.*

The converse of this proposition is false. For a simple counterexample, we can take the substitutions $\sigma_1 : a \to ab$, $\sigma_1 : b \to ab$ and $\sigma_2 : a \to aab$, $\sigma_2 : b \to aab$. Both define homeomorphic tiling spaces, but they are not conjugate as their associated shift spaces have different cardinalities [2]. Note that while this is not a complete invariant, these tiling spaces are still quite useful as determining conjugacy between shift spaces defined by substitutions is quite hard in general. For example, in [23], tiling spaces were used to study non-minimal shift spaces' structure associated with symbolic substitutions.

3.3. **Shift Spaces Associated to Primitive Symbolic Substitutions.** In this subsection, we shall focus on characterizing the shift spaces of primitive substitutions. To do this, we shall first introduce the concept of a hull.

Definition 3.10 ([1]). *Let $w \in \mathcal{A}^{\mathbb{Z}}$. The **two-sided (or symbolic) hull** of w is define $\mathbb{X}(w) := \overline{\{S^i(w) : i \in \mathbb{Z}\}}$*

$\mathbb{X}(w)$ is by definition a shift space. More specifically, it is the smallest shift space containing w. If $z \in \mathbb{X}(w)$, it may not be the smallest shift space containing z however. An example of this is seen later.

Definition 3.11 ([1]). *Let σ be a substitution rule on a finite alphabet $\mathcal{A}$. A finite word is called **legal** for σ if it occurs as a subword of $\sigma^k(a_i)$ for some $1 \leq i \leq n$, $k \in \mathbb{N}$*

Clearly legal words are mapped to legal words by the substitution.

Definition 3.12 ([1]). *A bi-infinite word w is called a **fixed point** of a primitive substitution σ if $\sigma(w) = w$ and $w_{-1}|w_0$ is a legal two-letter word of σ.*

Equivalently we can say that w is a fixed point if and only if $w_{-1}|w_0$ is a legal two-letter word and $\sigma(w_0)_{[0,0]} = w_0$ and $\sigma(w_{-1})_{[|\sigma(|w_{-1}|-1,|\sigma(w_{-1}|-1)} =$

w_{-1}. These definitions are equivalent as $\sigma(w_0)_{[0,0]} = w_0$ implies that the sequence $(\sigma^n(w_0))_{n\in\mathbb{N}}$ will converge to an infinite word w in $\mathcal{A}^{\mathbb{N}}$ with the property that $\sigma(w) = w$. Using compactness as well as the fact $\sigma^n(w_0)$ will agree with $\sigma^{n-1}(w_0)$ on the first $|\sigma^{n-1}(w_0)| - 1$ positions implies that $(\sigma^n(w_0))_{n\in\mathbb{N}}$ is convergent. The same idea can be used for $\sigma(w_{-1})$. Clearly not all primitive substitutions have fixed points. For example:

$$\sigma(a) = ba$$
$$\sigma(b) = aba$$

has no fixed points. Despite this all substitutions raised to an appropriate power will have a fixed point.

Proposition 3.13 ([1])**.** *If σ is a primitive substitution on a finite alphabet $\mathcal{A}$ with $n \geq 2$, there exists some $k \in \mathbb{N}$ and some $w \in \mathcal{A}^{\mathbb{Z}}$ such that w is a fixed point of σ^k (i.e. $w_{-1}w_0$ is a legal word and $\sigma^k(w) = w$)*

Proof. Assume without loss of generality that $|\sigma(a_i)| > 1$ for all $1 \leq i \leq n$ (if this is not the case take an appropriate power of σ) and define $g : \mathcal{A}^2 \to \mathcal{A}^2$ by $g(xy) = \sigma(x)_{|\sigma(a)|-1}\sigma(y)_0$. Select any legal two-letter word and iterate it. By the pigeonhole principle, there exists at least one legal two-letter word which repeats after enough iterations of g. i.e., there exists a legal word $x'y'$ and $k \in \mathbb{N}$ such that $g^k(x'y') = x'y'$. Note that this may not be the initial legal word. Choosing this legal two-letter word as our seed and iterating it under σ^k, we get our result. □

Definition 3.14 ([1])**.** *Let w be bi-infinite fixed point of σ (or an appropriate power of σ if σ has no fixed points). We call $\mathbb{X}(w)$ the **hull** of the substitution.*

With this, there is an obvious question of uniqueness. If σ has multiple fixed points, do the different fixed points lead to different hulls? Additionally, do the fixed points of σ^k and σ^l with $k \neq l$ lead to different hulls? The answer to all these questions is no and the results rely on the fact that σ is primitive. For a substitution that is not primitive, one could not associate a unique hull based on fixed points in general. As an example, with the irreducible but not primitive substitution σ_i defined in the previous subsection, $w_1 = ...aaaa.aaaa...$ and $w_2 = ...bbbb.bbbb...$ are both fixed points of σ_i^2, and clearly $\mathbb{X}(w_1) \neq \mathbb{X}(w_2)$. To show that the hull of a primitive substitution is unique, we first need to introduce the concept of local indistinguishably.

Definition 3.15 ([1])**.** *Two words u, v in the same alphabet are called **locally indistinguishable (LI)** (denoted $u \overset{LI}{\sim} v$) if each finite subword of u is also a subword of v and vice versa.*

Two finite words are locally indistinguishable if and only if they are equal. The LI class of a word $w \in \mathcal{A}^{\mathbb{Z}}$ is denoted as $LI(w) := \{z \in \mathcal{A}^{\mathbb{Z}} : z \overset{LI}{\sim} w\}$. It is easy to see that local indistinguishably defines an equivalence relation.

Proposition 3.16 ([1])**.** *If w is a bi-infinite word, its LI class is contained in the hull of w and* $\mathbb{X}(w) = \overline{LI(w)}$*. In particular* $u \overset{LI}{\sim} v \to \mathbb{X}(u) = \mathbb{X}(v)$*.*

The sketch of the proof for the above proposition goes as follows. If a word u is locally indistinguishable from w, then for all $n \in \mathbb{N}$, $u_{[-n,n]}$ is a subword of w. As a result we can find a $k \in \mathbb{Z}$ such that $S^k(w)_{[-n,n]} = u_{[-n,n]}$. As a result, we can construct a sequence in $\mathbb{X}(w)$ which converges to u. To show that $LI(w)$ is dense, we only need to note that all words of the form $S^k(w)$, $k \in \mathbb{Z}$ are locally indistinguishable from w. In general, the LI class of a word need not be closed. For a simple counter-example, consider $w = ...aaaa.baaa....$ Then $w' = ...aaaa.aaaa... \in \mathbb{X}(w)$ but it is not locally indistinguishable from w.

Proposition 3.17 ([1])**.** *Let σ be a primitive substitution rule on a finite alphabet. Then, any two bi-infinite fixed points u,v of σ are locally indistinguishable. The same conclusion holds if u and v are fixed points of possibly different positive powers of σ.*

The above proposition follows from three key observations:

1. If u is a fixed point of σ with seed $u_{-1}|u_0$, and w is a finite subword of u, then w will be a subword of $\sigma^n(u_{-1}u_0)$ for a large enough n.
2. Since σ is primitive for any letter a in the alphabet, $u_{-1}u_0$ will be a subword of $\sigma^m(a)$ for a large enough m.
3. If u is a fixed point of σ^k and v is fixed point of σ^l, then both u and v are fixed points of $\sigma^{lcm(k,l)}$.

From this proposition, it follows that our symbolic hulls are well defined for primitive substitutions. For the case of primitive substitutions, $\mathbb{X}(w)$ and $\mathbb{X}(\sigma)$ clearly coincide as any $u \in \mathbb{X}(\sigma)$ will be in the LI class of w and any $v \in \mathbb{X}(w)$ will only be composed of words allowed by $\mathcal{L}(\sigma)$. Due to our fixed point characterization, we end up with the following proposition.

Proposition 3.18. *Let σ be a primitive substitution. Then $\mathcal{L}(\sigma) = \mathcal{L}(\sigma^k)$ and $\mathbb{X}(\sigma) = \mathbb{X}(\sigma^k)$ for all $k \geq 1$.*

There still is one obvious question related to uniqueness. If $\mathbb{X}(w)$ is a symbolic hull for some substitution σ and $z \in \mathbb{X}(w)$, is it the case that $\mathbb{X}(z) = \mathbb{X}(w)$? For a general hull the answer is no. In the above example where we showed that the LI class is not always closed, it is clear that $\mathbb{X}(w) \neq \mathbb{X}(w')$, despite $w' \in \mathbb{X}(w)$. For our primitive substitutions however, the answer is yes. To show that this is the case, we will need the following well known proposition on minimal dynamical systems.

Proposition 3.19 ([20]). *Let (X, ψ) be a topological dynamical system such that X is a compact metric space. Then following are equivalent:*

1. *(X, ψ) is minimal.*
2. *Every orbit of X is dense.*
3. *ψ is surjective and every backwards orbit is dense.*

Note that (1.) if and only if (2.) does not require X to be compact nor a metric space. From (2.) and (3.), we get the following immediate corollary.

Corollary 3.20. *A two-sided shift space $\mathbb{X} \subseteq \mathcal{A}^{\mathbb{Z}}$ is minimal if and only if for all $w \in \mathbb{X}$, the shift orbit $\{S^i w \mid i \in \mathbb{Z}\}$ is dense in $\mathbb{X}$.*

For symbolic hulls, we are able to characterize minimality by the LI class.

Proposition 3.21 ([1]). *If w is a bi-infinite word in the finite alphabet $\mathcal{A}$, with LI class $LI(w)$ and hull $\mathbb{X}(w)$, then the following are equivalent:*

1. *$\mathbb{X}(w)$ is minimal*
2. *$LI(w)$ is closed*
3. *$\mathbb{X}(w) = LI(w)$*

Using the concept of repetitivity we get a second characterization.

Definition 3.22 ([1]). *A bi-infinite word w over a finite alphabet is called* ***repetitive*** *when every finite subword of w reappears in w with bounded gaps.*

For a bi-infinite word w and a finite subword u of w, define $T_u = \{i : S^i(w)_{[0,|u|-1]} = u\}$. Saying that w is repetitive is equivalent to saying that for every finite subword u of w there exists a finite set $I_u \subseteq \mathbb{Z}$ such that $T_u + I_u = \mathbb{Z}$.

Proposition 3.23 ([1]). *If w is a bi-infinite word in the finite alphabet $\mathcal{A}$, the hull $\mathbb{X}(w)$ is minimal if and only if w is repetitive.*

Proposition 3.24 ([1]). *Any bi-infinite fixed point of a primitive substitution on a finite alphabet is repetitive.*

Proof. Let w be a bi-infinite fixed point. Since the associated substitution is primitive, there exists $k \in \mathbb{N}$ such that a_1 will appear as a subword in $\sigma^k(a_j)$ for all j. As a result a_1 will appear with bounded gaps. Additionally, for each finite subword u of w there will exist $l \in \mathbb{N}$ such that u appears in $\sigma^l(a_1)$. As a result all finite subwords of w repeat with bounded gaps. Therefore w is bounded. □

Bi-infinite fixed points of primitive substitutions actually satisfy a stronger condition called linear repetitivity (or linear recurrent). Due to w being defined over a finite alphabet, for every $n \in \mathbb{N}$, there exists $N_n \in \mathbb{N}$ such that all finite subwords u of length n in w will be contained in any subword v in w of length at least N_n. A word $w \in \mathcal{A}^{\mathbb{Z}}$ is said to be linearly repetitive if $N_n = \mathcal{O}(n)$, i.e. N_n grows at an asymptotically linear rate. Moreover, if σ is any substitution such that $\mathbb{X}(\sigma)$ is minimal, then its elements are linearly repetitive [6]. This property turns out be quite useful as it implies important dynamical system properties of $\mathbb{X}(\sigma)$. We shall see a couple of examples of this later.

Minimality is not unique to the shift spaces of primitive substitutions.

Example 3.25 (A minimal shift space generated by a non-primitive substitution)**.** *The substitution $\sigma(0) \to 0010$, $\sigma(1) \to 1$ is an example of a non-primitive substitution that defines a minimal shift space* [13]*.*

We will see later that despite this, for any non-primitive substitution with a minimal shift space, we can always construct a primitive substitution whose shift space is conjugate to the non-primitive one. We do have the following proposition for irreducible substitutions however.

Proposition 3.26. *Let σ be an irreducible substitution over a finite alphabet $\mathcal{A}$ such that $(\mathbb{X}(\sigma), S)$ is minimal and for all $a \in \mathcal{A}$, $\lim_{n\to\infty} |\sigma^n(a)| = \infty$. Then σ is primitive.*

Proof. By copying the method in the proof of Proposition 3.13, we are able to generate a fixed point w of σ^k where $k \in \mathbb{N}$. By definition, $w \in \mathbb{X}(\sigma)$. By minimality $\mathbb{X}(w) = \mathbb{X}(\sigma)$. By irreducibility, all letters $a_i \in \mathcal{A}$ must occur inside w. Therefore $\sigma^k(a_i)$ is a subword of w for all $k \in \mathbb{N}, 1 \leq i \leq n$. By our previous proposition, w is repetitive, therefore all letters $a \in \mathcal{A}$ appear in w with bounded gaps. Since $\sigma^k(a_i)$ is a subword of w and since $\lim_{k\to\infty} |\sigma^k(a_i)| = \infty$, all letters $a \in \mathcal{A}$ must be in a subword of $\sigma^k(a_i)$ for a sufficiently large k. Therefore, σ is primitive. $\square$

There are two more natural questions that we will look at for our symbolic hulls:

1. How large is a given hull? Are they finite, countably infinite, or uncountable?
2. How many disjoint orbits are in a given hull?

To answer these questions depends on the particular nature of the primitive substitution. Namely, whether or not it is aperiodic.

Definition 3.27 ([1])**.** *Let w be a bi-infinite word. w is said to be* ***periodic*** *if there exists $k \in \mathbb{Z}$ such that $S^k(w) = w$. A primitive substitution* *is called periodic if its shift space only consists of periodic elements. w is called* ***(topologically) aperiodic*** *when $\mathbb{X}(w)$ contains no periodic words (i.e. there does not exist any $u \in \mathbb{X}(w)$ such that $S^k(u) = u$ for some $k \in \mathbb{Z}$. A primitive substitution is called aperiodic if its shift space contains no periodic elements.*

There are bi-infinite words which are neither periodic nor aperiodic. In this case, we simply say that they are non-periodic. An example of such a word is $w = ...aaaaa.baaaa....$ For the case of primitive substitutions, the minimality of the hull implies that the hull will either only contain periodic words or aperiodic words. As a result we only need to worry about aperiodic words. Also note that we say w is topologically aperiodic because there are other notions of aperiodicity.

If a primitive substitution has a periodic hull, then it clearly only has a finite number of elements and a single orbit. To deal with the aperiodic case we need to introduce a few measure-theoretic concepts.

Definition 3.28 ([1])**.** *Let $\mathbb{X}$ be a two-sided shift space. The set $\mathbb{P}_{\mathbb{Z}}(\mathbb{X})$ denotes the set of all shift-invariant probability measures on $\mathbb{X}$. i.e. the set of all measures μ such that for any measurable set $A \subseteq \mathbb{X}$, $\mu(A) = \mu(S(A))$, where $S(A)$ is the image of A under the shift operator.*

Proposition 3.29 ([1])**.** *Let $\mathbb{X}$ be a two-sided shift space. Then $\mathbb{P}_{\mathbb{Z}}(\mathbb{X})$ is nonempty.*

$\mathbb{P}_{\mathbb{Z}}(\mathbb{X})$ being nonempty follows from the fact that $\mathbb{P}(\mathbb{X})$, the set of probability measures on $\mathbb{X}$ is nonempty (it at least contains the point measures) and that $\mathbb{P}(\mathbb{X})$ is compact. Using this, one can construct a sequence of probability measures which converges to a shift-invariant probability measure. Also note that $\mathbb{P}_{\mathbb{Z}}(\mathbb{X})$ is convex.

Definition 3.30 ([1])**.** *We say that a Borel set $A \subseteq \mathbb{X}$ is an* ***invariant set*** *if $S^{-1}(A) = A$*

Definition 3.31 ([1])**.** *A shift-invariant probability measure μ on a two-sided shift space $\mathbb{X}$ is said to be* ***ergodic*** *if for any Borel invariant set A, $\mu(A)$ equals 0 or 1.*

A dynamical system having an ergodic measure just means that up to measure zero sets, nothing remains invariant with respect to the homeomorphism that defines the dynamical system

Proposition 3.32 ([1])**.** *A shift-invariant probability measure μ is ergodic if and only if it is extremal.*

Definition 3.33 ([1])**.** *Let $\mathbb{X}$ be a two-sided shift space. If $\mathbb{P}_{\mathbb{Z}}(\mathbb{X})$ consists of a single element, we say that $\mathbb{X}$ is **uniquely ergodic**. If $\mathbb{X}$ is also minimal, we say that it is **strictly ergodic**.*

Proposition 3.34 ([1])**.** *Let σ be a primitive substitution. Then its symbolic hull $\mathbb{X}$ is strictly ergodic.*

This proposition follows from the fact that linear repetitivity implies uniquely ergodic and the fact that the hull of a symbolic substitution is minimal.

Proposition 3.35 ([1])**.** *Let w be a repetitive, aperiodic word in a finite alphabet. Then $\mathbb{X}(w)$ is uncountable and contains uncountably many pairwise disjoint orbits.*

Proof. Let μ be any invariant probability measure. Then $\mathbb{X} = \bigcup_{u \in \mathbb{X}} \{u\}$ and $\mu(\{S^i(u)\}) = \mu(\{u\})$ for any $i \in \mathbb{Z}$. Since w is repetitive, $\mathbb{X}$ is minimal which implies that u is also non-periodic as it has a dense orbit. Therefore $S^i(u) = S^j(u)$ if and only if $i = j$. Therefore by σ-additivity, $\mu(u) = 0$ for all $u \in \mathbb{X}(w)$. Since $\mathbb{X}$ is a set of measure 1, and it is the union of measures zero sets, the union must be uncountable. Additionally as each orbit is countable, there must be uncountably many orbits. ☐

An alternative proof to the above is to show that $\mathbb{X}(w)$ is a Cantor set. With everything done so far, we would only need to show that $\mathbb{X}(w)$ is perfect. This is implied by repetitivity and the definition of the cylinder sets. Since $\mathbb{X}(w)$ is homeomorphic to an uncountable set, it too must be uncountable.

We shall go over three ways that one can differentiate between periodic and aperiodic hulls. The first method can be used for any hull so long as the word defining this hull is repetitive, while the second and third can only be used when that hull is defined by a primitive substitution.

Definition 3.36 ([1])**.** *Two elements u, v in a hull $\mathbb{X}(w)$ $u \neq v$ are called a **proximal pair** if $\lim_n(S^n(u), S^n(v)) = 0$ as n tends to plus or minus infinity*

Proposition 3.37 ([1])**.** *Let w be a repetitive bi-infinite word. Then $\mathbb{X}(w)$ is aperiodic if and only if it contains a proximal pair.*

For the second criteria, we need the following theorem

Theorem 3.38 (Perron-Frobenius theorem [1])**.** *If M is a primitive matrix, then M has a real eigenvalue that has multiplicity one and has modulus greater than any other eigenvalue of M. This eigenvalue is called the **PF-eigenvalue**, and is denoted λ_{PF}.*

Proposition 3.39 ([1]). *Let σ be a primitive substitution and let M_σ be the associated substitution matrix. If M_σ has an irrational PF-eigenvalue, then σ is aperiodic.*

Unfortunately, the converse of the above proposition is false. Note that one can normalize the right PF-eigenvalue to get the asymptotic frequencies of the different letters in the substitution.

To determine if any primitive substitution is periodic or aperiodic, we refer the interested reader to [26]. This paper answers a more general question of if a sequence generated by a morphism on a free monoid is eventually periodic, where eventually periodic means the sequence is of the form uv where v is a finite word and v an infinite repetitive word. Using ideas from formal language systems, the author provides an algorithm for determining if any sequence generated in this way is periodic or not.

4. Semi-Crossed Product Algebras

In this section, we shall introduce what is called the semi-crossed product algebra. We will see how the properties of these algebras are related to their associated dynamical systems and we shall get a nice characterization of these algebras in terms of their 2×2 upper triangular representations. At the end of this section, we will also look at a natural extension of the ideas below to multivariable dynamical systems.

4.1. **Topological Conjugacy Algebras.**

Definition 4.1 ([9]). *Let X be a compact Hausdorff space and let $\psi : X \to X$ be a continuous function. The* ***skew polynomial algebra****, denoted $\mathcal{P}(X, \psi)$, is all polynomials of the form:*

$$\sum_{i=1}^{n} f_i U^i, f_i \in C(X)$$

in the variable U where multiplication is defined by

$$Uf = (f \circ \psi)U$$

Now let $\mathcal{A}$ be any Banach algebra such that the following conditions hold:

1. $\mathcal{P}(X, \psi)$ is a dense subalgebra of $\mathcal{A}$ such that the units are the same.
2. $C(X) \subseteq P(X, \psi) \subseteq \mathcal{A}$ is closed
3. There exists an algebra homomorphism $E_0 : \mathcal{A} \to C(X)$ such that $E_0(f) = f$ for all $f \in C(X)$ and $ker E_0 = AU$
4. U is not a right divisor of 0

To any element $a \in \mathcal{A}$, we can associate to it a formal power series of the form $\sum_n E_n(a)U^n \in \mathcal{P}^\infty(X,\psi)$ where E_n is constructed to be the coefficient map. We do this by noting that $C(X)$ being closed in $\mathcal{A}$ implies that E_0 is continuous, which in turn implies that $\mathcal{A}U$ is a closed subset of $\mathcal{A}$. Therefore, by the inverse mapping theorem, the map $S(a) = aU$ where $a \in \mathcal{A}$ has a bounded left inverse which we shall denote as T. Using T and a simple induction, we can construct the coefficient maps $E_n = E_{n-1}T(I - E_0) = E_0(T(I - E_0))^n$. To see why this map is defined this way, for n=1, the fact that $a - E_0(a) \in ker E_0$, implies that there exists a unique $b \in \mathcal{A}$ such that $a = E_0(a) + bU$. As a result $(I - E_0)$ sends a to bU. T by definition sends bU to b and finally, evaluating $E_0(b)$ gives us our coefficient for U. Note that the coefficient maps are bounded. As a result, we can associate any $a \in A$ a formal power series via the mapping $\Delta : a \to \sum_n E_n(a)U^n \in \mathcal{P}^\infty(X,\psi)$. Also note that Δ is an algebra homomorphism.

Definition 4.2 ([9])**.** *Let (X,ψ) be a one-variable dynamical system where X is compact. Then a **topological conjugacy algebra** for (X,ψ) is an algebra $\mathcal{A}$ such that the above four conditions are satisfied and*

$$\limsup_n (\|E_n\|\|U^n\|)^{\frac{1}{n}} \leq 1$$

Imposing this norm condition is important as it means that our power series will be Cesaro summable (i.e. the sequence of arithmetic means of the partial sums converges). This fact will play an important role later on when we look at the characters of our algebra. With this we can define a topological conjugacy algebra for a non-compact locally compact set X. If we take its one-point compactification $\tilde{X} = X \bigcup \{w\}$, we can extend all proper continuous maps from X to $\tilde{X}$, by having $\{w\}$ be a fixed point. We then define the topological conjugacy algebra $\mathcal{A}$ for (X,ψ) as the norm closed subalgebra of the topological conjugacy algebra $\tilde{\mathcal{A}}$ for $(\tilde{X},\tilde{\psi})$ obtained by restricting the polynomial coefficients to elements of $C_0(X)$ (continuous functions which vanish at $\{w\}$).

By imposing appropriate restrictions on the norm of U, both $\mathcal{P}(X,\psi)$ and $\mathcal{P}^\infty(X,\psi)$, are almost topological conjugacy algebras minus issues of completeness/being a Banach algebra. Neither of these are very interesting for the purposes of dynamical systems however. Of particular interest to us will be the semi-crossed product algebra. This algebra itself is not a topological conjugacy algebra, but to it, we are able to associate a topological conjugacy algebra such that two semi-crossed product algebras are isomorphic as algebras if and only if there associated topological conjugacy algebras are isomorphic as algebras.

Example 4.3 (The semi-crossed product algebra $C_0(X) \times_\psi \mathbb{Z}^+$ associated to a dynamical system (X, ψ) [9])**.** *Let X be a locally compact Hausdorff space, ψ a proper continuous map on X, $H = l^2(\mathbb{N})$. Let $\xi = (\xi_n)_{n\in\mathbb{N}} \in l^2(\mathbb{N})$. For $x \in X, f \in C_0(X)$, define $\pi_x(f) : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ as*

$$\pi_x(f)(\xi_0, \xi_1, \xi_2, \dots) = (f(x)\xi_0, f \circ \psi(x)\xi_1, f \circ \psi^2(x)\xi_2, \dots)$$

Let U be the forward shift map. i.e.

$$U(\xi_0, \xi_1, \xi_2, \dots) = (0, \xi_0, \xi_1, \dots)$$

$C_0(X) \times_\psi \mathbb{Z}^+$ is defined to be the norm closed operator algebra acting on $\oplus_{x\in X} H$ generated by the operators $\oplus_{x\in X}\pi_x(f)$ and $\oplus_{x\in X} U\pi_x(g)$ where $f, g \in C_0(X)$

Viewed as matrices

$$\pi_x(f) = \begin{bmatrix} f(x) & 0 & 0 & 0 & \dots \\ 0 & f\circ\psi(x) & 0 & 0 & \dots \\ 0 & 0 & f\circ\psi^2(x) & 0 & \dots \\ 0 & 0 & 0 & f\circ\psi^3(x) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, U = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The semi-crossed product algebra is not a topological conjugacy algebra as the relation that it satisfies is $\pi_x(f)U = U\pi_x(f \circ \psi)$. This is remedied by defining the “opposite” algebra $(C_0(X) \times_\psi \mathbb{Z}^+)_{op}$. This algebra is the same as the semi-crossed product algebra, except we define a new multiplication $*$ to be $a * b := ba$. It is easy to check that the opposite algebra is a topological conjugacy algebra and that two semi-crossed product algebras are isomorphic (as algebras) if and only if their opposite algebras are isomorphic. One can also consider the algebra generated by using a left shift instead of a right shift. In this case, one ends up with a topological conjugacy algebra. One can also define the semi-crossed product algebra as the universal object where certain relations hold. One then can then show that this is equivalent to our definition above. The details of this alternate construction can be found in [27]. We will talk more about this universal construction at the end of the section when we touch on multivariable dynamical systems.

Similarly, one is also able to associate to a dynamical system a C^*-crossed product [21]. These crossed products are not as powerful however as there are examples of dynamical systems which are not conjugate but have isomorphic associated C^*−algebras [21].

4.2. **Characters and Representations of Topological Conjugacy Algebras.** Before we can discuss the main results related to the semi-crossed product algebras, we first need to discuss some results about the characters and representations of topological conjugacy algebras.

Definition 4.4. *Let $\mathcal{A}$ be a algebra over a field $\mathbb{F}$. A **character** of $\mathcal{A}$, ρ, is a homomorphism from $\mathcal{A}$ into $\mathbb{F}$. i.e. a character is a multiplicative linear functional on $\mathcal{A}$.*

For elements $f \in C_0(X)$, characters act as point evaluation maps (i.e. if ρ is a character, then for all $f \in C_0(X)$, there exists $x \in X$ such that $\rho(f) = f(x)$. As a result, for a topological conjugacy algebra, the space of all characters on $\mathcal{A}$, $M_{\mathcal{A}}$ can be partitioned as $M_{\mathcal{A}} = \bigcup_{x\in X} M_{\mathcal{A},x}$, where $M_{\mathcal{A},x} = \{\rho \in M_{\mathcal{A}} : \rho(f) = f(x), \forall f \in C_0(x)\}$. What $M_{\mathcal{A},x}$ looks like depends on if x is a fixed point of the dynamical system. For the case when x is not a fixed point, we need the following proposition.

Proposition 4.5 ([9])**.** *Let X be a locally compact Hausdorff space, $\mathcal{A}$ a topological conjugacy algebra, $\mathcal{B}$ an algebra and $\rho : \mathcal{A} \to \mathcal{B}$ an algebra homomorphism. If $C_0(X)U \subseteq ker\rho$, then $\rho(a) = \rho(E_0(a))$ for all $a \in \mathcal{A}$*

If x is not a fixed point and $\rho \in M_{\mathcal{A},x}$, then a simple argument with the skew relation gives us $\rho(fUg) = \rho(fU)g(x) = \rho(fU)g(\psi(x))$ for all $f, g \in C_0(X)$, which implies that $C_0(X)U \subseteq ker\rho$ which by our above proposition implies that $\rho(a) = \rho(E_0(a)) = E_0(a)(x)$ for all $a \in \mathcal{A}$. Thus $M_{\mathcal{A},x}$ consists of a single character. We shall denote this character as $\theta_{x,0}$. The case when x is a fixed point is quite different. In this case $M_{\mathcal{A},x}$ is isomorphic to a closed unit disk $\mathbb{D}_r$. Let $r = \lim_{n\to\infty} ||U^n||^{\frac{1}{n}}$. Since the mapping $S(a) = aU$ is bounded below, $r > ||T||^{-1} > 0$. By our definition of a topological conjugacy algebra, $\sum_n E_n(a)z^n$ will converge for all complex numbers $|z| < r$ with respect to its Cesaro mean. As a result for each $z \in \mathbb{C}, |z| < r$, we have a character $\theta_{x,z} \in M_{\mathcal{A},x}$ such that $\theta_{x,z}(U) = z$. Using this, we can define a map between $M_{\mathcal{A},x}$ and $\mathbb{D}_r$ that is continuous, dense and injective by sending $\theta_{x,z}$ to z. Since both spaces are compact, this map is automatically a homeomorphism.

We shall now discuss the 2×2 upper triangular representations of a topological conjugacy algebra.

Definition 4.6 ([9])**.** *Let $\mathcal{A}$ be an algebra. Denote the collection of representations of $\mathcal{A}$ onto $\mathfrak{T}_2$, the upper-triangular 2×2 matrices, as $rep_{\mathfrak{T}_2}\mathcal{A}$ and let*

$$\theta_{\pi,i}(a) = \langle \pi(a)\xi_i, \xi_i \rangle, a \in \mathcal{A}, i = 1, 2$$

be characters that correspond to compressions on the $(1,1)$ and $(2,2)$ entries where $\{\xi_1, \xi_2\}$ is the canonical basis of $\mathbb{C}^2$ and where $\langle , \rangle$ is the standard *complex inner product.*

Important for these characters and representations is the fact that if $\mathcal{A}_1, \mathcal{A}_2$ are algebras, $\{\xi_1, \xi_2\}$ is the canonical basis of $\mathbb{C}^2$ and $\gamma : \mathcal{A}_1 \to \mathcal{A}_2$ is an algebra isomorphism, then γ induces the isomorphisms

1. $\gamma_c : M_{\mathcal{A}_1} \to M_{\mathcal{A}_2}$ by $\gamma_c = \theta \circ \gamma^{-1}$
2. $\gamma_r : rep_{\mathfrak{T}_2}\mathcal{A}_1 \to rep_{\mathfrak{T}_2}\mathcal{A}_2$ by $\gamma_r(\pi) = \pi \circ \gamma^{-1}$

These isomorphisms are "compatible" in that $\gamma_c(\theta_{\pi,i}) = \theta_{\gamma_r(\pi),i}$. We can essentially think of the 2×2 upper triangular representations as taking matrix representations of our operators and ignoring all entries outside the upper left corner. For example, if we consider the algebra generated by

$$\pi_x(f) = \begin{bmatrix} f(x) & 0 & 0 & 0 & \dots \\ 0 & f \circ \psi(x) & 0 & 0 & \dots \\ 0 & 0 & f \circ \psi^2(x) & 0 & \dots \\ 0 & 0 & 0 & f \circ \psi^3(x) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and the left shift

$$V = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

then the 2×2 representations of these operators would be

$$\pi(\pi_x(f)) = \begin{bmatrix} f(x) & 0 \\ 0 & f \circ \psi(x) \end{bmatrix}$$

and

$$\pi(V) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Using these representations, we will show that if $\mathcal{A}$ is a topological conjugacy algebra, then we will be able to completely recover said algebra from these representations alone. To do this, we first need to consider a number of cases based on whether or not we have a fixed point. To this end, suppose that $\mathcal{A}$ is a topological conjugacy algebra for the dynamical system (X, ψ) and define:

$$rep_{x_1,x_2}\mathcal{A} = \{\pi \in rep_{\mathfrak{T}_2}\mathcal{A} : \theta_{\pi,1} \in M_{\mathcal{A},x_i}, i = 1, 2\}$$

Then $rep_{\mathfrak{T}_2}\mathcal{A} = \bigcup_{x,y} rep_{x,y}\mathcal{A}$. Note that not all $rep_{x,y}\mathcal{A}$ will be nonempty. The structure of these sets will be based on the relationship between x and y

in our dynamical system. We shall first deal with the situation where neither x nor y is a fixed point. The proof will be included as it helps illustrate how the dynamics of our system get encoded into our algebra.

Proposition 4.7 ([9]). *Let $\mathcal{A}$ be a topological conjugacy algebra for the dynamical system (X, ψ). If $x, y \in X$ with $\psi(x) \neq x$ and $\psi(y) \neq y$ and $\pi \in rep_{x,y}\mathcal{A}$, then $y = \psi(x)$*

Proof. By our hypothesis, $\theta_{\pi,1} = \theta_{x,0}$ and $\theta_{\pi,2} = \theta_{y,0}$. Therefore, for all $g \in C_0(X)$, we get that $\theta_{\pi,1}(gU) = \theta_{\pi,2}(gU) = 0$. As a result $\pi(gU) = \begin{bmatrix} 0 & c_g \\ 0 & 0 \end{bmatrix}$. There will be at least one $g \in C_0(X)$ such that $c_g \neq 0$. If this was not the case, then the range of our algebra would be commutative, contradicting the skew relation. For this g, using the skew relation we get the following:

$$\begin{bmatrix} 0 & c_g \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f(x) & t \\ 0 & f(y) \end{bmatrix} = \begin{bmatrix} f(\psi(x)) & t' \\ 0 & f(\psi(y)) \end{bmatrix} \begin{bmatrix} 0 & c_g \\ 0 & 0 \end{bmatrix} \to f(\psi(x)) = f(y)$$

□

Next, we shall deal with the case were x is not a fixed point, but y is. To do this, we need the following definition.

Definition 4.8 ([9]). *Let X be a locally compact Hausdorff space, and ψ a proper continuous map on X and $\mathcal{A}$ a topological conjugacy algebra for (X, ψ). Let $x, y \in X$ and assume that $\psi(x) \neq x$ but $\psi(y) = y$. A* ***pencil of nest representations*** *for $\mathcal{A}$ is a set $P_{x,y} \subseteq rep_{x,y}\mathcal{A}$ such that $\{\theta_{\pi,2} : \pi \in P_{x,y}\} = (M_{\mathcal{A},y})^\circ$*

Proposition 4.9 ([9]). *Let $\mathcal{A}$ be a topological conjugacy algebra for the dynamical system (X, ψ) and let $P_{x,y}$ be a pencil of representations for $\mathcal{A}$. Then $y = \psi(x)$*

The proof of the above proposition is identical to the proof of the previous one. The reason why we need to define a pencil of representations is that if π were discontinuous, then it is possible to have a situation where $\pi(U) = \begin{bmatrix} 0 & 0 \\ 0 & z \end{bmatrix}$ and for $a \in \mathcal{A}$, $\pi(a) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Such a representation may not exist, but we cannot assume that it does not either. The purpose of the pencil definition is to side-step this difficulty by defining a global object. For a fixed point x, it is not known if $rep_{x,x}\mathcal{A}$ is non-empty. This will not be an issue for us however as $rep_{x,x}\mathcal{A}$ will not play a role in the proof.

Theorem 4.10 ([9]). *Let $(X, \psi), (Y, \tau)$ be one-variable dynamical systems. Then they are conjugate if and only if there exists topological conjugacy algebras of (X, ψ) and (Y, τ) that are isomorphic algebras.*

The full proof of the above above theorem is relatively long, so we will only highlight the key points.

The forward implication is trivial. If $(X,\psi),(Y,\tau)$ are conjugate systems, take the topological conjugacy algebra generated by the π_x maps and the left shift map V. These algebras will clearly be isomorphic.

Conversely, let $\mathcal{A},\mathcal{B}$ be topological conjugacy algebras for $(X,\psi),(Y,\tau)$ which are isomorphic. Let γ be the isomorphism and γ_c be as before. γ_c will biject the maximal analytic disks of $\mathcal{A}$ and $\mathcal{B}$. This bijection extends to a bijection between $\{M_{\mathcal{A},x} : x \in X\}$ and $\{M_{\mathcal{B},y} : y \in Y\}$ i.e., there is a bijection $\gamma_s : X \to Y$ such that $\gamma_c(M_{\mathcal{A},x}) = M_{\mathcal{B},\gamma_s(x)}$. By the above γ_s maps fixed points to fixed point and satisfies the relation $f(\gamma_s(x)) = (\theta_{x,0} \circ \gamma^{-1})(f)$. Using nets, one can show that γ_s is a homeomorphism. To show that γ_s implements conjugacy, we have to check three cases:

1. x is a fixed point
2. $\psi(x) \neq x$ and $\psi^2(x) \neq \psi(x)$
3. $\psi(x) \neq x$, but $\psi^2(x) = \psi(x)$

The first case is trivial as fixed points are mapped to fixed points. Case 2 follows immediately from Proposition 2.9. Finally case 3 follows Proposition 2.11 and the fact that γ_c preserving maximal analytic disks implies that γ_r must preserve pencils of representations.

Corollary 4.11 ([9])**.** *Let $(X,\psi),(Y,\tau)$ be one-variable dynamical systems. Then they are conjugate if and only if $C_0(X) \times_\psi \mathbb{Z}^+$ and $C_0(Y) \times_\tau \mathbb{Z}^+$ are isomorphic as algebras.*

The semicrossed product algebra essentially works by encoding the information of the forward orbits into the algebra structure, so in the case of shift spaces, if we understand the structure of the forward shift of all of our points, then we should be able to determine conjugacy. However, there are many redundancies regarding how this information gets encoded, particularly for minimal shift spaces. With a minimal system, we can essentially determine everything about the system by looking at a single orbit. As a result, encoding the information of the orbit of every point in the space is very likely unnecessary.

4.3. **Generalization to Multivariable Dynamical Systems.** We shall end this section by briefly touching on multivariable dynamical systems and two natural generalizations of semi-crossed product algebras to this setting.

Definition 4.12. *A **multivariable dynamical system** (X,ψ_i), $1 \leq i \leq n$ is a locally compact Hausdorff space X together with n proper continuous maps ψ_i.*

When talking about multivariable dynamical systems, we shall denote (X, ψ_i) $1 \leq i \leq n$ simply by (X, ψ). There is a natural extension of conjugacy from dynamical system to multivariable dynamical systems.

Definition 4.13. *Two multivariable dynamical systems $(X, \psi), (Y, \tau)$ are said to be **conjugate** if there exists a homeomorphism γ of X onto Y and a permutation $\alpha \in S_n$ such that $\tau_i \circ \gamma = \gamma \circ \psi_{\alpha(i)}$ for all $1 \leq i \leq n$*

For multivariable dynamical systems, we have a second natural extension of conjugacy that is weaker then conjugacy.

Definition 4.14 ([8])**.** *Two multivariable dynamical systems $(X, \psi), (Y, \tau)$ are said to be **piecewise conjugate** if there is a homeomorphism γ of X onto Y and an open cover $\{U_\alpha : \alpha \in S_n\}$ of X such that for each $\alpha \in S_n$, $\gamma^{-1} \circ \tau_i \circ \gamma|_{U_\alpha} = \psi_{\alpha(i)}|_{U_\alpha}$ for all $1 \leq i \leq n$*

This second version of conjugacy is weaker in that the permutations depend on the particular open set and in general, more than one permutation is needed. There are cases where conjugacy and piecewise conjugacy do coincide, but this definitely does not hold in general. As an example, consider the multivariable dynamical systems $(\mathbb{Z}_2, \psi_i)$, $(\mathbb{Z}_2, \tau_i)$, $i = 1, 2$ defined by:

$$\begin{aligned} \psi_1(1) &= 1 & \psi_1(2) &= 2 \\ \psi_2(1) &= 2 & \psi_2(2) &= 1 \\ \tau_1(1) &= 1 & \tau_1(2) &= 1 \\ \tau_2(1) &= 2 & \tau_2(2) &= 2 \end{aligned}$$

Where $\mathbb{Z}_2$ is given the discrete topology. Let our open cover be $U_{Id} = \{1\}$, $U_\alpha = \{2\}$, where Id denotes the identity permutation in S_2 and α the swap permutation in S_2. Taking γ to be the identity transformation, we get

$$\begin{aligned} \gamma^{-1} \circ \tau_1 \circ \gamma|_{U_{Id}}(1) &= \psi_1|_{U_{Id}}(1) = 1 \\ \gamma^{-1} \circ \tau_1 \circ \gamma|_{U_\alpha}(2) &= \psi_{\alpha(1)}|_{U_\alpha}(2) = \psi_2|_{U_\alpha}(2) = 1 \\ \gamma^{-1} \circ \tau_2 \circ \gamma|_{U_{Id}}(1) &= \psi_2|_{U_{Id}}(1) = 2 \\ \gamma^{-1} \circ \tau_2 \circ \gamma|_{U_\alpha}(2) &= \psi_{\alpha(2)}|_{U_\alpha}(2) = \psi_1|_{U_\alpha}(2) = 2 \end{aligned}$$

Thus, the systems are piecewise conjugate, but they are clearly not conjugate. Piecewise conjugacy can in a sense be thought of as a local conjugacy.

In the case of these multivariable systems, [8] looks at two operator algebras. These operator algebras are natural generalizations of the semi-crossed product algebra in the one-variable case, as there definitions both coincide with it for $n = 1$. These algebras are defined by a universal property. It is universal in that it is the largest algebra generated by these relations subject to some reasonable constraints such as a cardinalitity constraint, a constraint on the norms of the generators, and the constraint that the algebra is realizable as operators on a Hilbert space. The importance of these constraints is that they guarantee that the universal algebra is a well-defined object. A brief introduction to universal operator algebras (more specifically, universal C^*-algebras) can be found in [3].

Definition 4.15 ([8])**.** *Let (X, ψ_i) $1 \leq i \leq n$ be a multivariable dynamical system.*

1. *The **tensor algebra** is defined to be the universal operator algebra $\mathcal{A}(X, \psi)$ generated by $C_0(X)$ and generators $s_1, ..., s_n$ satisfying the covariance relations $fs_i = s_i(f \circ \psi_i)$ for $f \in C_0(X)$ and $1 \leq i \leq n$ and satisfying the row contractive condition $\|[s_1\ s_2\ \ldots\ s_n]\| \leq 1$*
2. *The **semicrossed product algebra** is defined to be the universal algebra $C_0(X) \times_\psi \mathbb{F}_n^+$ generated by $C_0(X)$ and generators $s_1, ..., s_n$ satisfying $\|s_i\| \leq 1$ for $1 \leq i \leq n$*

It is clear that these algebras coincide with the $n = 1$ case.

Proposition 4.16 ([8])**.** *Let $(X, \psi), (Y, \tau)$ be two multivariable dynamical systems and let $\mathcal{A}, \mathcal{B}$ denote either the tensor algebras and semi-crossed product algebras of (X, ψ) and (Y, τ) respectively. If $\mathcal{A}$ and $\mathcal{B}$ are isomorphic as algebras, then (X, ψ) and (Y, τ) are piecewise conjugate.*

The converse has been shown to hold for $n = 2, 3$ and 4 and special types of systems [8, 30]. For instance, if we assume that X is totally disconnected, then the converse holds at least for the tensor algebra. It is suspected by the authors that the converse is true and in the paper, they provide a technical conjecture about $U(n)$ which if true would imply the converse in full generality.

5. Strongly Maximal TAF-Algebras

In this section, we shall introduce what are called strongly maximal triangular approximately finite (TAF) algebras. By necessity we shall also introduce approximately finite (AF) algebras, but these will not be the focus. Strongly maximal TAF-algebras are more specific in that we can use them to tell apart conjugacy for a type of dynamical system called a Cantor minimal systems, of which infinite minimal shift spaces are an example of. We shall only focus on infinite minimal shift spaces in this section are we already addressed how to tell apart periodic and aperiodic substitution rules. At the end of this section, we will also show that all minimal shift spaces defined by a substitution are conjugate to a shift space defined by a primitive substitution.

5.1. **AF-Algebras, TAF-Algebras, and Bratteli Diagrams.**

Definition 5.1. *Suppose we have a sequence of algebras $\mathcal{A}_n$, and homomorphisms $\phi_n : \mathcal{A}_n \to \mathcal{A}_{n+1}$.*
*$(\mathcal{A}_n, \phi_n)$ is called an **algebraic chain system** and ϕ_n are called connecting homomorphisms.*

Definition 5.2 ([4])**.** *An algebra $\mathcal{A}$ is said to be a **AF-algebra** if there exists a sequence of finite dimensional C^*-algebras $\{\mathcal{A}_n\}$ and a sequence of $*$-homomorphisms $\{\phi_n\}$ such that $\mathcal{A}$ is the closure of the algebraic direct limit of the chain system $(\mathcal{A}_i, \phi_i)_{i\in\mathbb{N}}$. i.e. we can write $\mathcal{A} = \overline{\bigcup_{i=1}^{\infty} \mathcal{A}_i}$.*

Finite-dimensional C^*-algebras have a nice characterization which gives a natural way to associate AF-algebras, and by extension TAF-algebras to a given substitution.

Theorem 5.3 ([7])**.** *Let $\mathcal{A}$ be a finite dimensional C^*-Algebra. Then there exist positive integers $K, N_1, ..., N_K$ such that:*

$$\mathcal{A} \cong M_{N_1}(\mathbb{C}) \oplus, ..., \oplus M_{N_K}(\mathbb{C})$$

Furthermore, K is uniquely determined, and $N_1, ..., N_K$ are unique up to permutation.

Proposition 5.4 ([7])**.** *Let $\mathcal{A} \cong M_{N_1}(\mathbb{C}) \oplus, ..., \oplus M_{N_K}(\mathbb{C})$, $\mathcal{B} \cong M_{N_1'}(\mathbb{C}) \oplus, ..., \oplus M_{N_l'}(\mathbb{C})$ be finite-dimensional C^*-algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism. Then ϕ is inner equivalent to the embedding of copies of the full matrix algebras of $\mathcal{A}$ into the full matrix algebras of $\mathcal{B}$ as block diagonal entries along with zero matrices when needed. $*$-homomorphisms of this form are said to be canonical.*

Going forward, we will assume that all $*$-homomorphisms are canonical. A simple example of a canonical $*$-homomorphism would be $\phi(\mathbb{C} \oplus M_2) \to M_2 \oplus M_4$ given by:

$$\phi(\mathbb{C} \oplus M_2) = M_2 \oplus \begin{bmatrix} M_2 & 0 & 0 \\ 0 & \mathbb{C} & 0 \\ 0 & 0 & \mathbb{C} \end{bmatrix}$$

where the 0's represent zero matrices of appropriate sizes.

Definition 5.5. *Let $\mathcal{A}_1 \cong \bigoplus_{i=1}^{n_1} M_{m_i}$ and $\mathcal{A}_2 \cong \bigoplus_{i=1}^{n_2} M_{m_i'}$ be two finite-dimensional C^*-algebras and let ϕ be a canonical $*$-homomorphism from*

$\mathcal{A}_1$ *into* $\mathcal{A}_2$*. We define the* ***transition matrix*** B *to be an* $n_1 \times n_2$ *matrix where the* (i,j)*-entry denotes the number of copies of* M_{m_i} *inside of* $M_{m'_j}$*.*

Definition 5.6 ([4])**.** *A* ***labelled Bratteli diagram*** *is an infinite directed graph which satisfies the following properties:*

1. *. The vertex set* V *and edge set* E *can be written as the disjoint countable union of nonempty finite sets* $V = \bigcup_{n=0}^{\infty} V_n$*. and* $E = \bigcup_{n=0}^{\infty} E_n$ *respectively.*

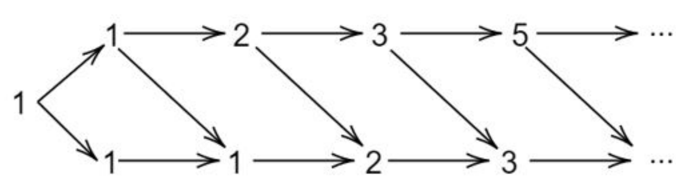
2. $r(E_n) \subseteq V_n$ *and* $s(E_n) \subseteq V_{n-1}$ *were* r *and* s *are the range and source map respectively. Additionally, for all* $v \in V$ $s^{-1}(v) \neq \varnothing$

3. d*, the "labelling" of the Bratteli diagram is a mapping* $d : V \to \mathbb{N}$ *such that* $d(v) \geq \sum_{r(e)=v} d(s(e))$ *for all* $v \in V \backslash V_0$

To every Bratteli diagram, we can associate an AF-algebra and to every AF-algebra we can associate a Bratteli diagram. If $\mathcal{A}_n \cong M_{N_1}(\mathbb{C}) \oplus, ..., \oplus M_{N_K}(\mathbb{C})$ and $\mathcal{A}_{n+1} \cong M_{N'_1}(\mathbb{C}) \oplus, ..., \oplus M_{N'_l}(\mathbb{C})$, and ϕ_n is a $*$-homomorphism, Then we can build a Bratteli diagram by associating to each M_{N_i} and each $M_{N'_j}$ a vertex. Then the number of edges between M_{N_i} and $M_{N'_j}$ equals the number of copies of M_{N_i} that get embedded inside M_{N_j}. As a result, to each level of the Bratteli diagram, we can associate a transition matrix which is defined in the same way as above. Also note that by convention, it is typically assumed that $\mathcal{A}_1 = \mathbb{C}$.

To associate a Bratteli diagram to a symbolic substitution, we can simply let the transition matrix equal the substitution matrix at each level. Equivalently, at each level, we can associate to each vertex a letter from our alphabet. Then the number of edges between the vertex corresponding to a on level n and b on level $n+1$ is equal to the number of copies of a that occur in b. So long as we are consistent with our labeling, the transition matrices at each level will equal the substitution matrices

Example 5.7. *The Bratteli Diagram associated to the Fibonacci substitution. Note that its substitution matrix is* $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$



Note that Bratteli diagrams be convention are often drawn starting at the top and going down. For the sake of saving space, we will use the convention of going left to right.

Properties of the substitutions do get encoded into the AF-algebras. More information about this and further resources can be found in [14]. Despite this, they are not very useful for dynamical system purposes. As an example, AF-algebras do not preserve aperiodicity. This is because many different substitutions get associated to the same AF-algebra. Namely, every substitution with the same transition matrix as well as many others. For instance the famous Thue-Morse substitution given by:

$$a \to ab$$

$$b \to ba$$

and the substitution given by

$$a \to ab$$

$$b \to ab$$

both have the same substitution matrix. The first substitution is aperiodic while the second is periodic, as a result the shift systems that they define are not conjugate.

It is clear that this issues in part comes from the fact that AF-algebras do not take into account the order of letters as they appear in the substitution rule. To get around this, we shall introduce a partial order on the edges. By extending this partial order to the set of infinite paths, we are able to define a successor map which will turn our Bratteli diagram into a partial dynamical system.

5.2. **Bratteli Diagrams as Partial Dynamical Systems.**

Definition 5.8 ([19]). *An* **ordered Bratteli diagram** $(V, E, r, s, d, \leq)$ *is a Bratteli diagram* (V, E, r, s, d) *with a partial order on the edges such that two edges* e, e' *are comparable if and only if* $r(e) = r(e')$*.*

We can extend this to a partial order on the set of all finite paths where two paths $p_1 = (e_k, e_{k+1}, ..., e_{k+l})$, $p_2 = (e'_k, e'_{k+1}, ..., e'_{k+n})$ are comparable if and only if they are between the same levels of vertices and $r(e_{k+n}) = r(e'_{k+n})$. In this case, we say that $p_1 < p_2$ if and only for some i with $k+1 \leq i \leq k+n$, we have

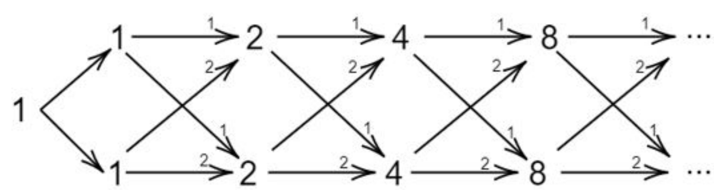
$$e_i < e'_i \quad \text{and} \quad e_j = e'_j,\ i < j \leq k+n.$$

To these ordered Bratteli diagrams, we are able to associate what is called a strongly maximal TAF-algebra, which is a subalgebra of the AF-algebra.

Definition 5.9 ([29]). *Let $\mathcal{A}$ be an AF-algebra. A subalgebra $\mathcal{T}$ of $\mathcal{A}$ is said to be a **TAF algebra** if $\mathcal{T} \cap \mathcal{T}^* = \mathcal{D}$ were $\mathcal{D}$ is a maximal abelian self-adjoint subalgebra of $\mathcal{A}$. $\mathcal{T}$ is said to be a **maximal TAF algebra** if it is the only AF subalgebra containing $\mathcal{T}$. $\mathcal{T}$ is said to be **strongly maximal** if a sequence $\mathcal{A}_n$ can be chosen such that $\mathcal{A} = \overline{\bigcup_{n=1}^{\infty} \mathcal{A}_n}$ and $\mathcal{T} \cap \mathcal{A}_n$ is maximal triangular in every $\mathcal{A}_n$.*

Proposition 5.10 ([29]). *Let $\mathcal{T}$ be a strongly maximal TAF subalgebra of $\mathcal{A} = \overline{\bigcup_{n=1}^{\infty} \mathcal{A}_n}$ such that $\mathcal{T} \cap \mathcal{A}_n$ is maximal in every $\mathcal{A}_n$. Then $\mathcal{T} \cap \mathcal{A}_n \cong \bigoplus_{k=1}^{m} \mathcal{T}_{n_k}$ were $\bigoplus_{k=1}^{m} \mathcal{T}_{n_k}$ is a block upper triangular algebra and $n_1 + n_2 + \ldots n_m = n$*

This characterization of strongly maximal TAF-algebras allows us to assign to it an ordered Bratteli diagram where the order of the number of edges between levels is based on the number of copies of $\mathcal{T}_{n_i}$ that get embedded into $\mathcal{T}_{m_j}$ and the order on the edges is determined by the order of the embeddings of $\mathcal{T}_{n_1}, \ldots \mathcal{T}_{n_k}$ into $\mathcal{T}_{m_i}$. For example, if we had an embedding of $\mathcal{T}_{2^n} \oplus \mathcal{T}_{2^n}$ into $\mathcal{T}_{2^{n+1}} \oplus \mathcal{T}_{2^{n+1}}$ defined by $a \oplus b \to (a \oplus b) \oplus (a \oplus b)$ where $a, b \in \mathcal{T}_{2^n}$, then the ordered Bratteli diagram that corresponds to this embedding would be:



Naively, we can associate an ordered Bratteli diagram to a substitution by letting the order of the edges correspond to the order of the letters as they appear in the substitution. For instance, if we had the substitution given by

$$a \to ab$$

$$b \to ab$$

Then this would correspond to the above ordered Bratteli diagram.

Note that in the above diagram, we are assuming that the edge labeled one is greater than the edge labeled two and so on. This is done to emphasize the lexicographic order. This type of association does not actually preserve conjugacy however except for in special circumstances. To show why this is the case we first need to turn the path space of the ordered Bratteli diagram into a partial dynamical system and then explain how this partial dynamical system corresponds to the TAF-algebra.

Definition 5.11 ([19]). *Let B be an ordered Bratteli diagram and let $X_B = \{(e_1, e_2, ...) : e_i \in E, r(e_i) = e_{i+1}\}$ be the set of all infinite paths. Let $(e_1, e_2, ..., e_k)$ be a finite path in our Bratteli diagram. The **cylinder set** of $(e_1, e_2, ..., e_k)$ is defined to be $U(e_1, e_2, ..., e_k) = \{(f_1, f_2, ...) : f_i \in E, r(f_i) = f_{i+1}, e_i = f_i \forall 1 \leq i \leq k\}$*

The cylinder sets define a topological basis on the set of all infinite paths and so long as X_B is infinite, it is easy to see that this turns the set of all infinite paths into a Cantor set.

Definition 5.12 ([19]). *Let $e = (e_1, e_2, ...)$ be an infinite path in an ordered Bratteli diagram B. Let $e(n) = e_n$. The set of all infinite maximal paths X_{max} is defined to be the set of paths e such that $e(n)$ is a maximal edge for all n. The set of all minimal paths X_{min} is defined similarly.*

The set of maximal and minimal paths are always non-empty, but they may not necessarily be disjoint.

Definition 5.13. *Let (V, E, r, s, d) be a labeled Bratteli diagram and let $m_0 < m_1 < ...$ be an increasing sequence non-negative integers. The **telescoping** (or contraction) of (V, E, r, s, d) with respect to m_n is the labeled Bratteli diagram (V', E', r', s', d') where $V'_n = V_{m_n}, E'_n = E_{m_{n-1}+1} \circ E_{m_{n-1}+2} \circ ... \circ E_{m_n}$, d' is the restriction of d to V'. r' and s' are the extensions of r and s restricted to the paths E'_n.*
Microscoping is the inverse of telescoping.

When (V, E, r, s, d) gets telescoped into (V', E', r', s', d'), the incidence matrices of (V', E', r', s', d') will correspond to the matrices of (V, E, r, s, d) multiplied together. i.e., if A_1, A_2 were the incidence matrices of the first two levels of (V, E, r, s, d), the incidence matrix (V', E', r', s', d') where the first two levels are telescoped together would be $A_1 A_2$ (or $A_2 A_1$ we adopted the convention of going top down).

Definition 5.14 ([11]). *An ordered Bratteli diagram V is said to be **simple** if there exists a telescoping V' of V such that the incidence matrices at each level consist of strictly nonzero entries. V is said to be properly ordered if it additionally has a unique minimal and maximal path.*

It is easy to see that simple ordered Bratteli diagrams have disjoint minimal and maximal paths. Additionally, it is easy to see that a substitution is primitive if and only if its associated ordered Bratteli diagram is simple.

Definition 5.15 (Successor (Vershik) map). *Let $(V, E \leq)$ be a properly ordered Bratteli diagram. Define the following homeomorphism T on X:*
Let $T(x_{max}) = x_{min}$. Let $x = (e_1, e_2, ...) \in X$ not equal to x_{max}. Then

$T(x) = (f_1, f_2, ..., f_k, e_{k+1}, ...)$ were e_k is the first non-maximal edge, f_k is the successor of e_k and $(f_1, ..., f_{k-1})$ is the minimal path from v_0 to $s(f_k)$.

The successor map was introduced by Vershik in [32], but the terminology is different. Markov Partitions are Bratteli diagrams and adic transformations are the dynamics on those diagrams.

Properly ordered Bratteli diagrams are an example of Cantor minimal systems. In [19], it was shown that all Cantor minimal systems are conjugate to a properly ordered Bratteli diagram. Since our aperiodic primitive substitutions are examples of Cantor minimal systems, there will always exist an ordered Bratteli diagram that it is conjugate to. This does not imply that the hull associated to our substitution is conjugate to the ordered Bratteli diagram associated to our substitution in the way that we did it above.

The successor map can be extended to non-properly ordered Bratteli diagrams by restricting the domain to no include the maximal paths and the range to not include the minimal path. In the first case, this turns our path space into a dynamical system and in the second a partial dynamical system. The relationship between ordered Bratteli diagrams, partial dynamical systems and TAF-algebras is summarized in the following theorem.

Proposition 5.16 ([28]). *Let $\mathcal{T}_1, \mathcal{T}_2$ be two strongly maximal TAF-algebras and let B_1, B_2 be ordered Bratteli diagrams that correspond to $\mathcal{T}_1, \mathcal{T}_2$ respectively. Then the following are equivalent:*

1. *$\mathcal{T}_1$ is isometrically isomorphic to $\mathcal{T}_2$.*
2. *B_1 is telescope equivalent to B_2.*
3. *The partial dynamical system defined on the path space of B_1 is conjugate to the partial dynamical system defined on the path space of B_2.*

From this characterization, we can introduce a simple invariant which allows us to construct a counterexample.

Proposition 5.17. *Let $(V, E, \leq)$ be an ordered stationary Bratteli diagram. Let $(V', E', \leq')$ be a telescoping of $(V, E, \leq)$. Then $(V, E, \leq)$ and $(V', E', \leq')$ have the same number of maximal and minimal infinite paths.*

Proof. Let $0 < n_1 < n_2 < ...$ define the telescoping of $(V', E', \leq')$ Define the bijection $F : X_{(V,E)} \to X_{(V',E')}$ by $F : (e_1, e_2, ...) = ((e_1, ..., e_{n_1}), (e_{n_1+1}, ..., e_{n_2}), ...)$. Clearly if $(e_1, e_2, ...)$ is maximal or minimal, then so is $((e_1, ..., e_{n_1}), (e_{n_1+1}, ..., e_{n_2}), ...)$. Conversely, if $((e_1, ..., e_{n_1}), (e_{n_1+1}, ..., e_{n_2}),...)$ is maximal or minimal then each finite path $(e_{n_i}, e_{n_i+1}, ...e_{n_{i+1}})$ must also be maximal or minimal, which implies that $(e_1, e_2, ...)$ is also maximal or minimal. □

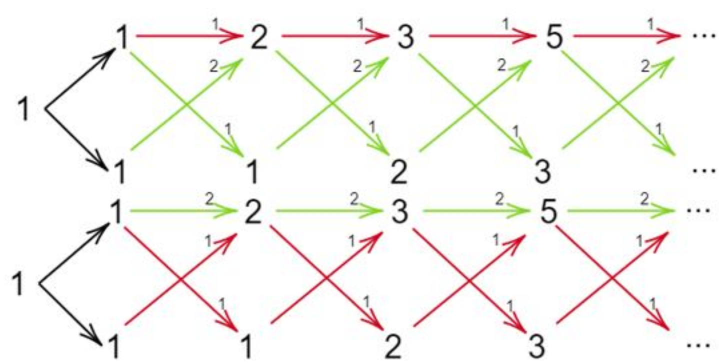
For our counter-example, consider the following substitutions:

$$\sigma_1 : a \to ab$$
$$\sigma_1 : b \to a$$
$$\sigma_2 : a \to ba$$
$$\sigma_2 : b \to a$$

These substitutions define the same hull. This follows from the following proposition.

Proposition 5.18 ([1])**.** *Let σ be a primitive substitution over an alphabet $\{a_1, ..., a_n\} = \mathcal{A}$ and let $u \in \mathcal{A}^*$ be a finite word. Define σ_u as $\sigma_u(a_i) = u^{-1}\sigma(a_i)u, 1 \leq i \leq n$, where u^{-1} is the formal inverse of u. If σ_u defines a non-negative substitution, then σ_u is primitive and σ and σ_u define the same hull.*

Taking $u = a$, we can see that σ_1, σ_2 define the same hull and therefore define conjugate systems. They do not have the same TAF-algebras however and the number of maximal and minimal paths differ. Note that the maximal paths are coloured red while the minimal paths are coloured green.



We can still use TAF-algebras to determine conjugacy. Using what are called Kakutani-Rohlin partitions, to any Cantor minimal system we can associate a conjugate ordered Bratteli diagram, so in principle, we can use TAF-algebras to determine conjugacy of our shift space. More details about these partitions can be found in [19, 28]. For our system defined by primitive substitutions, we will use a different method for associating ordered Bratteli diagrams. The advantage of this method is that it provides a simple way for determining what is called the dimension group associated to our dynamical system. The dimension group of a dynamical system is a conjugacy invariant. More details about this can be found in [11]. To do this, we shall now introduce what are called proper substitutions.

5.3. **Proper Substitutions and Properly Ordered Bratteli Diagrams.**

Definition 5.19 ([11]). *A substitution σ on an alphabet $\mathcal{A}$ is said to be* ***proper*** *if there exists an integer $p > 0$ and letters $r, l \in \mathcal{A}$ such that for every $a \in \mathcal{A}$, r is the first letter of $\sigma^p(a)$ and l is the last letter of $\sigma^p(a)$.*

The motivation for this definition comes from the following proposition.

Proposition 5.20 ([11]). *A primitive substitution is proper if and only if its associated ordered Bratteli diagram is properly ordered.*

In the case of our properly ordered primitive substitutions our "naive" association ends being able to determine conjugacy between proper primitive substitutions.

Proposition 5.21 ([11]). *Let σ be a proper primitive substitution and let V be the associated ordered Bratteli diagram. If σ is aperiodic, its hull is conjugate to the path space of the ordered Bratteli diagram*

With the following proposition, we can extend this to all primitive substitutions.

Theorem 5.22 ([11]). *Let σ be any primitive aperiodic substitution. Then there exists a primitive proper aperiodic substitution τ such that they define conjugate shift spaces.*

The full proof of the above theorem while not overly challenging is long and requires a lot of set up, so we shall only give an outline here. For the reader interested in symbolic dynamics, reading through the full proof and its setup will be of interest since it uses ideas important to the unique decomposition of sequences in shift spaces. For this we need the following definition.

Definition 5.23 ([11]). *A word w of the alphabet $\mathcal{A}$ is said to be a return word of $u.v$ in x if there exists two consecutive occurrences j, k of $u.v$ in x such that $w = x_{[j,k)}$*

For any primitive substitution σ, there are only a finite number of return words inside the hull. This follows from the linear recurrence of elements inside the hull. Denote the set of return words for $u.v$ as $\mathcal{R}_{u.v}$. Let $R_{u.v} = \{1, ..., card(\mathcal{R}_{u.v})\}$. Order these return words based on there first occurrence inside $x_{[0,\infty)}$ where $x \in \mathbb{X}(\sigma)$. Then we define $\phi_{u.v} : R_{u.v} \to \mathcal{R}_{u.v}$ where $\phi_{u.v}(i)$ equals the i'th return word. All elements of $\mathcal{A}^{\mathbb{Z}}$ can be uniquely decomposed into elements of $\mathcal{R}_{u.v}$. This allows us to extend $\phi_{u.v}$ into a bijective mapping from $\mathcal{R}_{u.v}^{\mathbb{Z}}$ into $\mathcal{A}^{\mathbb{Z}}$. This leads to the following definition:

Definition 5.24 ([11]). *Let $x \in \mathcal{A}^{\mathbb{Z}}$. The u.v* ***derivative*** *of x, denoted $D_{u.v}(x)$ is the unique sequence in $R_{u.v}$ such that $\phi_{u.v}(D_{u.v}(x)) = x$*

Finally, we defined our desired map τ as follows. For any return word $w \in \mathcal{R}_{u.v}$, there exists a unique word $u \in R_{u.v}^{+}$ and a unique $j \in R_{u.v}$ such that $\sigma(w) = \phi_{u.v}(u)$ and $\phi_{u.v} = u$. Therefore, we define τ as $\tau(j) = w$. This map is proper, primitive and aperiodic and defines a conjugate shift space.

While the focus of this section has been on primitive substitutions, due to all Cantor minimal systems being conjugate to ordered Bratteli diagrams, TAF-algebras can be used for any substitution that defines an infinite minimal shift space. This leads to the obvious question of given a non-primitive substitution that defines a minimal infinite shift space, does there exist a primitive substitution such that the shift spaces that they define are conjugate? The answer to this question is yes. Moreover, we can explicitly construct said substitution.

5.4. Non-Primitive Substitutions Associated to Minimal Shift Spaces.

We shall start of by introducing tame and wild substitutions. Substitutions that define an infinite minimal shift space will end up being tame. This tameness property will end up allowing us to define a primitive substitution which will be conjugate to our original substitution.

Definition 5.25 ([23]). *Let $\sigma : \mathcal{A} \to \mathcal{A}^*$. A word $u \in \mathcal{A}^*$ is said to be* ***bounded*** *with respect to σ if there exists $M \in \mathbb{N}$ such that $|\sigma^n(u)| \leq M$ for all $n \in \mathbb{N}$ and* ***expanding*** *if it is not bounded. The set of all bounded letters of σ is denoted $\mathcal{A}_B$ and the set of expanding letters is denoted $\mathcal{A}_\infty$.*

Definition 5.26 ([23]). *Let σ be a substitution and $\mathcal{B}$ the set of bounded legal words for σ. If $\mathcal{B}$ is finite, we say that σ is* ***tame****. If $\mathcal{B}$ is infinite, we say that σ is* ***wild****.*

Note that tameness cannot be seen by looking at the shift space. For instance the substitution given by:

$$a \to ab$$
$$b \to b$$

and

$$a \to abb$$
$$b \to bbb$$

Both define the same subshift, namely the subshift containing only $...bbb.bbb...$. The first substitution is wild while the second one is tame.

Definition 5.27 ([23])**.** *Let $\mathcal{A}_{right} \subseteq \mathcal{A}_\infty$ denote the set of all expanding letters such that for every $a \in \mathcal{A}_{right}$, the rightmost letter of $\sigma(a)$ is a bounded letter. Define $\mathcal{A}_{left}$ similarly.*

As an example, if we had the substitution on the three letter alphabet $\{a, b, c\}$:

$$\sigma(a) = cab$$
$$\sigma(b) = b$$
$$\sigma(c) = a$$

Then $a \in \mathcal{A}_{right}$ but not b or c

Definition 5.28 ([23])**.** *Suppose there exists a letter $a \in \mathcal{A}_{right}$ and an increasing sequence of integers N_i such that the rightmost expanding letter appearing in $\sigma^{N_i}(a)$ is also in $\mathcal{A}_{right}$ for all $i \geq 1$ or else there exists a letter $a \in \mathcal{A}_{left}$ and an increasing sequence of integers N_i such that the leftmost expanding letter appearing in $\sigma^{N_i}(a)$ is also in $\mathcal{A}_{left}$ for all $i \geq 1$. Then we say that σ has **property** $(*)$.*

Lemma 5.29 ([23])**.** *Let σ be a substitution on $\mathcal{A}$ with property $(*)$. Then $X(\sigma)$ contains a periodic sequence, the letters of which are bounded.*

Proof. The proof will be for the case of $\mathcal{A}_{right}$, but $\mathcal{A}_{left}$ will follow similarly. Let $a \in \mathcal{A}_{right}$ be a letter such that there exists a sequence of integers $\{N_i\}_{i \in \mathbb{N}}$ such that the rightmost expanding letter of $\sigma^{N_i}(a)$ is in $\mathcal{A}_{right}$. Without loss of generality, we can assume that there exists $N \in \mathbb{N}$ such that rightmost expanding letter is also a. We can do this by noting that the right most expanding letters of $\sigma^{N_k}(a)$ must also have the same property as a by taking the shifted sequence $M_i = N_{i-k}$. Then we can write $\sigma^N(a) = vau$, where u is a bounded word. Then by induction we can write:

$$\sigma^{(k+1)N}(a) = \sigma^{kN}(v)...\sigma^N(V)vau\sigma^N(u)...\sigma kN(u)$$

Since u is a bounded word, there exists K such that $|\sigma^{(K+1)N}(u)| = |\sigma^{KN}(u)|$. Since there are only finitely many words of this length, by potentially replacing σ with an appropriate power, we can choose K so that $\sigma^{(K+1)N}(u) =$

$\sigma^{KN}(u)$. Thus for all $j \geq K$, $(\sigma^{KN}(u))^j$, where $(\sigma^{KN}(u))^j$ denotes $(\sigma^{KN}(u))$ concatenated to itself j times, appears as a subword of $\sigma^n(a)$ for a large enough n. Thus, the infinite periodic word $...\sigma^{KN}(u)\sigma^{KN}(u)\sigma^{KN}(u)...$ appears inside $\mathbb{X}(\sigma)$. $\square$

Taking our above example and iterating σ on a, we see that $...bbb.bbb... \in \mathbb{X}(\sigma)$

$$a \to ca\underline{b} \to aca\underline{bb} \to cabaca\underline{bbb} \to$$
$$acabbcabaca\underline{bbbb} \to cabacabbbacabbcabbcabaca\underline{bbbbb} \to \cdots$$

This lemma allows us characterize wild substitutions in terms of this $(*)$ property.

Theorem 5.30 ([23])**.** *Let σ be a substitution on an alphabet $\mathcal{A}$. Then σ is wild if and only if it has property $(*)$*

From this theorem, it is immediate that substitutions which define aperiodic shift spaces are tame. The following lemma on tame substitutions is the last result that we shall need before we begin defining our primitive substitution. This lemma follows immediately from the definition of tameness.

Lemma 5.31 ([23])**.** *Let σ be a tame substitution on $\mathcal{A}$. If $\mathbb{X}(\sigma)$ is nonempty, then it contains a bi-infinite sequence $w \in \mathbb{X}(w)$ with the property that there exists $M \in \mathbb{N}$ such that every subword u of w of length greater than M contains an expanding letter. In particular, w contains infinitely many expanding letters.*

We shall now define our substitution. To this end, let σ be a non-periodic substitution such that $\mathbb{X}(\sigma)$ is minimal. Then σ is tame. By minimality and tameness, there exists $b \in \mathcal{A}_\infty$, $w \in \mathbb{X}(\sigma)$ and $N \in \mathbb{N}$ such that $\sigma^N(w) = w$ such that $\sigma^N(b)$ will be a subword of w. By linear recurrence, for sufficiently large N, and for any legal word u, there will exist $k_u \in \mathbb{N}$ such that $\sigma^N(b)$ contains u and two copies of b. Define the return words of b as $\mathcal{B} \coloneqq \{bu : u$ does not contain b and bub is legal$\}$. Note that this definition coincides with the definition given before. The return words are finite therefore we can enumerate $\mathcal{B}\backslash\{b\} = \{v_1, ..., v_k\}$. where we denote $v_0 = b$ if $b \in \mathcal{B}$. Using this, we can decompose $\sigma^N(b)$ as:

$$uv_{01}...v_{0r_0}$$

Since we assumed that $\sigma^N(b)$ contains at least two b's, $r_0 \geq 2$. Since $\sigma^N(b)$, then for all $j < r$, $v_{0j}b$ is also legal, thus $v_{0j} \in \mathcal{B}$.
Then for all $i \geq 1$, we can now write:
$\sigma^N(v_i) = \sigma^N(b)w_i v_{i1}...v_{ir_i}$, where w_i does not contain b, v_{ij} is of the form bv, where v does not contain b and $r_i \geq 0$. Similar to before, if $r_i > 0$, then

for $j < r$, $v_{ij} \in \mathcal{B}$. While v_{ir_i} may not appear in $\mathcal{B}$, it will be the case that $v_{ir_i}u \in \mathcal{B}$ for $i > 0$. This follows from the fact that $\sigma^N(v_i b)$ is a legal word containing $v_{ir_i}ub$. For the case of $i = 0$, by the form of $\sigma^N(v_i)$ shown above, it follows that $v_{0r_0}w_i \in \mathcal{B}$ if $i > 0$ and $v_{0r_0}w_i u \in \mathcal{B}$ if $i = 0$. By denoting $v_{ir_i}u = v'_{ir_i}$ for all i and $v_{0r_0 w_i u} = w'_i$ for $i > 0$, we can now define a new substitution which will be primitive and define a hull conjugate to $\mathbb{X}(\sigma)$

Definition 5.32 ([23]). *Let $\mathcal{C}$ be an alphabet such that it is disjoint from $\mathcal{A}$ and $\mathcal{B}$ and such that $|\mathcal{C}| = |\mathcal{B}|$. Let $\alpha : \mathcal{B} \to \mathcal{C}$ be any bijection between $\mathcal{B}$ and $\mathcal{C}$. Denote the image of $v \in \mathcal{B}$ and $\tilde{v}$. Then the substitution $\psi : \mathcal{C} \to \mathcal{C}^*$ is defined by:*

$$\psi(\tilde{v_0}) = \tilde{v_{01}} ... \tilde{v_{0r_0-1}} \tilde{v'_{0r_0}}$$

if $b \in \mathcal{B}$ and

$$\psi(\tilde{(v_i)}) = \tilde{v_{01}} ... \tilde{v_{0r_0-1}} \tilde{w_i}' \tilde{v_{i1}} ... \tilde{v_{ir_i-1}} \tilde{v'_{ir_i}}$$

if $r_i > 0$ or

$$\psi(\tilde{v_i}) = \tilde{v_{01}} ... \tilde{v_{0r_0-1}} \tilde{w_i}'$$

if $r_i = 0$ for all $i > 0$

Proposition 5.33 ([23]). *The substitution defined above is primitive.*

Proof. For any $v \in \mathcal{B}$, we can find $n_v \in \mathbb{N}$ such that vb is a subword of $\sigma^{n_v N}(b)$. Therefore choose $l = max_{v \in \mathcal{B} n_v}$. Because all words of the form vb where $v \in \mathcal{B}$ can be found in $\sigma^{lN}(b)$ and because these words can only overlap at most there first or last letters, all elements of $\mathcal{B}$ are subwords of $\sigma^{lN}(b)$ where no two share any common indices. Additionally, since for all $w \in \mathcal{B}$, $\sigma(w)$ starts with uv_{01} where b is a subword of v_{01}, $\sigma^{(l+1)N}(w)$ contains all $v \in \mathcal{B}$, thus for all $w \in \mathcal{B}$ since $\psi(\tilde{w})$ starts with $\tilde{v_{01}}$, ψ^{l+1} will contain $\tilde{v}$ for all $v \in \mathcal{B}$, therefore ψ is primitive. □

Theorem 5.34 ([23]). *Let ϕ be a minimal substitution with non-empty minimal shift space $\mathbb{X}(\phi)$. Then there exists an alphabet $\mathcal{Z}$ and a primitive substitution σ on $\mathcal{Z}$ such that that $\mathbb{X}(\sigma)$ and $\mathbb{X}(\phi)$ are conjugate.*

This theorem relies on the following definition and proposition.

Definition 5.35 ([10]). *A bi-infinite word w over an alphabet $\mathcal{A}$ is called **substitutive** if there exists a substitution ψ on an alphabet $\mathcal{B}$ and a map $\phi : \mathcal{B} \to \mathcal{A}$ such that $\phi(v) = w$ where v is a fixed point of ϕ. If ψ is primitive, then w is called **substitutive primitive**.*

Proposition 5.36 ([10]). *Let ψ be a primitive substitution over an alphabet $\mathcal{C}$ and let X_ψ denote on of its fixed points (if no such fixed point exists, pass to a appropriate power). Let $g : \mathcal{C} \to \mathcal{A}^+$ be a map. Then $g(X_\psi)$ is substitutive primitive.*

Proof. Let $\mathcal{Z} := \{(\tilde{v}, k) : \tilde{V} \in \mathcal{C}, 1 \leq k \leq |g(\tilde{v})|\}$ and define $\sigma : \mathcal{C} \to \mathcal{Z}^+$ by $\sigma(\tilde{v}) = (\tilde{v}, 1)...(\tilde{v}, |g(\tilde{v})|)$ Since ψ is primitive, we can assume without loss of generality that $|\psi(\tilde{v})| \geq |g(\tilde{v})|$ Define $\theta : \mathcal{Z} \to \mathcal{Z}^+$ by:

$$\theta((\tilde{v}, k)) = \sigma(\psi(\tilde{v})_{[k,k]})$$

for $k < |g(\tilde{v})|$ and $\theta((\tilde{v}, k)) = \sigma(\psi(\tilde{v})_{[|g(a)|, |\psi(a)|]})$ for $k = |g(a)|$ Then we get for all $\tilde{v} \in \mathcal{C}$ we get:

$$\theta(\sigma(\tilde{v})) = \sigma(\psi(\tilde{v}))$$

Therefore if w is a fixed point of ψ, $\sigma(w)$ will be a fixed point of θ. Since $\theta^n \sigma = \sigma \psi^n$, the primitivity of ψ implies the primitivity of θ. Finally define: $h : \mathcal{Z} \to \mathcal{A}$ by $h(g(\tilde{v})_{[k,k]}) = (\tilde{v}, k)$. Then $h(\sigma(w)) = g(w)$ □

Corollary 5.37 ([23])**.** *Let $\psi : \mathcal{C} \to \mathcal{C}^*$ be a primitive substitution and let g be a map from $\mathcal{C}$ to $\mathcal{A}^*$. Let $X_g \subset \mathcal{A}^{\mathbb{Z}}$ be defined as $X_g := \{S_A^n(g(x)) : x \in X_\psi, n \in \mathbb{Z}\}$. Then there exists a an alphabet $\mathcal{Z}$, a primitive substitution $\theta : \mathcal{Z} \to \mathcal{Z}^+$ and a map $h : \mathcal{Z} \to \mathcal{A}$ such that $h(X_\theta) = X_g$*

Proof. Let ψ be defined as above and define $g : \mathcal{C} \to \mathcal{A}^+$ to be $g(\tilde{v}_i) = v_i$ where $v_i \in \mathcal{A}^+$. Then as all elements of X_ϕ can be uniquely decomposed into it return words, by the construction of ψ, it is easy to see that $X_g = X_\psi$. By our above proposition there exists a primitive substitution θ and a map h such that $h(\sigma(\tilde{v})) = g(\tilde{v})$. h is easily checked to be a factor map between X_θ and X_ϕ. More over, it is a conjugacy map. Elements of $\mathcal{Z}$ are the set of all pairs $(\tilde{v}, k)$ where $v \in \mathcal{B}$ and $1 \leq k \leq |v|$. Every sequence in X_ψ can be uniquely represented as return words from $\mathcal{B}$. The we define $p : X_\phi \to \mathcal{Z}^{\mathbb{Z}}$ as follows:
Let $w \in X_\phi$. If w_j falls at position k in the return word v_i, then $p(w)_j = (\tilde{v}_i, k)$. This is a sliding block code with block size $max_{v \in \mathcal{B}}|v|$ and is the inverse of h. Since h is an invertible factor map whose inverse is also a factor map, it is a conjugacy. □

5.5. **Conclusions.** To summarize, we defined topological dynamical systems and shift dynamical systems. We then defined shift dynamical systems associated to symbolic substitutions and then characterized the shift spaces of primitive substitutions. Moving on to operator algebras, we saw how to any topological dynamical system we can associate a semi-crossed product algebra which was a complete conjugacy invariant. We then defined TAF-algebras and showed that they were a complete conjugacy invariant for the shift spaces defined by primitive substitutions and showed that all primitive substitutions define shift spaces conjugate to proper primitive substitutions in the process. Finally, we extended this result to all minimal shift spaces defined by substitutions that are conjugate to the shift space of a primitive substitution.

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