Lorenz's System - Analysis of a Sensitive System

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Abstract

Meteorology is a branch of geophysics concerned with atmospheric processes and phenomena and atmospheric effects on our weather. Edward Lorenz was a devoted meteorologist who made several significant contributions to this field. We first describe the fascinating history of Lorenz's discoveries and his revolutionary additions to the area of meteorology. In particular, he noted the extremely sensitive dependence on the initial conditions of a Chaotic system in the atmosphere, which is commonly referred to as the Butterfly Effect and pertains to Lorenz's system of three Ordinary Differential Equations (ODEs) that models the atmospheric convection in the atmosphere. We conducted a novel, in-depth mathematical analysis of the Theorem of Existence and Uniqueness for a system of ODEs in general, and addressed how it applies to Lorenz's system. Further, we exhibited how Lorenz's system is ill-posed using an application where we varied the initial parameters by minimal variations and noted relatively quick and drastic differences in the trajectories of the system.

Contents

1 Introduction

1.1 Short History of Edward N. Lorenz's Scientific Contributions

Based on the articles [3] and [4], we provide a short history of Edward N. Lorenz's scientific contributions. Edward Lorenz was an American meteorologist born in 1917 in the state of Connecticut. Lorenz obtained his bachelor's degree at Dartmouth College and his master's degree at Harvard University, both in mathematics. Despite Lorenz's extensive studies in mathematics, upon serving in World War II as a weather forecaster, Lorenz developed an immense passion for meteorology.

While Lorenz was very competent in mathematics, he was very inspired by meteorology. With this considered, Lorenz pursued a doctorate degree in meteorology in 1948. This signified the beginning of his fifty-eight years of contributions to the field of meteorology. Among all of Lorenz's contributions to meteorology there are four major ones, which he discovered on his own, that are still widely used in modern day meteorology.

Contrary to popular technique, and worth noting, Lorenz worked primarily with ordinary differential equations (ODEs) rather than partial differential equations (PDEs). This was because ODEs could be derived directly from the physics of a phenomenon, specifically, geophysical thermodynamics (GFD), as opposed to an approximation to a PDE. While there is an obvious leap from ODEs to PDEs, Lorenz's system of ODEs had startling parallels to the habits of nature.

One of Lorenz's discoveries, quite central to meteorology in the 1950s, was the general circulation of the atmosphere. Lorenz defined the issue as the way weather patterns vary on a global-scale. Lorenz contributed with two important concepts to this issue, these being the clarification of the kinetic energy cycle given the notion of available potential energy (APE) and the better understanding of general circulation models (GCMs), respectively. APE is the portion of total potential energy which can be converted to kinetic energy, through the adiabatic rearrangements of air parcels. In short, Lorenz sought to explain how thermal energy acts as a source of potential energy, which adiabatically converts kinetic energy often in the form of winds. Some criticized APE, claiming that the adiabatic conversion of air parcels must be complete for the kinetic energy to be generated. Otherwise, all APE would rapidly be exhausted, and the kinetic energy would cease. Lorenz countered this critique by asserting that the general circulation likely operates to its maximal intensity. Another critique asserts that air circulation complexities can be regional, and therefore the adiabatic conversion will be local and not global; this raised the problem of domain-size dependence in defining APE. Nonetheless, Lorenzs conditions seemed not to be understood. Lorenz opted for a more general view, where hemispheric atmospheric energy balance is dominated by the broad pattern of westerly winds and its associated eddy fluxes, [3]. APE had one last criticism, by Lorenz himself. Lorenz stated that a considerable amount of the total potential energy in the atmosphere arises from water and air, he called it moist available energy, and is not dominated by the energy created from the adiabatic rearrangements of air parcels.

Lorenz also introduced weather prediction, which is extremely relevant in modern meteorology. Humans heavily rely on the weather forecast. In discussing predictability, Lorenz posed three major perspectives: dynamical, empirical, and dynamical-empirical. In other words, Lorenz focused on initial value problems, historical correlates, and measuring errors through statistical modeling. Predictability can be discussed in terms of the strange attractor, given by Lorenz's third-order ODE system. The renowned chaotic solutions of Lorenz system, who induce the so-called butterfly effect, reveal the impossibility of long-term prediction of weather. One of the main characteristics of chaos is the exponential divergence of the neighbouring trajectories, and Lorenz noticed the following:

1. If the closeness of the initial states does not vanish, then in finite time there is an expo-

nential divergence of the neighboring trajectories, which leads to inaccurate predictions.

2. If the closeness of the initial states vanishes slowly, then the predictability time for loss of accuracy becomes infinitely long.

Upon using a separate dynamical system to model prediction error growth and weather maps at pressures up to 850,000 Pascals, Lorenz found that weather predictions can be made accurately within 12 days, although not entirely.

Another remarkable idea introduced by Lorenz is the slow manifold. Relating to GFD, Lorenz discussed the force balance in a changing flow in which the acceleration is weak. Balanced dynamics in the atmosphere usually refers to large-scale winds and currents which have progressed rather slowly. Lorenz used two parameters to discuss the flow; Rossby (Ro) number and Froude (Fr) number. The Ro number is determined by taking the magnitude of the horizontal velocity, V, divided by the product of the Coriolis frequency of Earth's rotation, f, and the vertical scale length, L. The Fr number is found by taking the magnitude of the horizontal velocity, V, divided by the product between the stable density stratification frequency, N, and the horizontal length scale, H. When the dynamics of large-scale atmospheric phenomena is dominated by the influence of the Earths rotation and stratification, we observe small values for Ro and Fr. The small values of Ro and Fr indicate a quasi-geostrophic (QG) system of motion of the fluid in the atmosphere, when the Coriolis force and pressure gradient forces are almost in balance. An exact balance between the Coriolis force and the pressure gradient forces indicates a geostrophic flow. Solutions to QG provide a representation of an invariant slow manifold in the phase space of all possible flows, [6].

One of his most cherished discoveries, and the topic of this work, is the butterfly effect. Initially, Saltzman solved a system of ODEs for the thermal convection in 2D. Lorenz built on this work by obtaining time-dependent solutions to this system by means of numerical integration, finding that all but three of the dependent variables converged to zero (for the most part). The solution of the system was non-periodic and difficult to distinguish. The lack of pattern in the solutions implied chaos in the system. In addition, there were issues with the continuous dependency of the Cauchy data, i.e., the initial value problem (IVP) for the system of ODEs is extremely sensitive on the variation of the Cauchy data. Very slight differences made the solutions diverge from one another. Given this sensitivity, parametrization must be introduced involving not only deterministic terms, but stochastic terms as well. Regardless, either deterministic formulation or stochastic formulation yield nearly indistinguishable results, and hence, either formulation may be used. But as we know, Lorenz chose to work with deterministic models, as deterministic models rely solely on parameter values and initial values.

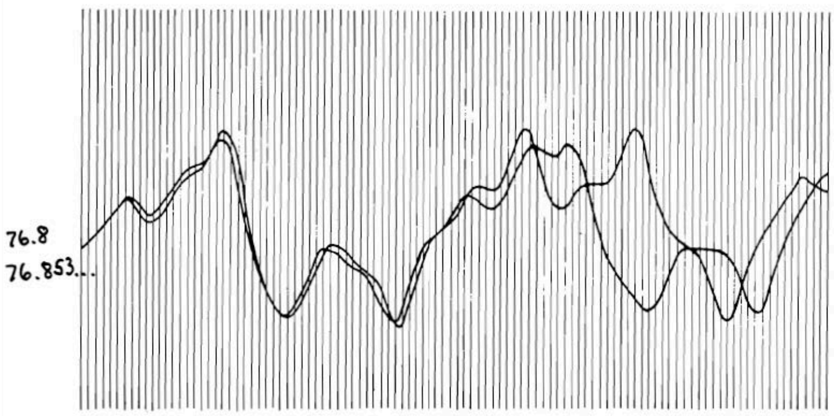
1.2 The Butterfly Effect

As described in [4], Lorenz, in his book The Essence of Chaos, described how, in 1961, he became aware of the high sensitivity to small variations of the Cauchy data for the IVP of the GCM that he was modeling at the time, a system of twelve ODEs describing the air circulation in the atmosphere. Lorenz decided on his system of twelves ODEs only after several failed attempts experimenting with other systems of equations with no success in finding constants which lead to chaos.

Lorenz already had the results obtained from the first run of his model. Running it for the second time, the results started similarly as the ones from the first run, but towards the end of the simulation, the new results showed to be completely divergent from the results obtained in the first run, i.e., the second simulation completely lost the resemblance to the first simulation, see Figure 1. Upon inspection, Lorenz soon discovered this occurred as a result of round-off errors.

Here is a paragraph from [4] regarding Lorenz's "discovery"

Wishing to repeat his simulation, he restarted it with numbers that had been printed out for the start conditions, and left it to go down the hall to fetch a cup of coffee. On his return, he found that the result was nothing like the previous one. He soon identified the reason: the numbers from the printout were rounded off. In the course of a coffee break, that small error had propagated with exponential speed to change the result completely.



How two weather patterns diverge. From nearly the same starting point, Edward Lorenz saw his computer weather produce patterns that grew farther and farther apart until all resemblance disappeared. (From Lorenz's 1961 printouts.)

Figure 1: Lorenz's 1961 Printouts indicating the eventual divergence in weather patterns.

1.2.1 Atmospheric Convection

The air in the lower layer of the Troposphere, i.e., the air moving along the ground, approximately 0 - 11 kilometers from the Earth's surface, absorbs heat non-homogeneously. Due to the temperature differences during the heating process, warm air rises while cool air descends, replacing the volume previously occupied by the warm air. This process is done through a vertical movement of the air, and it is called atmospheric convection, i.e., it is the process of thermal conductivity in meteorology. The ascending and descending convective motion, i.e., the convection currents, create a convection cell. The main characteristic of a convection cell is the density differences induced by the temperature differences within an air parcel, and its size is determined by the air's thermophysical properties; temperature, density, pressure, conductivity, and heat capacity, to name but a few of these properties. Convection cells are a key component in GCMs for the atmosphere, as they describe many of the energy changes we observe in the atmosphere. Convection cells often appear in the form of clouds, which is how much of the energy in the atmosphere is transferred and released.

1.2.2 The Lorenz Attractor

The simplified mathematical model for the atmospheric convection was developed by Lorenz in 1963, [2], and it is a three-dimensional dynamical system expressed as follows:

$$\dot{x} = \sigma(y - x) \tag{1}$$

$$\dot{y} = x(\rho - z) - y \tag{2}$$

$$\dot{z} = xy - \beta z \tag{3}$$

Explanations of the symbols in the system (1 - 3)

- $x(t)$ represents the intensity of convective currents
- $y(t)$ represents the temperature difference between the ascending and descending air currents
- $z(t)$ represents the normal temperature deviation; the distortion of the vertical temperature profile from linearity
- σ is the Prandtl number; the ratio between the momentum diffusivity (kinematic viscos-

ity) and the thermal diffusivity

- ρ is the Rayleigh number; determines the conduction or convection dominance in the heat transfer process

- β is a geometric factor related to the physical dimensions of the air parcel; it is proportional to the ratio between the width and the height of the air parcel

The constants σ, ρ and β are assumed positive with $\sigma > \beta + 1$, [1], and they will determine the behaviour of the trajectory.

The initial parameters that Lorenz used, which led him to his discovery, were $\sigma = 10$, $\beta = \frac{8}{3}$ and $\rho = 28$. The system (1 - 3) exhibits chaotic behaviour for these (and nearby) values.

Based on the Cauchy data of the IVP associated to the system (1 - 3), the trajectories that start from almost all initial points will converge to an invariant set; a particular strange attractor called the Lorenz attractor. The plot of each of these trajectories looks like a butterfly that never repeats itself exactly (thus proving to be a strange attractor), but it preserves fractal symmetry. The Lorenz attractor is also called the Strange Attractor.

Using Lorenz's initial parameters, we used the MAPLE software to solve the system (1 - 3) with the initial data ($x_0 = -8,\ y_0 = 8,\ z_0 = 27$). In Figures 2-4 below we represent the variation in time of the solution of the IVP for the system of ODEs (1 - 3). As well, in Figures 5-7 we represent, in the phase space, the orbit of the system (1 - 3) corresponding to the initial data ($x_0,\ y_0,\ z_0$), and the projections of this orbit in the xy-, xz-, and yz-plane.

```
restart;
with(plots):
with(DETools):

#Vector of Variables of the Lorenz System
v:=[x(t),y(t),z(t)]:

#Captions for ODEPlots and Phase Portraits
Captions_ODE:=["Time evolution of the intensity of convective
currents","Time evolution of the temperature difference between
the ascending and descending air currents","Time evolution of
the normal temperature deviation"]:
Captions_PP:=["Lorenz System - Phase Portrait","xy-Projection
Lorenz System Orbit","yz-Projection Lorenz System Orbit","xz-
Projection Lorenz System Orbit"]:

#Lorenz System
for i from 1 to 3 do
print('The Lorenz system:  equation',i);
ODE[i]:=readstat("Input the Lorenz system");
od;

#Parameters for Lorenz System
print('The values assigned for the parameters of Lorenz sys-
tem');
sigma:=readstat("Input the value of sigma");
rho:=readstat("Input the value of rho");
beta:=readstat("Input the value of beta");

#Initial Cauchy Data
print('The initial Cauchy data for the Lorenz system
ics:=readstat("Input initial Cauch data");

#Solving the Lorenz System
max_range:=readstat("Input maximum range to solve Lorenz sys-
tem"):
sol:= dsolve(seq(ODE[i],i=1..3),ics,numeric,range=0..max_range,
output=listprocedure):
#ODEPlots
c:=readstat("Input curve colour"):
f:=readstat("Input maximum frames for odeplot"):
for i from 1 to 3 do
odeplot(sol,[t,v[i]],color=c,frames=f,caption=Captions_ODE[i]):
od;

#Phase Portraits
max_pts:=readstat("Input maximum points for odeplot"):
dt:=readstat("Input time step to solve Lorenz system"):
t_max:=readstat("Input maximum range for phaseportrait"):
c:=readstat("Input linecolour style"):
odeplot(sol,v,numpoints=max_pts,caption=Captions_PP[1]);
```

```
for i from 1 to 3 do
if i+1<= 3 then
p[i]:=phaseportrait([seq(ODE[k],k=1..3)],v,t=0..t_max,[[ics]],
stepsize=dt,scene=[v[i],v[i+1]],linecolour=c,numpoints=max_pts,
title=Captions_PP[i+1]);
else
p[i]:=phaseportrait([seq(ODE[k],k=1..3)],v,t=0..t_max,[[ics]],
stepsize=dt,scene=[v[i-2],v[i]],linecolour=c,numpoints=max_pts,
title=Captions_PP[i+1]);
fi;
od;
for i from 1 to 3 do
display(p[i]);
od;

#Lorenz Atrraction - Animation
#Isolate each solution from sol.
for i from 1 to 3 do
v[i]:=sol[i+1]:
v[i]:=rhs(v[i]):
od:

#Animation
max_pts_a:=readstat("Input maximum points for animation"):
w:=readstat("Input curve thickness"):
c_a:=readstat("Input spacecurve colour"):
T_max:=readstat("Input maximum range for animation"):
f_max:=readstat("Input maximum frames for animation"):
opts:=thickness=w,numpoints=max_pts_a,color=c_a:
animate(spacecurve, [[v[1](t),v[2](t),v[3](t)],t=0..T,opts],T
=0..T_max,frames=f_max);
```

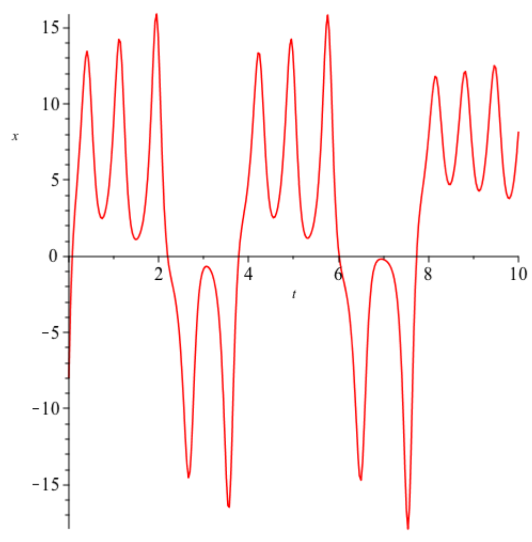


Figure 2: Time evolution of $x(t)$, i.e. the intensity of convective currents.

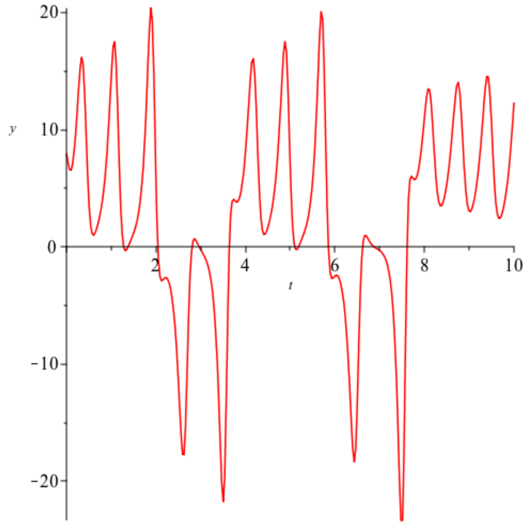


Figure 3: Time evolution of $y(t)$, i.e. the temperature difference between the ascending and descending air currents.

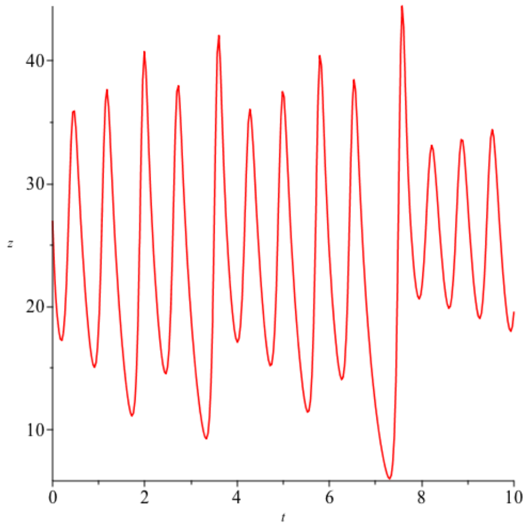


Figure 4: Time evolution of $z(t)$, i.e. the normal temperature deviation.

1.2.3 Lorenz System Sensitivity

As we mentioned at the beginning of Section 1.2, Lorenz, in 1961, became aware of the high sensitivity to small variations of the Cauchy data for the IVP of the GCM he was modeling at the time. Considering only the time-variation of the intensity of convective currents, the temperature difference between the ascending and descending air currents, and the normal temperature deviation, respectively, the simplified model for the atmospheric convection is the one described in Section 1.2.2, i.e. the system of ODEs (1 - 3), which consequently will not be a well-posed system of ODEs.

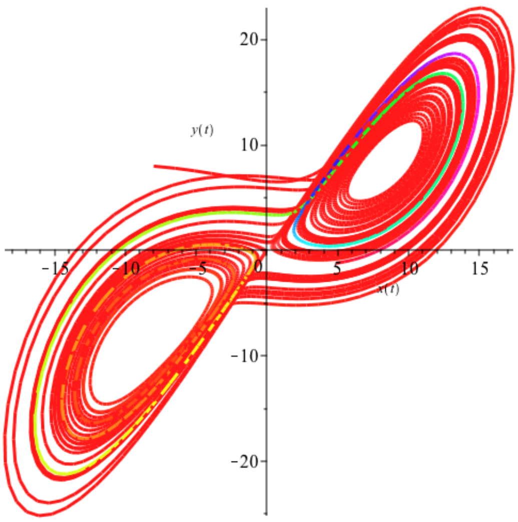


Figure 5: The xy-projection of the orbit of the Lorenz system.

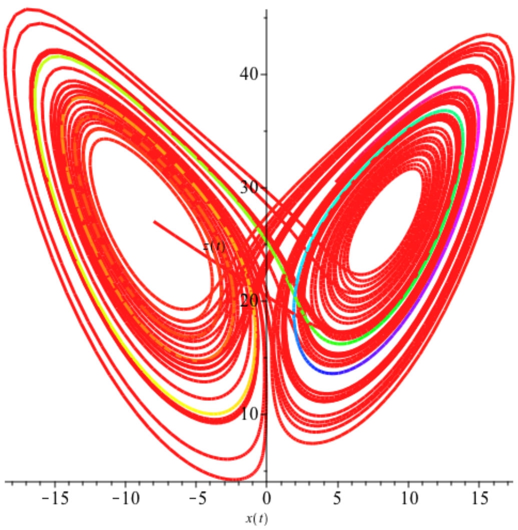


Figure 6: The xz-projection of the orbit of the Lorenz system.

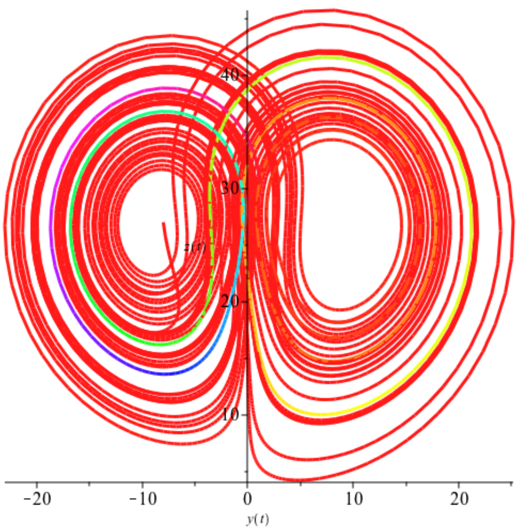


Figure 7: The yz-projection of the orbit of the Lorenz system.

While the well-posedness concept is typically associated to partial differential equations (PDEs), we usually extend its meaning to ODEs as well.

Definition We say that a PDE is well-posed if

1. A solution exists

2. The solution is determined uniquely by the Cauchy data

3. The solution depends continuously on the Cauchy data, i.e., small variations in the initial conditions induce small variations in the solution.

Extrapolating the latter definition to the system of ODEs (1 - 3), the system fails to be well-posed due to the fact that the third condition in the definition fails to be satisfied. This is what Lorenz discovered in 1961! This discovery prompted Lorenz to view atmospheric instability in a new uncharted territory, and on December 29, 1972, at 10:00 a.m., he gave the famous lecture

> Predictability; Does the Flap of a Butterfly's wings in Brazil set off a Tornado in Texas?

The lecture was presented at the 139^{th} American Association for the Advancement of Science at the Sheraton Park in Washington, D.C., as part of the AAAS section on Environmental Sciences New Approaches to Global Weather: GARP (The Global Atmospheric Research Program).

Lorenz's analogy to a "Flap of a Butterfly's wings" came from the fact that when small variations are induced to the initial conditions, the initial difference between the two solution curves is so small that it could be compared with the flap of a butterfly's wings. The sensitivity of Lorenz's system (1 - 3) to the small variations of the initial Cauchy data is known as the Butterfly Effect. Starting from this idea, Lorenz affirmed in his lecture that it is impossible to predict weather with 100% accuracy.

In general, to predict the time evolution of a system, we need to know its state at a given moment in time, which implies being able to measure all the parameters of the system. Once we have them measured, we introduce the initial conditions (in our case the initial Cauchy data) in the equations that describe the system to obtain the time evolution of the states of the system.

Lorenz system has a dynamic of type "chaotic", and one of the common characteristics of chaotic dynamics, which can be considered as suggestive for the qualitative study of it, is that the time evolution of a chaotic system is "the sensitivity to the initial conditions" and implicitly to the parameters of the system, and any external perturbations. The meaning of sensitivity to the initial conditions is that the phase portraits starting from initial conditions no matter how close are diverging, but without approaching infinity.

Another common characteristic of chaotic dynamics is that the extraction of information from it is characterized through the fact that over time, the states of the system highlight more and more precisely the initial conditions and the parameters used.

Using Lorenz's initial parameters, we used the MAPLE software to exemplify Lorenz's system sensitivity, see Figures 8-10. We used the following Cauchy data sets, which differ only by a small variation $\varepsilon = 0.001$: $(x_0 = -8, y_0 = 8, z_0 = 27)$ and $(x_0 = -7.999, y_0 = 8.001, z_0 = 27.001)$. Let us notice how the solutions start very similar with respect to each other, but as time progresses they will become divergent towards each other.

Lorenz's System Sensitivity - MAPLE Simulation

```
restart;
with(plots):
with(DETools):
```

```
# Vector of Equation of the Lorenz Systems
v := [ODE[i, 1], ODE[i, 2], ODE[i, 3]];
v1 := [x[i](t), y[i](t), z[i](t)];

# Captions for ODE Plots
Captions_ODE := ["Time evolution of the difference of inten-
sity of convective currents between the two systems", "Time evo-
lution of the difference of the temperature difference between
the ascending and descending air currents of the two systems",
"Time evolution of the difference of the normal temperature de-
viation of both systems"]

# Lorenz Systems
for i to 2 do
print('Lorenz system, system of ODEs', i);
ODE[i, 1] := diff(x[i](t), t) = sigma*(y[i](t) - x[i](t));
ODE[i, 2] := diff(y[i](t), t) = rho*x[i](t) - y[i](t) -
x[i](t)*z[i](t);
ODE[i, 3] := diff(z[i](t), t) = x[i](t)*y[i](t) -
beta*z[i](t);
end do

# Parameters for Lorenz Systems
print('The values assigned for the parameters of Lorenz sys-
tem');
sigma := readstat("Input the value of sigma");
rho := readstat("Input the value of rho");
beta := readstat("Input the value of beta");

# Describe the Cauchy data for the systems
epsilon := 0;
for i to 2 do
print('Initial Cauchy data for Lorenz system', i);
if 1 < i then
epsilon := readstat("epsilon will be assigned");
end if;
ics[i] := x[i](0) = -8 + epsilon, y[i](0) = 8 + epsilon, z[i](0)
= 27 + epsilon;
end do;

# Solving the Lorenz Systems
for i to 2 do sol[i] := dsolve(ODE[i, 1], ODE[i, 2], ODE[i, 3],
ics[i], numeric,
range = 0 ..  10, output = listprocedure);
end do

# Isolating each solution for sol[i]
for i to 3 do
v1[i] := sol[1][i + 1];
v1[i] := rhs(v1[i]);
v1[i] := sol[2][i + 1];
v1[i] := rhs(v1[i]);
```

```
end do

# Animation
r_t := readstat("Input the t-range");
w := readstat("Input curve thickness");
c_a := readstat("Input curve colour");
opts := thickness = w, color = c_a;
animate(plot, [x_sol[1](t) - x_sol[2](t), t = 0..T, opts], T
= r_t);
animate(plot, [y_sol[1](t) - y_sol[2](t), t = 0..T, opts], T
= r_t);
animate(plot, [z_sol[1](t) - z_sol[2](t), t = 0..T, opts], T
= r_t);
```

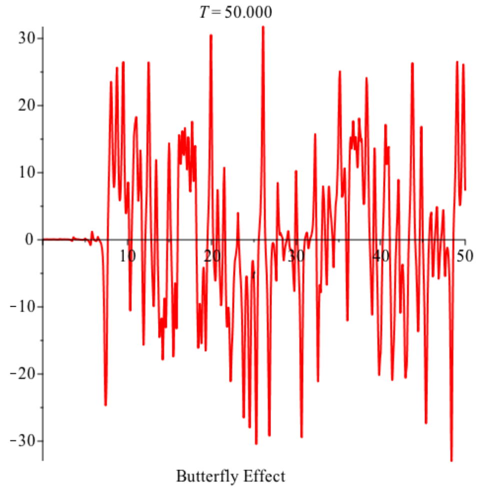


Figure 8: Butterfly Effect - Difference between the intensity of convective currents for both Lorenz systems.

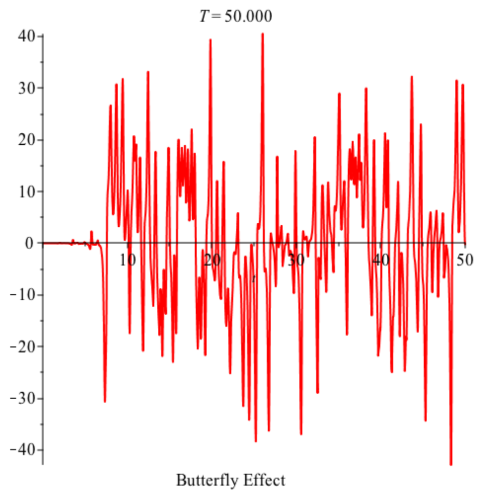


Figure 9: Butterfly Effect - Difference between the temperature difference between ascending and descending air currents for both Lorenz systems.

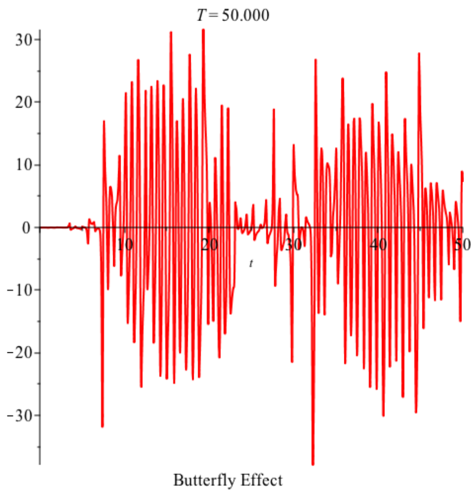


Figure 10: Butterfly Effect - Difference between the normal temperature deviation for both Lorenz systems.

2 Mathematical Analysis on the Existence and Uniqueness of Systems of Ordinary Differential Equations

While in the first chapter we showed the sensitivity of Lorenz's system, which makes his system of ODEs not to be well-posed, still, the first two conditions in the well-posedness of a system of ODEs are satisfied. In this chapter, we show that Lorenz's system does satisfy the two conditions, and this is provided by the following Theorem:

Theorem Consider the following initial value problem (IVP)

$$\frac{dy}{dt} = f(t, y(t)) \tag{4}$$

$$y(t_0) = y_0 \tag{5}$$

Assume that f satisfies the following conditions:

1. It is continuous on the domain

$$D = \{(t, y) \mid t_0 \leq t \leq t_0 + \lambda, \|y(t) - y_0\| \leq \mu, \lambda > 0, \mu > 0\} \subset \mathbb{R}^{m+1},\ m \geq 1$$

2. It is Lipschitz in the second argument, i.e., there exists $L > 0$ such that $\|f(t,y) - f(t,z)\| \leq L\|y - z\|$ for every (t,y), $(t,z) \in D$

Then for $t_0 \leq t \leq t_0 + \delta$ where $\delta = \min(\lambda, \frac{\mu}{M})$, $M = \max\limits_{(t,y)\in D} \|f(t,y)\|$, the IVP (4 - 5) has a unique solution

$$y = y(t)$$

Proof

Let us integrate (4):

$$dy = f(t, y(t))dt$$

$$\int dy = \int f(t, y(t))dt$$

$$y = \int f(t, y(t))dt + C, C \in \mathbb{R} \tag{6}$$

Let us denote the indefinite integral in (6) by $F(t)$. Then we have

$$y = F(t) + C, C \in \mathbb{R} \tag{7}$$

From (7), imposing the initial condition (5), we obtain:

$$y(t_0) = F(t_0) + C, C \in \mathbb{R}$$

$$y_0 = F(t_0) + C, C \in \mathbb{R}$$

$$\Downarrow$$

$$C = y_0 - F(t_0) \tag{8}$$

Then from (7) and (8), we get

$$y = F(t) + y_0 - F(t_0)$$

$$y = y_0 + F(t) - F(t_0)$$

$$\Downarrow$$

$$y = y_0 + \int_{t_0}^{t} f(s, y(s))ds \tag{9}$$

which follows immediately from the Fundamental Theorem of Calculus II. Now let us consider the Picard sequence in $\mathbb{R}^{m+1}$, [5].

$$\begin{aligned}
y_0 &= y_0(t) \\
y_1 &= y_0 + \int_{t_0}^{t} f(s, y_0(s))ds \\
y_2 &= y_0 + \int_{t_0}^{t} f(s, y_1(s))ds \\
&\vdots \\
y_n &= y_0 + \int_{t_0}^{t} f(s, y_{n-1}(s))ds, \quad n \geq 0
\end{aligned} \tag{10}$$

We want to show that the sequence $(y_n(t))_{n\geq 0}$ (when convenient we will use the flexible notations $y_n := y_n(t)$) converges uniformly to $y(t)$, where $y(t)$ is the unique solution of the IVP (4 - 5). First, using Mathematical Induction, we show that the sequence $(y_n)_{n\geq 0}$ is well-defined, i.e., $\|y_n - y_0\| \leq \mu \; \forall \; n \geq 0$.

For $n = 0$, it is obvious:

$$\|y_0 - y_0\| = \|0\| = 0 \leq \mu$$

Note: Notation wise, the symbol 0 plays dual role as a zero vector and as the number zero.

For $n = 1$, we have:

$$\|y_1 - y_0\| = \left\| \int_{t_0}^{t} f(s, y_0(s)) ds \right\| \leq \int_{t_0}^{t} \|f(s, y_0(s))\| ds$$

Because $(s, y_0(s)) = (s, y_0) \in D$, then $\|f(s, y_0(s))\| \leq M$ and we have:

$$\|y_1 - y_0\| \leq \int_{t_0}^{t} M ds = M(t - t_0) \leq M\delta \leq M\frac{\mu}{M} = \mu$$

Hence $\|y_1 - y_0\| \leq \mu$, so y_1 resides in D. Assume $P(n)$ is true and let us prove that $P(n+1)$ is true.

$$P(n) : \|y_n - y_0\| \leq \mu$$

$$P(n+1) : \|y_{n+1} - y_0\| \leq \mu$$

For $n := n + 1$, we have

$$\|y_{n+1} - y_0\| = \left\| \int_{t_0}^{t} f(s, y_n(s)) ds \right\| \leq \int_{t_0}^{t} \|f(s, y_n(s))\| ds$$

Because $(s, y_n(s)) = (s, y_n) \in D$, then $\|f(s, y_n(s))\| \leq M$ and we have:

$$\|y_{n+1} - y_0\| \leq \int_{t_0}^{t} M ds = M(t - t_0) \leq M\delta \leq M\frac{\mu}{M} = \mu$$

Hence $\|y_{n+1} - y_0\| \leq \mu$, so $P(n+1)$ is true. Therefore, proven by Mathematical Induction, the sequence $(y_n)_{n\geq 0}$ is well-defined.

Secondly, let us show that the sequence $(y_n)_{n\geq 0}$ converges uniformly. Consider the following series of vectors

$$y_0 + \sum_{n=0}^{\infty} (y_{n+1} - y_n) \tag{11}$$

and consider the general term of the sequence of partial sums

$$S_{n+1} = y_0 + \sum_{i=0}^{n-1} (y_{i+1} - y_i) \tag{12}$$

Then, by noting that the series is telescopic, we simply end up with

$$S_{n+1} = y_0 + (y_1 - y_0) + (y_2 - y_1) + \cdots + (y_{n-1} - y_{n-2}) + (y_n - y_{n-1})$$

$$\Downarrow$$

$$S_{n+1} = y_n \tag{13}$$

Consider as well the series

$$\|y_0\| + \sum_{n=0}^{\infty} \|y_{n+1} - y_n\| \tag{14}$$

and consider the general term of the sequence of partial sums

$$P_{n+1} = \|y_0\| + \sum_{i=0}^{n-1} \|y_{i+1} - y_i\| \tag{15}$$

We have then

$$\begin{aligned} \|y_1 - y_0\| &= \left\| \int_{t_0}^{t} f(s, y_0(s)) ds \right\| \\ &\leq \int_{t_0}^{t} \|f(s, y_0(s))\| ds \\ &\leq M(t - t_0) \\ &= \frac{M}{L} \frac{L(t - t_0)}{1!} \end{aligned}$$

$$\begin{aligned} \|y_2 - y_1\| &= \left\| \int_{t_0}^{t} f(s, y_1(s)) ds - \int_{t_0}^{t} f(s, y_0(s)) ds \right\| \\ &= \left\| \int_{t_0}^{t} (f(s, y_1(s)) - f(s, y_0(s))) ds \right\| \end{aligned}$$

$$
\begin{aligned}
&\leq \int_{t_0}^{t} \|f(s, y_1(s)) - f(s, y_0(s))\| ds \\
&\leq \int_{t_0}^{t} L\|y_1(s) - y_0(s)\| ds \\
&\leq L \int_{t_0}^{t} M(s - t_0) ds \\
&= ML\frac{(t-t_0)^2}{2!} \\
&= \frac{M}{L}\frac{L^2(t-t_0)^2}{2!}
\end{aligned}
$$

$$
\begin{aligned}
\|y_3 - y_2\| &= \left\| \int_{t_0}^{t} f(s, y_2(s)) ds - \int_{t_0}^{t} f(s, y_1(s)) ds \right\| \\
&= \left\| \int_{t_0}^{t} (f(s, y_2(s)) - f(s, y_1(s))) ds \right\| \\
&\leq \int_{t_0}^{t} \|f(s, y_2(s)) - f(s, y_1(s))\| ds \\
&\leq \int_{t_0}^{t} L\|y_2(s) - y_1(s)\| ds \\
&\leq L^2 \int_{t_0}^{t} M\frac{(s-t_0)^2}{2!} ds \\
&= ML^2\frac{(t-t_0)^3}{3!} \\
&= \frac{M}{L}\frac{L^3(t-t_0)^3}{3!}
\end{aligned}
$$

Let us extend this result inductively. Assume P(n) is true and let us prove P(n+1) is true.

$$
\begin{aligned}
P(n) &: \|y_n - y_{n-1}\| \leq \frac{M}{L}\frac{L^n(t-t_0)^n}{n!} \\
P(n+1) &: \|y_{n+1} - y_n\| \leq \frac{M}{L}\frac{L^{n+1}(t-t_0)^{n+1}}{(n+1)!}
\end{aligned}
$$

Then for $n := n+1$, we have

$$
\begin{aligned}
\|y_{n+1} - y_n\| &= \left\| \int_{t_0}^{t} f(s, y_n(s)) ds - \int_{t_0}^{t} f(s, y_{n-1}(s)) ds \right\| \\
&= \left\| \int_{t_0}^{t} (f(s, y_n(s)) - f(s, y_{n-1}(s))) ds \right\| \\
&\leq \int_{t_0}^{t} \|f(s, y_n(s)) - f(s, y_{n-1}(s))\| ds
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_{t_0}^{t} L\|y_n(s) - y_{n-1}(s)\|ds \\
&\le L^n \int_{t_0}^{t} M\frac{(s-t_0)^n}{n!}ds \\
&= ML^n\frac{(t-t_0)^{n+1}}{(n+1)!} \\
&= \frac{M}{L}\frac{L^{n+1}(t-t_0)^{n+1}}{(n+1)!}
\end{aligned}
$$

So $P(n+1)$ is true. Because $t_0 \le t \le t_0 + \delta$, we have $0 \le t - t_0 \le \delta$. Thus:

$$\|y_{i+1} - y_i\| \le \frac{M}{L}\frac{L^{i+1}(t-t_0)^{i+1}}{(i+1)!} \le \frac{M}{L}\frac{(L\delta)^{i+1}}{(i+1)!} \tag{16}$$

Using (16), let us show that

$$P_{n+1} \le \|y_0\| - \frac{M}{L} + \sum_{i=0}^{n} \frac{M}{L}\frac{(L\delta)^i}{i!} \tag{17}$$

We have

$$
\begin{aligned}
P_{n+1} &= \|y_0\| + \sum_{i=0}^{n-1} \|y_{i+1} - y_i\| \\
&\le \|y_0\| + \sum_{i=0}^{n-1} \frac{M}{L}\frac{(L\delta)^{i+1}}{(i+1)!} \\
&= \|y_0\| + \sum_{i=1}^{n} \frac{M}{L}\frac{(L\delta)^i}{i!}
\end{aligned}
$$

$$= \|y_0\| - \frac{M}{L} + \sum_{i=0}^{n} \frac{M}{L}\frac{(L\delta)^i}{i!} \tag{18}$$

Let us show that the series

$$\|y_0\| - \frac{M}{L} + \sum_{i=0}^{\infty} \frac{M}{L}\frac{(L\delta)^i}{i!} \tag{19}$$

is convergent, with the sum

$$\|y_0\| + \frac{M}{L}(e^{L\delta} - 1) \tag{20}$$

Noting that the series $\sum_{i=0}^{\infty} \frac{(L\delta)^i}{i!}$ is a convergent Maclaurin series converging to $e^{L\delta}$, then the series (19) will be convergent with the sum

$$\begin{aligned} &\|y_0\| - \frac{M}{L} + \sum_{i=0}^{\infty} \frac{M}{L} \frac{(L\delta)^i}{i!} \\ &= \|y_0\| - \frac{M}{L} + \frac{M}{L} e^{L\delta} \\ &= \|y_0\| + \frac{M}{L}(e^{L\delta} - 1) \end{aligned}$$

Therefore the series (14) is convergent for any $t \in [t_0, t_0 + \delta]$. This means precisely that the series (11) is absolutely convergent for any $t \in [t_0, t_0 + \delta]$, which means that it is convergent for any $t \in [t_0, t_0 + \delta]$, which means that it is uniformly convergent, i.e.,

$$\lim_{n\to\infty} S_{n+1} = \lim_{n\to\infty} y_n = y(t)\ \forall\, t \in [t_0, t_0 + \delta] \tag{21}$$

Finally, we show now that $y(t)$ obtained above for $t \in [t_0, t_0 + \delta]$ is the solution of the IVP (4 - 5). From (21), we get immediately that $(y_n)_{n\geq 0}$ converges uniformly to $y(t)$ on the interval $[t_0, t_0 + \delta]$. Then we have $(y_n)_{n\geq 0}$ is uniformly convergent to $y(t)$ on $[t_0, t_0 + \delta]$ if and only if $\forall\, \varepsilon > 0, \exists\, N_\varepsilon > 0$ such that $\|y_n(t) - y(t)\| < \varepsilon\ \forall\, t \in [t_0, t_0 + \delta]$, which is equivalent to

$$\lim_{n\to\infty} \|y_n(t) - y(t)\| = 0 \tag{22}$$

Remark If $\|y_n(t) - y(t)\| < \varepsilon\ \forall\ t \in [t_0, t_0 + \delta]$, that means that ε is an upper bound of $\|y_n(t) - y(t)\|$ when $n \geq 0$. If we assume that this infinite norm is such that $\|y_n(t) - y(t)\| \geq \varepsilon$, i.e., the supremum is $\geq \varepsilon$, this is a contradiction given that we have the supnorm since by definition, the supnorm is the least upper bound, so ε would be an even smaller upper bound, which is not possible.

We have

$$y_n = y_0 + \int_{t_0}^{t} f(s, y_{n-1}(s))ds$$

Then taking a limiting process as $n \to \infty$, we have

$$\begin{aligned}
\lim_{n\to\infty} y_n &= \lim_{n\to\infty} \left(y_0 + \int_{t_0}^{t} f(s, y_{n-1}(s))ds \right) \\
&= \lim_{n\to\infty} y_0 + \lim_{n\to\infty} \int_{t_0}^{t} f(s, y_{n-1}(s))ds \\
&= y_0 + \lim_{n\to\infty} \int_{t_0}^{t} f(s, y_{n-1}(s))ds \qquad (23)
\end{aligned}$$

To justify the use of the addition limit law, we will prove that the limit in (23) exists and is equal to

$$\lim_{n\to\infty} \int_{t_0}^{t} f(s, y_{n-1}(s))ds = \int_{t_0}^{t} f(s, y(s))ds \;\; \forall\, t \in [t_0, t_0 + \delta] \qquad (24)$$

We have

$$\begin{aligned}
&\left\| \int_{t_0}^{t} f(s, y_{n-1}(s))ds - \int_{t_0}^{t} f(s, y(s))ds \right\| \\
&\leq \int_{t_0}^{t} \|f(s, y_{n-1}(s)) - f(s, y(s))\| ds \\
&\leq \int_{t_0}^{t} L\|y_{n-1}(s) - y(s)\| ds \qquad (25)
\end{aligned}$$

Then from (22) we have $\forall\;\; \varepsilon > 0 \;\; \exists\;\; N_\varepsilon > 0$ such that for $n \geq N_\varepsilon$ we have

$$\|y_n(t) - y(t)\| < \frac{\varepsilon}{L\delta}$$

and (25) becomes

$$\begin{aligned}
&\left\|\int_{t_0}^{t} f(s, y_{n-1}(s))ds - \int_{t_0}^{t} f(s, y(s))ds\right\| \\
&\quad \leq \int_{t_0}^{t} L\|y_{n-1}(s) - y(s)\|ds \\
&\quad < \int_{t_0}^{t} L\frac{\varepsilon}{L\delta} \\
&\quad = \frac{\varepsilon}{\delta}(t - t_0) \\
&\quad \leq \frac{\varepsilon}{\delta}\delta = \varepsilon
\end{aligned}$$

Hence $\forall\ \varepsilon > 0\ \exists\ N_\varepsilon > 0$ such that for $n \geq N_\varepsilon$ we have

$$\left\|\int_{t_0}^{t} f(s, y_{n-1}(s))ds - \int_{t_0}^{t} f(s, y(s))ds\right\| < \varepsilon$$

which means

$$\lim_{n\to\infty} \int_{t_0}^{t} f(s, y_{n-1}(s))ds = \int_{t_0}^{t} f(s, y(s))ds$$

Then (23) becomes

$$\lim_{n\to\infty} y_n = y_0 + \int_{t_0}^{t} f(s, y(s))ds \tag{26}$$

and from (21) and (26) we obtain

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds \tag{27}$$

which means that $y(t)$ for $t \in [t_0, t_0 + \delta]$ is a solution of the IVP (4-5).

Now, we want to prove that the solution $y(t)$, given by (27) for the IVP (4-5) is unique. Let us assume that there is another solution, $z(t)$, for the IVP (4-5). Then we have

$$z(t) = y_0 + \int_{t_0}^{t} f(s, z(s))ds \tag{28}$$

for any $t \in [t_0, t_0 + \delta]$. We want to prove that $z(t) = y(t) \; \forall \; t \in [t_0, t_0 + \delta]$. Using (28), we show that

$$\|z(t) - y_0\| \leq M(t - t_0) \tag{29}$$

We have

$$\begin{aligned}
\|z(t) - y_0\| &= \left\| y_0 + \int_{t_0}^{t} f(s, z(s))ds - y_0 \right\| \\
&= \left\| \int_{t_0}^{t} f(s, z(s))ds \right\| \\
&\leq \int_{t_0}^{t} \|f(s, z(s))\| ds \\
&\leq \int_{t_0}^{t} M ds \\
&= M(t - t_0)
\end{aligned}$$

since $(s, z(s)) = (s, z) \in D$, and so we have that the inequality (29) holds, as desired. Let us now show that

$$\|y_n(t) - z(t)\| \leq \int_{t_0}^{t} L\|y_{n-1}(s) - z(s)\| ds \tag{30}$$

We have

$$\begin{aligned}
\|y_n(t) - z(t)\| &= \left\| y_0 + \int_{t_0}^{t} f(s, y_{n-1}(s))ds - y_0 - \int_{t_0}^{t} f(s, z(s))ds \right\| \\
&= \left\| \int_{t_0}^{t} (f(s, y_{n-1}(s)) - f(s, z(s)))ds \right\| \\
&\leq \int_{t_0}^{t} \|f(s, y_{n-1}(s)) - f(s, z(s))\| ds \\
&\leq \int_{t_0}^{t} L\|y_{n-1}(s) - z(s)\| ds
\end{aligned}$$

as desired. Thus the inequality (30) holds. By repeating this process n times, we obtain the following:

$$\begin{aligned}\|y_n(t)-z(t)\| &\leq \int_{t_0}^{t} L\left(\int_{t_0}^{t} L\left(\ldots\left(\int_{t_0}^{t} L\|y_0(s)-z(s)\|ds\right)\ldots\right)ds\right)ds \\ &= \int_{t_0}^{t} L\left(\int_{t_0}^{t} L\left(\ldots\left(\int_{t_0}^{t} L\|y_0-z(s)\|ds\right)\ldots\right)ds\right)ds\end{aligned} \tag{31}$$

Then by (29) and (31), we will show that

$$\|y_n(t)-z(t)\| \leq ML^n\frac{(t-t_0)^{n+1}}{(n+1)!} \tag{32}$$

And so we have

$$\begin{aligned}\|y_n(t)-z(t)\| &\leq \int_{t_0}^{t} L\left(\int_{t_0}^{t} L\left(\ldots\left(\int_{t_0}^{t} L\|y_0-z(s)\|ds\right)\ldots\right)ds\right)ds \\ &\leq \int_{t_0}^{t} L\left(\int_{t_0}^{t} L\left(\ldots\left(\int_{t_0}^{t} ML(t-t_0)\right)\ldots\right)ds\right)ds \\ &= \int_{t_0}^{t} L\left(\int_{t_0}^{t} L\left(\ldots\left(\int_{t_0}^{t} ML^2\frac{(t-t_0)^2}{2!}\right)\ldots\right)ds\right)ds \\ &= \int_{t_0}^{t} L\left(ML^{n-1}\frac{(t-t_0)^n}{n!}\right)ds \\ &= ML^n\frac{(t-t_0)^{n+1}}{(n+1)!}\end{aligned}$$

which is precisely the inequality seen in (32). Now using the inequality (32) and the fact that $t \in [t_0, t_0+\delta]$, we have

$$\begin{gathered}\|y_n(t)-z(t)\| \leq ML^n\frac{\delta^{n+1}}{(n+1)!} \\ \Updownarrow \\ \|y_n(t)-z(t)\| \leq \frac{M}{L}\frac{(L\delta)^{n+1}}{(n+1)!}\end{gathered} \tag{33}$$

We now prove that

$$\lim_{n\to\infty}\frac{(L\delta)^{n+1}}{(n+1)!} = 0 \tag{34}$$

Note that $L > 0, \delta > 0$ are fixed. Recall that $\delta = \min(\lambda, \frac{\mu}{M})$ where both $\lambda > 0$, $\mu > 0$ and $M = \max_{(t,y)\in D} \|f(t,y)\|$. Bearing this in mind, $L\delta > 0$ always. Then there exists a smallest positive integer k such that $L\delta < k$. As n gets arbitrarily large, i.e., $n \to \infty$, we have

$$\begin{aligned}(n+1)! &= 1 \cdot 2 \cdot 3 \cdots k(k+1) \cdots n(n+1) \\ &> 1 \cdot 2 \cdot 3 \cdots kk \cdots kk \\ &= k! \cdot k^{n+1-k}\end{aligned}$$

Then we have

$$\frac{1}{(n+1)!} < \frac{1}{k! \cdot k^{(n+1)-k}}$$

$$\Downarrow$$

$$\frac{(L\delta)^{n+1}}{(n+1)!} < \frac{(L\delta)^{n+1}}{k! \cdot k^{n+1-k}} = \frac{(L\delta)^k)(L\delta)^{n+1-k})}{k!k^{n+1-k}} = \frac{(L\delta)^k}{k!}\left(\frac{L\delta}{k}\right)^{n+1-k}$$

But $0 < \frac{L\delta}{k} < 1$ since $L\delta < k$, thus

$$\lim_{n\to\infty}\left(\frac{L\delta}{k}\right)^{n+1-k} = 0 \Rightarrow \lim_{n\to\infty}\frac{(L\delta)^k}{k!}\left(\frac{L\delta}{k}\right)^{n+1-k} = 0 \tag{35}$$

We have

$$0 < \frac{(L\delta)^{n+1}}{(n+1)!} < \frac{(L\delta)^k}{k!}\left(\frac{L\delta}{k}\right)^{n+1-k} \tag{36}$$

Because $\lim_{n\to\infty} = 0$, and from (35), using the Squeeze Theorem on the double inequality (36) we obtain

$$\lim_{n\to\infty}\frac{(L\delta)^{n+1}}{(n+1)!} = 0 \tag{37}$$

Then from (33) and (34), we get

$$\lim_{n\to\infty} y_n(t) = z(t) \tag{38}$$

But we have $\lim_{n\to\infty} y_n(t) = y(t)$, therefore $z(t) = y(t)$. Hence the solution $y(t)$, given by (27), of the IVP (4-5) is unique.

Referring to Lorenz's system (1-3), one can choose a domain D on which the vectorial function

$$f(x, y, z) = (\sigma(y-x), x(\rho-z)-y, xy-\beta z)^T,\ x := x(t),\ y := y(t),\ z := z(t) \qquad (39)$$

satisfies the conditions of the Theorem of Existence and Uniqueness presented in this section, which will assure the existence and uniqueness of the solution for Lorenz's system under given Cauchy data.

Note: Notation wise, the symbols x, y and z, play dual roles as vectors for the Theorem of Existence and Uniqueness, and as variables for the vectorial function (39).

3 Application: Lorenz System Behaviour on Variable Parameter β

As an application, we present an interesting behavior of the Lorenz system when the parameter β is varied with a certain pattern. The work presented in this chapter is an empirical study based on direct observations obtained through the variation of the parameter β, which will reveal the formation of a circular air current, like an "air eddy".

In the system of ODEs (1-3) we used the same Prandtl number (σ) and Rayleigh number (ρ) as Lorenz did in 1961, i.e., $\sigma = 10$ and $\rho = 28$, and we varied β as follows

$$\beta := \beta + i + 1,\ i = 1..5 \qquad (40)$$

starting from the initial value for β, $\beta = \frac{8}{3}$ (Lorenz's choice in 1961). It is important to notice that under the variation of β given in (40), we satisfy Lorenz's assumption $\sigma > \beta + 1$, [1]; the last value of β in the increasing variation sequence is $9.6 < \sigma = 10$.

Figure 11 shows a circular air current when the data is collected from the solution of the Lorenz

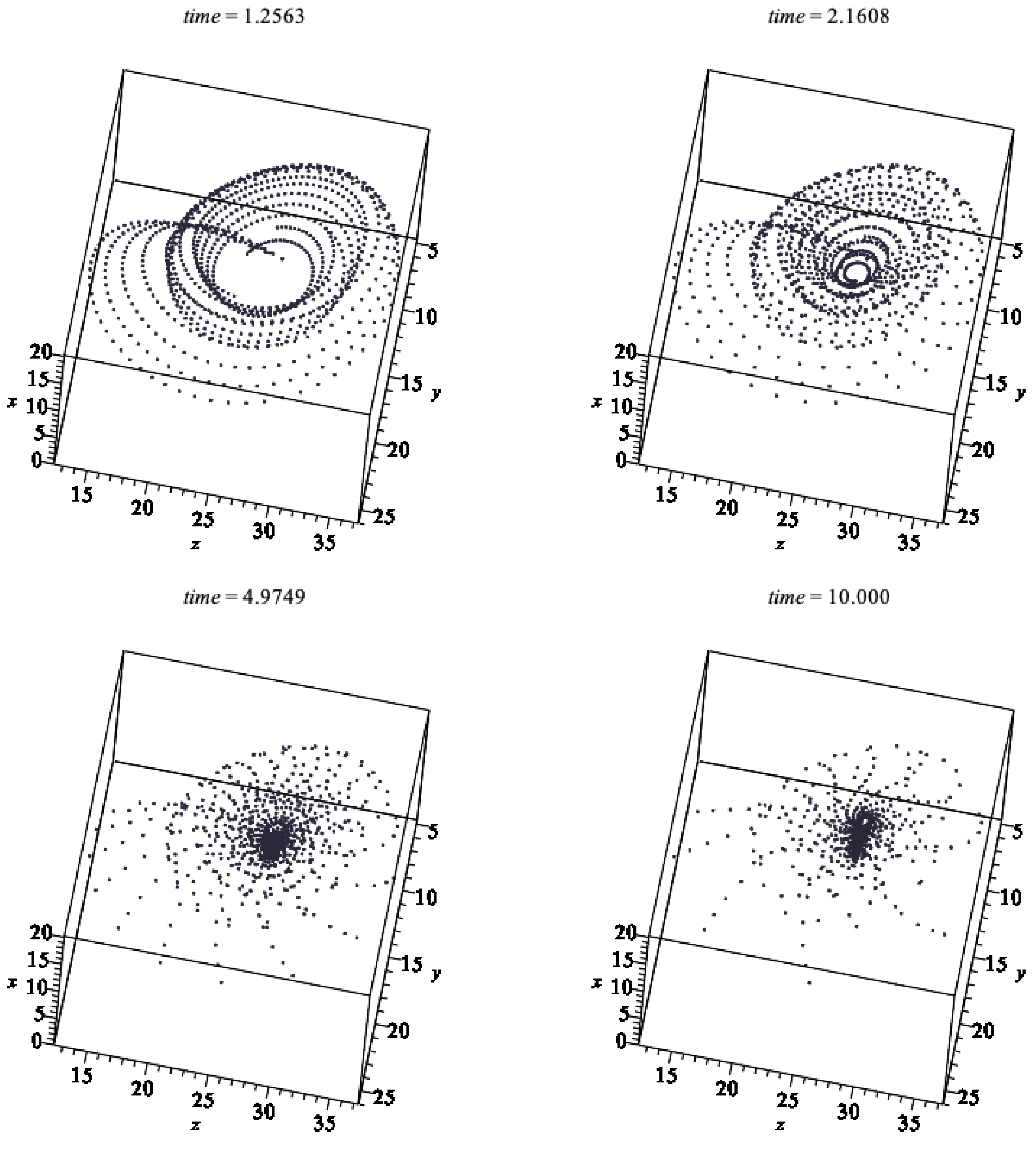


Figure 11: "Air Eddy" - Solution of Lorenz System under the variation of β given in (40).

system (1-3) using the variation of β given in (40).

Lorenz was an extraordinary scientist who discovered crucial concepts in modern mathematics and meteorology. It is worth pointing to Lorenz's use of ODEs over PDEs, expressing how critical it is to simplify our mathematical models where possible because there is a wonderful simplicity in our surroundings. With this considered, we should not undermine the complexity of Lorenz's system. Nor should we make the false assumption that the Lorenz Attractor is random, as it is not. The nature of the chaotic solutions of Lorenz's system is very much deterministic, although it may not appear as such. To an extent, the essence of our atmosphere is very much chaotic. We have seen how the most minuscule "flap of a butterfly's wings" can cause vast disturbances in our atmosphere, which is perhaps the critical takeaway. Mainly, these "flaps" are the reason why the weather can only be predicted so accurately in advance, which leaves many of us frustrated due to the apparent influence the weather has on our everyday activities. Lorenz's work in Chaos Theory and his novel Butterfly Effect is a beautiful demonstration of how we can always find beauty in chaos.

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