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A Note on Positive Bilinear Maps

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Abstract

This paper concerns positive maps between C*-algebras, particularly when those positive maps are multilinear. We construct examples of positive bilinear maps that are not 2-positive, and therefore are not completely positive bilinear maps. Paulsen and Smith showed that completely bounded bilinear maps are in one-to-one correspondence with completely bounded linear maps. We show that a similar correspondence does not hold for positive bilinear and positive linear maps. In particular, we observe that a similar correspondence does not hold if we replace "completely bounded" with "completely positive".

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1 Introduction

When describing the tensor product of vector spaces, we naturally identify bilinear maps defined on the cartesian product of vector spaces with linear maps defined on the tensor product of the given vector spaces. In fact, this correspondence is equivalent with the universal property of the tensor product. If we allow more structure on the vector spaces, say C*-algebra structure, then it makes sense to discuss positive linear maps and positive bilinear maps. It is then natural to ask whether this natural identification carries over between positive bilinear maps and positive linear maps.

In this note we show that this is not the case. Furthermore, we notice that there is no correspondence between completely positive maps and completely positive bilinear maps. This may be surprising in light of a result from Paulsen and Smith which shows that the one-to-one correspondence between bilinear maps and linear maps *does* carry over to a one-to-one correspondence between bilinear completely bounded and linear completely bounded. This in turn shows that there are bilinear completely positive maps which are not bilinear completely bounded. Finally, we shall construct examples of positive bilinear maps that are not 2-positive, and therefore are not completely positive bilinear maps.

2 Preliminaries

2.1 Cartesian - Tensor Product Correspondence

Let X and Y be vector spaces, then the cartesian product of X and Y, denoted $X \times Y$, is the vector space of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$. Note that addition will be component-wise and scalar multiplication scales both elements in our pair:

$$(x, y) + (v, w) = (x + v, y + w) \tag{2.1}$$

$$\lambda(x, y) = (\lambda x, \lambda y) \tag{2.2}$$

We say that a map $\varphi : X \times Y \to Z$ is bilinear if it is linear in each variable and we shall write that $B(X \times Y, Z)$ is the vector space of bilinear maps from $X \times Y$ to Z.

The tensor product of X and Y, denoted $X \otimes Y$, can be constructed as a space of linear maps on $B(X \times Y, Z)$. Let $x \in X$ and $y \in Y$, we can define a linear map $f = x \otimes y$, also called an elementary tensor, to be the evaluation at the point (x, y). In other words, for each $\varphi \in B(X \times Y, Z)$ we have the following:

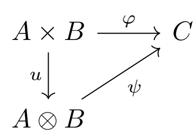
$$f(\varphi) = \langle \varphi, f \rangle = \langle \varphi, x \otimes y \rangle = \varphi(x, y) \tag{2.3}$$

We can now state a proposition found in [**?**], which is a well-known result. The proof is detailed below so that it is more accessible to undergraduate students.

Proposition. *There exists an isomorphism between the vector space of all bilinear maps from $A \times B \to C$ and the vector space of all linear maps from $A \otimes B \to C$. In other words, we can say:*

$$B(A \times B, C) \cong L(A \otimes B, C) \tag{2.4}$$

We can also illustrate this proposition by the commutative diagram shown below:



Proof. Let us create a map $\Phi : L(A \otimes B, C) \to B(A \times B, C)$. We define $\Phi(f) = f \circ u$ where u is the universal bilinear map from $A \times B$ into $A \otimes B$, $u(x, y) = x \otimes y$. For the reader's convenience, we will recall the properties of the bilinear map u, shown below:

$$u(x_1 + x_2, y) = (x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y = u(x_1, y) + u(x_2, y) \tag{2.5}$$

$$u(x, y_1 + y_2) = x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2 = u(x, y_1) + u(x, y_2) \tag{2.6}$$

$$u(\lambda x, y) = \lambda x \otimes y = x \otimes \lambda y = \lambda(x \otimes y) = \lambda u(x, y) \tag{2.7}$$

We shall verify that Φ is a linear bijection. We need to check that for $f_1, f_2 \in L(A \otimes B, C)$, we have $\Phi(f_1 + f_2) = \Phi(f_1) + \Phi(f_2)$.

$$\Phi(f_1 + f_2) = (f_1 + f_2) \circ u \tag{2.8}$$

$$(f_1 + f_2) \circ u(x, y) = f_1\big(u(x, y)\big) + f_2\big(u(x, y)\big) = f_1 \circ u(x, y) + f_2 \circ u(x, y);\ \forall (x, y) \in A \times B \tag{2.9}$$

Therefore, we have that:

$$(f_1 + f_2) \circ u = f_1 \circ u + f_2 \circ u \Longrightarrow \Phi(f_1 + f_2) = \Phi(f_1) + \Phi(f_2) \tag{2.10}$$

Hence, we have that Φ is a linear map.

Next, we shall check injectivity for Φ. Assume $\Phi(f) = 0$. Then $\Phi(f) = f \circ u = 0$ which means that $f \circ u(x, y) = 0$; $\forall (x, y) \in A \times B$. Therefore, we can say that $f(x \otimes y) = 0$; $\forall x \otimes y \in A \otimes B$. It follows that we must have $f = 0$, and hence Φ is injective.

We will now check that Φ is surjective. We shall pick an element $b \in B(A \times B, C)$ and we want a linear map f such that $\Phi(f) = b$. Recall that the elements in $A \otimes B$ are finite sums of elementary tensors; i.e. $\sum x_i \otimes y_i$.

Define f as shown below:

$$f\Big(\sum (x_i \otimes y_i)\Big) := \sum b(x_i, y_i) \tag{2.11}$$

In particular, $f(x \otimes y) = b(x, y)$. Next we check f is linear:

$$f(x \otimes y + z \otimes w) = b(x, y) + b(z, w) = f(x \otimes y) + f(z \otimes w) \tag{2.12}$$

$$f\big(\lambda(x \otimes y)\big) = \lambda b(x, y) = \lambda f(x \otimes y) \tag{2.13}$$

We need to check f is well-defined, i.e., if we have $\sum x_i \otimes y_i = \sum a_i \otimes b_i$, then $b\big(\sum x_i \otimes y_i\big) = b\big(\sum a_i \otimes b_i\big)$. To confirm this we recall that the tensor product of vector spaces is the quotient of the free product of vector spaces where two expressions are equal if and only if one can be simplified to the other using bilinearity. In particular, we should mention that if $x \otimes y = 0$ then either $x = 0$ or $y = 0$. Hence, f is well-defined and $\Phi(f) = b$. Therefore, Φ is surjective. Hence, we have that $B(A \times B, C) \cong L(A \otimes B, C)$. QED

3 Basic C*-algebras Results

In this section we will review some well-known results that will be used later; for more details, see Chapter 4 in [**?**]. By a C*-algebra, we mean a Banach $*$-algebra which satisfies the so-called C*-norm identity: $\|xx^*\| = \|x\|^2$; for any element x in the algebra. The set of all bounded linear operators $T : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, is an example of a C*-algebra, denoted $B(\mathcal{H})$. In fact, any C*-algebra is a closed $*$-algebra of $B(\mathcal{H})$, for some Hilbert space $\mathcal{H}$.

If A is a C*-algebra, then we construct matrix algebras over A, which are then new C*-algebras, denoted $M_n(A)$, obtained as the set of all square matrices $n \times n$ with entries in the given C*-algebra A. It is well known, see [**?**], that $M_n(A)$ is isomorphic to the tensor product $M_n(\mathbb{C}) \otimes A$. We note that $M_n(A)$ has a natural $*$-algebra structure extending to that on $M_n(\mathbb{C})$. For instance, there is a unique norm making $M_n(A)$ a C*-algebra. First, start with a faithful, non-degenerate representation $\pi : A \to B(\mathcal{H})$. Then, we have that:

$$\pi_n : M_n(A) \to M_n\big(B(\mathcal{H})\big) \cong B(\mathcal{H}^n) \tag{3.1}$$

is an injective representation. Pull back the norm from $M_n\big(B(\mathcal{H})\big)$:

$$\big\|[a_{i,j}]\big\| = \Big\|\pi_n\big([a_{(i,j)}]\big)\Big\| \tag{3.2}$$

to get the norm on $M_n(A)$.

Forming matrix algebras has the functorial property that if A and B are C*-algebras and if $\phi : A \to B$ is a $*$-homomorphism, then there is a natural $*$-homomorphism, called the n-amplification, $\phi_n : M_n(A) \to M_n(B)$ given by the following:

$$\phi_n\big([a_{i,j}]\big) = \big[\phi(a_{i,j})\big] \tag{3.3}$$

C*-algebras are vector spaces with additional structure; in particular, a partial order is present. First, we say that an element x of a C*-algebra is positive if the element is self-adjoint (i.e., $x = x^*$) and its spectrum is contained in $[0, \infty)$. Equivalently, x is positive if there exists y in the C*-algebra such that $x = y^*y$. The collection of all positive elements of a C*-algebra, which we denote as A_+, form a positive cone; i.e. $x, y \in A_+$, then $x + y \in A_+$ and $\lambda x \in A_+$, if $\lambda \geq 0$. The positive cone A_+ defines a partial order on the set of self-adjoint elements of the C*-algebra:

$$x \leq y \longleftrightarrow y - x \in A_+ \tag{3.4}$$

The partial order allows us to define positive maps: $\phi : A \to B$ is called positive if $\phi(A_+) \subset B_+$. The map ϕ is said to be completely positive if $\phi_n : M_n(A) \to M_n(B)$ is positive for all n $\big($in the order structure induced by the C*-algebras $M_n(A)$ and $M_n(B)\big)$.

Remark. *On $M_n(\mathbb{C})$, we define the normalized trace $tr : M_n(\mathbb{C}) \to \mathbb{C}$ by:*

$$tr\big([a_{i,j}]\big) = \frac{1}{n}\sum_{j=1}^{n} a_{j,j};\ \forall [a_{i,j}] \in M_n(\mathbb{C}) \tag{3.5}$$

To see that tr is positive linear functional, we do the following calculation:

$$tr(A^*A) = tr\left(\left[\sum_{k=1}^{n} \overline{a_{k,i}} a_{k,j}\right]\right) = \frac{1}{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\big|a_{k,j}\big|^2 \geq 0 \tag{3.6}$$

If we have two normed algebras X and Y and $\psi : X \to Y$, then we say that ψ is bounded if $\exists c > 0$ such that $\forall x \in X$, and therefore we have the following:

$$c\|x\|_X \geq \big\|\psi(x)\big\|_Y \tag{3.7}$$

For each $n \in \mathbb{N}$, if ψ_n is the n-amplification we define $\|\psi\|_n = \|\psi_n\|$. Then the completely bounded norm is defined as $\|\psi\|_{cb} = \sup_n \|\psi\|_n$. The linear map ψ is said to be completely bounded if $\|\psi\|_{cb} < \infty$.

If the reader wishes to know more about bounded and completely bounded maps as it pertains to this paper, they should reference the paper by Paulsen and Smith [**?**].

4 Main Result

In Section 2, we showed that there is a one-to-one correspondence between bilinear maps and linear maps defined on the cartesian product and the tensor product; respectively. We now show that the correspondence **??** does not carry over when considering positive bilinear maps and positive linear maps. Recall that a linear map is one that preserves addition and scalar multiplication; a bilinear map is linear in each variable separately while holding the other fixed.

Following [**?**], for $k = 2$, we say that the bilinear map

$$\varphi : \mathcal{A} \times \mathcal{A} \to B(\mathcal{H}) \tag{4.1}$$

is positive if

$$\varphi(A, A^*) \geq 0; \forall A \in \mathcal{A} \tag{4.2}$$

4.1 Positive maps that do not correspond to positive bilinear maps

In this section we consider the special case $\mathcal{A} = M_2(\mathbb{C})$. For $M_2(\mathbb{C})$, the bilinear map, as per the definition **??** and **??**, is the map $\phi : M_2(\mathbb{C}) \times M_2(\mathbb{C}) \to M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$, where we identify $M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) = M_4(\mathbb{C})$ using the Kronecker tensor of matrices. Next, we show there is a positive linear map whose corresponding bilinear map is not positive.

We begin by establishing the following useful lemma.

Lemma. *Let $\mathcal{A} = M_2(\mathbb{C})$ be the C^*-algebra of 2×2 matrices with complex entries. There exists $A \in \mathcal{A}$ such that $A \otimes A^*$ is not positive.*

Proof. Let $A \in M_2(\mathbb{C})$ where:

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \text{then } A^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{4.3}$$

Next, we have the tensor of the two elements above to be the following:

$$A \otimes A^* = \begin{bmatrix} 0 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & 0 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ & \\ 1 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & 0 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.4}$$

We will verify that this matrix is in fact not positive, i.e., not a positive definite matrix.

$$\left\langle \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 0 \\ 0 \\ x_2 \\ 0 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right\rangle = x_2 x_3;\ \forall x_i \in \mathbb{C} \tag{4.5}$$

Clearly, any two complex numbers multiplied together may not necessarily be positive, hence the element $A \otimes A^*$ is not positive. QED

Now, let $A \in \mathcal{A}$ (as above) where $A \otimes A^* \in \mathcal{A} \otimes \mathcal{A}$ is not positive. Since $A \otimes A^*$ is not positive there exists a positive linear functional ψ such that when evaluated on $A \otimes A^*$, is not positive [**?**]; i.e. $\psi \geq 0$ but $\psi(A \otimes A^*) \ngeq 0$. Since the associated bilinear map has the property that $\varphi(A, A^*) = \psi(A \otimes A^*)$, then we have that φ is not positive even though ψ is positive.

Therefore, we just proved the following proposition:

Proposition. *There exists a positive linear map whose corresponding bilinear map (**??**) is not positive.*

Furthermore, we are able to state the following:

Proposition. *There exists a completely positive map whose corresponding bilinear map (**??**) is not completely positive.*

Proof. Positive linear functionals are completely positive, see [**?**]. Hence, the linear functional used in the previous proposition is completely positive. The corresponding bilinear map cannot be completely positive since it is not even positive. QED

To better appreciate our results, we recall the following proposition of Paulsen and Smith [**?**]:

Proposition (Paulsen and Smith)**.** *Let E and F subspaces of $B(\mathcal{H})$, where $B(\mathcal{H})$ is the bounded linear operators on a Hilbert space $\mathcal{H}$. Let $V : E \times F \to B(\mathcal{H})$ be a bilinear map and $\phi : E \otimes F \to B(\mathcal{H})$ be the associated linear map. Then V is completely bounded if and only if ϕ is completely bounded and $\|V\|_{cb} = \|\phi\|_{cb}$.*

Remark. *The reader should be cautious since, a priori, there are many possible norms that can be defined on $E \otimes F$. With this in mind, the Paulsen and Smith result is true only when the matrix norm is selected to ensure the tensor product is the same as the Haagerup tensor product. We briefly describe the Haagerup tensor product below:*

If E and F subspaces of $B(\mathcal{H})$, then the algebraic tensor product $E \odot F$ may be given a norm as follows. If $v = \sum_{i=1}^{n} e_i \otimes f_i$ is an element of $E \odot F$ then:

$$\|v\|_H = \inf\left(\left\|\sum_i e_i e_i^*\right\|^{\frac{1}{2}} \left\|\sum_i f_i f_i^*\right\|^{\frac{1}{2}}\right) \tag{4.6}$$

where the infimum is taken over all representations of v as a finite sum of elementary tensors. Then $\|v\|_H$ is called the Haagerup norm, and is used on the algebraic tensor product to define the Haagerup tensor product.

Remark. *We see that the naive replacement of bounded and completely bounded by positive and completely positive is not possible. This is interesting because the completely positive linear maps are completely bounded and completely positive bilinear maps are not necessarily completely bounded, see [**?**]. Therefore, our result is reinforcing the relation just described.*

Finally, we finish with two examples of constructions showing that there exists positive bilinear maps that are not completely positive.

4.1.1 Example of a Positive Bilinear Map that is not Completely Positive

It is well known that the transposition map $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ given by $\phi(A) = A^T$ is a positive linear map but not completely positive. It is natural to ask whether a similar statement can be made about bilinear maps.

Lemma. *We say $\phi : M_n(\mathbb{C}) \times M_n(\mathbb{C}) \to \mathbb{C}$, given by $\phi(A, B) = tr(AB)$, is a positive bilinear map that is not a 2-positive bilinear map and hence is not completely positive bilinear map. Here, by $tr(A)$, we mean the unique normalized trace on matrices; see Remark in Section* 3*.*

Proof. Let $\phi_2 : M_2\big(M_n(\mathbb{C})\big) \times M_2\big(M_n(\mathbb{C})\big) \to M_2(\mathbb{C})$ be the 2-amplification map.

If we have that:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \tag{4.7}$$

then ϕ_2 is positive as a bilinear map if $\phi_2(A, B)$ is positive whenever $A^* = B$; see equation **??** or [**?**].

The condition $A^* = B$ means that $b_{11} = a_{11}^*$, $b_{12} = a_{21}^*$, $b_{21} = a_{12}^*$, and $b_{22} = a_{22}^*$, where $a_{ij}, b_{ij} \in M_n(\mathbb{C})$. Assume that:

$$\phi_2(A, A^*) = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \tag{4.8}$$

We will show that $\phi_2(A, A^*) = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$ is not always positive.

We shall note the following below:

$$c_1 = \phi(a_{11}, b_{11}) + \phi(a_{12}, b_{21}) = \phi(a_{11}, a_{11}^*) + \phi(a_{12}, a_{12}^*) = tr(a_{11}a_{11}^*) + tr(a_{12}a_{12}^*) \geq 0 \tag{4.9}$$

$$c_4 = \phi(a_{21}, b_{12}) + \phi(a_{22}, b_{22}) = \phi(a_{21}, a_{21}^*) + \phi(a_{22}, a_{22}^*) = tr(a_{21}a_{21}^*) + tr(a_{22}a_{22}^*) \geq 0 \tag{4.10}$$

Additionally, we note that:

$$c_2 = \phi(a_{11}, b_{12}) + \phi(a_{12}, b_{22}) = \phi(a_{11}, a_{21}^*) + \phi(a_{12}, a_{22}^*) = tr(a_{11}a_{21}^*) + tr(a_{12}a_{22}^*) \tag{4.11}$$

$$c_3 = \phi(a_{21}, b_{11}) + \phi(a_{22}, b_{21}) = \phi(a_{21}, a_{11}^*) + \phi(a_{22}, a_{12}^*) = tr(a_{21}a_{11}^*) + tr(a_{22}a_{12}^*) \tag{4.12}$$

and observe by inspection that $\overline{c_2} = c_3$.

Therefore, the matrix $\phi_2(A, A^*)$ is self-adjoint, i.e., $\phi_2(A, A^*)^* = \phi_2(A, A^*)$. To say that the self-adjoint matrix $\phi_2(A, A^*)$ is positive is equivalent to saying that both eigenvalues are positive. The characteristic polynomial of this matrix is $\lambda^2 - \lambda(c_1 + c_4) + (c_1c_4 - c_2c_3) = 0$. It is clear that having both a positive trace and a positive determinant is equivalent to saying that the eigenvalues are positive.

We already said that c_1 and c_4 are positive and hence then the trace is positive; however, the determinant may not be positive. For instance, choose c_3 such that $|c_3| > \max\{|c_1|, |c_4|\}$. Recall that $\overline{c_2} = c_3$; hence the eigenvalues are not both positive. QED

Remark. *If we let $n = 1$ in the above Lemma, we obtain an example of a positive bilinear functional that is not completely positive. This should be compared to the case of positive linear functionals which are necessarily completely positive.*

Remark. *If we let $n = 1$ and view $\mathbb{C} \times \mathbb{C}$ as an abelian C*-algebra, then we have an example of a positive bilinear map that is not completely positive. This should be compared to the linear case, where if the domain is an abelian C*-algebra, then any positive linear map is completely positive, see [**?**].*

Remark. *If we only know that the trace is positive, there is the possibility that one eigenvalue is positive and the other is negative, summing to a value greater than or equal to zero. In this case, we would need to look at the product of the eigenvalues and the determinant as a second criterion to determine if the matrix is positive.*

4.1.2 Multilinear Example II

This example is very similar to the one outlined above. The main difference is that the bilinear maps in the previous are replaced by 4-linear maps here, however, the arguments follow the same strategy established in the previous example. We will assume $\mathcal{A}$ and $\mathcal{B}$ to be any C*-algebras. However, to be able to apply the normalized trace tr, we need to restrict $\mathcal{A} = M_n(\mathbb{C})$ and $\mathcal{B} = \mathbb{C}$.

Let $\mathcal{A}$ and $\mathcal{B}$ denote C*-algebras and $\mathcal{A}^4 = \mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A}$ be the cartesian product. Let $M_2(\mathcal{A})$ be the C*-algebra of all matrices of size 2×2 with elements in $\mathcal{A}$ and $\varphi : \mathcal{A}^4 \to \mathcal{B}$ be a 4-linear map where

$\varphi(A,B,C,D) = tr(ABCD)$; where $A,B,C,D \in \mathcal{A}$. We will consider the 4-linear map $\varphi_2 : M_2(\mathcal{A})^4 \to M_2(\mathcal{B})$, defined as follows:

$$\varphi_2(A_1,A_2,A_3,A_4) = \left[\sum_{\ell,r,t=1}^{2} \varphi\big(a_{1i\ell}, a_{2\ell r}, a_{3rt}, a_{4tj}\big) \right]_{i,j=1}^{2} \tag{4.13}$$

where the following are elements of $M_2(\mathcal{A})$:

$$A_1 = \begin{bmatrix} a_{111} & a_{112} \\ a_{121} & a_{122} \end{bmatrix},\ A_2 = \begin{bmatrix} a_{211} & a_{212} \\ a_{221} & a_{222} \end{bmatrix},\ A_3 = \begin{bmatrix} a_{311} & a_{312} \\ a_{321} & a_{322} \end{bmatrix},\ A_4 = \begin{bmatrix} a_{411} & a_{412} \\ a_{421} & a_{422} \end{bmatrix} \tag{4.14}$$

Let us denote by $\mathcal{C}$ the matrix $\varphi_2(A_1,A_2,A_3,A_4)$ and by c_1, c_2, c_3, c_4 the entries of this matrix:

$$\varphi_2(A_1,A_2,A_3,A_4) = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \mathcal{C}. \tag{4.15}$$

As described in [?], we say that a 4-linear map $\varphi : \mathcal{A}^4 \to \mathcal{B}$ is completely positive if $\varphi_n(A_1,A_2,A_3,A_4) \geq 0$; $\forall (A_1,A_2,A_3,A_4) \in M_n(\mathcal{A})^4$ with $(A_1,A_2,A_3,A_4) = (A_4^*,A_3^*,A_2^*,A_1^*)$. We will restrict our attention to the case $n = 2$. Therefore, we have the following:

$$A_1 = A_4^*,\ A_4 = A_1^*,\ A_2 = A_3^*,\ A_3 = A_2^* \tag{4.16}$$

Note that if $A_1 = A_4^*$ then $A_4 = A_1^*$ and if $A_2 = A_3^*$ then $A_3 = A_2^*$. Therefore, we have:

$$A_1 = \begin{bmatrix} a_{111} & a_{112} \\ a_{121} & a_{122} \end{bmatrix} = \begin{bmatrix} a_{411}^* & a_{412}^* \\ a_{421}^* & a_{422}^* \end{bmatrix} = A_4^* \tag{4.17}$$

where $a_{111} = a_{411}^*$, $a_{112} = a_{421}^*$, $a_{121} = a_{412}^*$ and $a_{122} = a_{422}^*$. Additionally, we have:

$$A_2 = \begin{bmatrix} a_{211} & a_{212} \\ a_{221} & a_{222} \end{bmatrix} = \begin{bmatrix} a_{311}^* & a_{312}^* \\ a_{321}^* & a_{322}^* \end{bmatrix} = A_3^* \tag{4.18}$$

where $a_{211} = a_{311}^*$, $a_{212} = a_{321}^*$, $a_{221} = a_{312}^*$ and $a_{222} = a_{322}^*$.

We can use this information to find out more about the entries of the matrix $\mathcal{C}$. First, let us fully write out what each entry is when expanded out.

$$\begin{aligned} c_1 &= \sum_{\ell,r,t=1}^{2} \varphi\big(a_{11\ell}, a_{2\ell r}, a_{3rt}, a_{4t1}\big) \\ &= \varphi\big(a_{111}, a_{211}, a_{311}, a_{411}\big) + \varphi\big(a_{112}, a_{221}, a_{311}, a_{411}\big) \\ &+ \varphi\big(a_{111}, a_{212}, a_{321}, a_{411}\big) + \varphi\big(a_{111}, a_{211}, a_{312}, a_{421}\big) \\ &+ \varphi\big(a_{112}, a_{222}, a_{321}, a_{411}\big) + \varphi\big(a_{112}, a_{221}, a_{312}, a_{421}\big) \\ &+ \varphi\big(a_{111}, a_{212}, a_{322}, a_{421}\big) + \varphi\big(a_{112}, a_{222}, a_{322}, a_{421}\big) \end{aligned} \tag{4.19}$$

$$
\begin{aligned}
c_2 = & \sum_{\ell,r,t=1}^{2} \varphi\big(a_{11\ell}, a_{2\ell r}, a_{3rt}, a_{4t2}\big) \\
& = \varphi\big(a_{111}, a_{211}, a_{311}, a_{412}\big) + \varphi\big(a_{112}, a_{221}, a_{311}, a_{412}\big) \\
& + \varphi\big(a_{111}, a_{212}, a_{321}, a_{412}\big) + \varphi\big(a_{111}, a_{211}, a_{312}, a_{422}\big) \\
& + \varphi\big(a_{112}, a_{222}, a_{321}, a_{412}\big) + \varphi\big(a_{112}, a_{221}, a_{312}, a_{422}\big) \\
& + \varphi\big(a_{111}, a_{212}, a_{322}, a_{422}\big) + \varphi\big(a_{112}, a_{222}, a_{322}, a_{422}\big)
\end{aligned} \tag{4.20}
$$

$$
\begin{aligned}
c_3 = & \sum_{\ell,r,t=1}^{2} \varphi\big(a_{12\ell}, a_{2\ell r}, a_{3rt}, a_{4t1}\big) \\
& = \varphi\big(a_{121}, a_{211}, a_{311}, a_{411}\big) + \varphi\big(a_{122}, a_{221}, a_{311}, a_{411}\big) \\
& + \varphi\big(a_{121}, a_{212}, a_{321}, a_{411}\big) + \varphi\big(a_{121}, a_{211}, a_{312}, a_{421}\big) \\
& + \varphi\big(a_{122}, a_{222}, a_{321}, a_{411}\big) + \varphi\big(a_{122}, a_{221}, a_{312}, a_{421}\big) \\
& + \varphi\big(a_{121}, a_{212}, a_{322}, a_{421}\big) + \varphi\big(a_{122}, a_{222}, a_{322}, a_{421}\big)
\end{aligned} \tag{4.21}
$$

$$
\begin{aligned}
c_4 = & \sum_{\ell,r,t=1}^{2} \varphi\big(a_{12\ell}, a_{2\ell r}, a_{3rt}, a_{4t2}\big) \\
& = \varphi\big(a_{121}, a_{211}, a_{311}, a_{412}\big) + \varphi\big(a_{122}, a_{221}, a_{311}, a_{412}\big) \\
& + \varphi\big(a_{121}, a_{212}, a_{321}, a_{412}\big) + \varphi\big(a_{121}, a_{211}, a_{312}, a_{422}\big) \\
& + \varphi\big(a_{122}, a_{222}, a_{321}, a_{412}\big) + \varphi\big(a_{122}, a_{221}, a_{312}, a_{422}\big) \\
& + \varphi\big(a_{121}, a_{212}, a_{322}, a_{422}\big) + \varphi\big(a_{122}, a_{222}, a_{322}, a_{422}\big)
\end{aligned} \tag{4.22}
$$

We recall that φ is given by the normalized trace. This in turn shows that c_1 can be written as:

$$
\begin{aligned}
c_1 = & \, tr\big(a_{111}a_{211}a_{211}^*a_{111}^*\big) + tr\big(a_{112}a_{221}a_{211}^*a_{111}^*\big) \\
& + tr\big(a_{111}a_{212}a_{212}^*a_{111}^*\big) + tr\big(a_{111}a_{211}a_{221}^*a_{112}^*\big) \\
& + tr\big(a_{112}a_{222}a_{212}^*a_{111}^*\big) + tr\big(a_{112}a_{221}a_{221}^*a_{112}^*\big) \\
& + tr\big(a_{111}a_{212}a_{222}^*a_{112}^*\big) + tr\big(a_{112}a_{222}a_{222}^*a_{112}^*\big) \\
& = tr\big((a_{111}a_{211})(a_{111}a_{211})^*\big) + tr\big((a_{111}a_{212})(a_{111}a_{212})^*\big) \\
& + tr\big((a_{112}a_{221})(a_{112}a_{221})^*\big) + tr\big((a_{112}a_{222})(a_{112}a_{222})^*\big) \\
& + tr\big(a_{112}a_{222}a_{212}^*a_{111}^*\big) + tr\big(a_{111}a_{212}a_{222}^*a_{112}^*\big) \\
& + tr\big(a_{112}a_{221}a_{211}^*a_{111}^*\big) + tr\big(a_{111}a_{211}a_{221}^*a_{112}^*\big)
\end{aligned} \tag{4.23}
$$

We will use the fact that an element y can be expressed as $y = xx^*$ if it is positive. Note that in our case $x = ab$ and $x^* = (ab)^* = b^*a^*$. Therefore, we deduce that some of the terms comprising c_1 and c_4 are positive because they may be expressed as $\varphi(a, b, b^*, a^*) = tr(abb^*a^*) = tr\big((ab)(ab)^*\big)$. Since $(ab)(ab)^*$ is positive and trace tr is a positive map, we can conclude that those components of c_1 and c_4 are positive. However, the last four terms in the equation 4.23 are not necessarily positive since the product within the trace may not be positive. Similar calculations show that c_4 is also not necessarily positive. This in turn shows that the sum of the eigenvalues is not positive.

Finally, we see from the last four terms of equation 4.23, the following below:

$$\varphi\big(a_{112}, a_{221}, a_{211}^*, a_{111}^*\big) = \Big(\varphi\big(a_{111}, a_{211}, a_{221}^*, a_{112}^*\big)\Big)^* \tag{4.24}$$

$$\varphi\big(a_{112}, a_{222}, a_{212}^*, a_{111}^*\big) = \Big(\varphi\big(a_{111}, a_{212}, a_{222}^*, a_{112}^*\big)\Big)^* \tag{4.25}$$

Hence, it is clear that $c_2 = c_3^*$ in our matrix $\mathcal{C}$. Therefore, we have that φ_2 is not a positive map, and therefore we have that φ is not a completely positive map.

5 Conclusion

Based on our results, we conclude that the one-to-one correspondence between linear and bilinear maps does not carry over to a one-to-one correspondence between completely positive linear and completely positive bilinear maps. Furthermore, having a positive linear map does not imply that we would have a positive bilinear map as it pertains to the commutative diagram. Finally, the examples we provided show the existence of positive bilinear maps which are not completely positive bilinear maps, a fact that is familiar to us when speaking of positive linear maps.

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