A NOTE ON TEMPERED MEASURES

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ABSTRACT. The relation between tempered distributions and measures is analysed and clarified. While this is straightforward for positive measures, it is surprisingly subtle for signed or complex measures.

1. INTRODUCTION

Tempered distributions are the objects of choice for many problems in harmonic analysis on $\mathbb{R}^d$, with manifold applications for instance in mathematical physics. Sometimes, however, one has to deal with unbounded Radon measures in full generality. While the relations between them are fairly straightforward for *positive* tempered measures, things become more subtle for signed or complex measures.

Though this complication is well known in principle [1], it is a bit hidden in the literature and continues to lead to some typical mistakes. This is our motivation for this little note, which is meant to provide the general connection in full generality, stated as explicitly and concretely as possible.

In fact, we begin with the general case in Section 2, where we treat the positive and the general measures separately. For the critical statement that a signed or complex tempered measure need not be slowly increasing, we provide constructive counterexamples in Section 3. Finally, in Section 4, we consider the special situation of measures with uniformly discrete support, which is of particular relevance in the spectral theory of aperiodic order [2, 3].

2. THE GENERAL CASE

Throughout, $\mathcal{S}(\mathbb{R}^d)$ denotes the space of Schwartz functions on $\mathbb{R}^d$ and $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions, all in the sense of [15]. Clearly, $\mathcal{S}(\mathbb{R}^d)$ contains $C_c^\infty(\mathbb{R}^d)$, the space of C^∞-functions with compact support. For general background and results on Radon measures, we refer to [6]. If μ is a positive measure on $\mathbb{R}^d$, we write $L^1(\mu)$ for $L^1(\mathbb{R}^d, \mu)$.

Definition 2.1. Let μ be a Radon measure on $\mathbb{R}^d$. It is a *tempered measure* if there exists some $T \in \mathcal{S}'(\mathbb{R}^d)$ such that $\mu(\varphi) = T(\varphi)$ holds for all $\varphi \in C_c^\infty(\mathbb{R}^d)$.

Further, μ is called *strongly tempered* when, for all $\psi \in \mathcal{S}(\mathbb{R}^d)$, we have $|\psi| \in L^1(|\mu|)$ together with the property that $\psi \mapsto \int_{\mathbb{R}^d} \psi(x) \, d\mu(x)$ defines a tempered distribution.

Here, the first part is the definition of [1, 15], while the second essentially is the definition from [7], though some care has to be exercised when it comes to general Radon measures in comparison to positive ones.

Definition 2.2. A Radon measure μ on $\mathbb{R}^d$ is called *slowly increasing* if

$$\int_{\mathbb{R}^d} \frac{\mathrm{d}|\mu|(x)}{1 + |P(x)|} < \infty$$

holds for some polynomial $P \in \mathbb{C}[x_1, \ldots, x_d]$.

The second notion in Definition 2.1 was originally introduced in [16] in a different way, by saying that a measure is strongly tempered when it is slowly increasing. We shall later show that these two definitions are equivalent.

Let us begin with a straightforward consequence of our definitions.

Lemma 2.3. *Let μ be a Radon measure on $\mathbb{R}^d$. If μ is slowly increasing, it is also strongly tempered. Any strongly tempered measure is also tempered.*

Proof. The first claim follows from the observation that

$$\big|\mu(\psi)\big| \leqslant \int_{\mathbb{R}^d} |\psi(x)| \, \mathrm{d}|\mu|(x) = \int_{\mathbb{R}^d} \big|\psi(x)\big| \big(1 + |P(x)|\big) \frac{\mathrm{d}|\mu|(x)}{1 + |P(x)|}$$

$$\leqslant \big\|\big(1 + |P|\big)\psi\big\|_\infty \int_{\mathbb{R}^d} \frac{\mathrm{d}|\mu|(x)}{1 + |P(x)|}$$

holds for any $\psi \in \mathcal{S}(\mathbb{R}^d)$, where the last integral is a finite constant.

The second claim is obvious. Indeed, if μ is strongly tempered, $\psi \mapsto T(\psi) = \int_{\mathbb{R}^d} \psi(x) \, \mathrm{d}\mu(x)$ defines a tempered distribution. Moreover, for any $\varphi \in C_{\mathsf{c}}^\infty(\mathbb{R}^d)$, we have $T(\varphi) = \mu(\varphi)$ by definition. $\qquad\square$

Let us continue with one result that looks technical, but is actually fundamental. Its proof follows the arguments from [7, Thm. 2.1], but we present it in full detail here for improved readability and because various of our deviations will become significant later. To increase its readability, we split it into a preliminary lemma and the main result as Proposition 2.5.

For simplicity, for multi-indices α and β, we denote by $\|.\|_{\alpha,\beta}$ the Schwartz norm,

$$\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^d} \big|x^\beta D^\alpha f(x)\big|,$$

with $x^\beta = x_1^{\beta_1} \cdots x_d^{\beta_d}$ and $D^\alpha = \big(\frac{\partial}{\partial x_1}\big)^{\alpha_1} \cdots \big(\frac{\partial}{\partial x_d}\big)^{\alpha_d}$ as usual.

Lemma 2.4. *Let $k_n \in \mathbb{N}$ and $c_n \in (0, \infty)$ define two sequences with the following properties,*

(1) $k_1 \geqslant 4$ *and* $k_{n+1} \geqslant k_n + 4$ *for all* $n \geqslant 1$,

(2) *for all* $N \in \mathbb{N}$, *the sequence* $\big(c_n 2^{(k_n - 3)N}\big)_{n \in \mathbb{N}}$ *is bounded.*

Then, there exists some non-negative $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $\psi(x) = c_n$ holds for all $n \in \mathbb{N}$ and all x with $2^{k_n - 1} \leqslant |x|_2 \leqslant 2^{k_n + 1}$, where $|.|_2$ refers to the Euclidean norm on $\mathbb{R}^d$.

Proof. Select a non-negative function $\varphi \in C_{\mathsf{c}}^{\infty}(\mathbb{R}^d)$ with $\varphi(x) = 1$ for all $4 \leqslant |x|_2 \leqslant 16$ and with $\mathrm{supp}(\varphi) \subset \{x : 2 < |x|_2 < 32\}$, and set $\varphi_n(x) = \varphi(x/2^{k_n-3})$ for each n, which all are non-negative functions. Also, one has $\varphi_n(x) = 1$ for all $2^{k_n-1} \leqslant |x|_2 \leqslant 2^{k_n+1}$ together with

$$\mathrm{supp}(\varphi_n) \subset \{x : 2^{k_n-2} < |x|_2 < 2^{k_n+2}\}.$$

In particular, the functions φ_n have pairwise disjoint supports. Next, consider the non-negative function $\psi := \sum_{n=1}^{\infty} c_n \varphi_n$ which satisfies the properties guaranteed by Lemma 2.4. Therefore, if we show that $\psi \in \mathcal{S}(\mathbb{R}^d)$, we are done.

Let α and β be arbitrary multi-indices, and set $N = |\beta| - |\alpha|$, where $|\beta| = \beta_1 + \ldots + \beta_d$ as usual. By (2), there exist constants $C_{\alpha,\beta} = C_{\alpha,\beta}(N)$ such that

$$c_n 2^{(k_n-3)N} \leqslant C_{\alpha,\beta} \qquad \text{holds for all } n \in \mathbb{N}.$$

For arbitrary but fixed $x \in \mathbb{R}^d$, one of the following two cases applies.

Case 1. There is no $n \in \mathbb{N}$ such that $2^{k_n-2} \leqslant |x|_2 \leqslant 2^{k_n+2}$. Then, we have $\psi \equiv 0$ in a neighbourhood of x by construction, and hence

$$\left| x^\beta D^\alpha \psi(x) \right| = 0.$$

Case 2. There is some $n \in \mathbb{N}$ such that $2^{k_n-2} \leqslant |x|_2 \leqslant 2^{k_n+2}$. Then, since $k_{j+1} > k_j + 4$ for all j, this n is unique. The pairwise disjoint supports of the functions φ_n then imply that $\psi = c_n \varphi_n$ holds in a neighbourhood of x. Therefore, we get

$$\left| x^\beta D^\alpha \psi(x) \right| = \left| x^\beta D^\alpha c_n \varphi_n(x) \right| = c_n \left| x^\beta D^\alpha \big(\varphi(x/2^{k_n-3}) \big) \right| = \frac{c_n |x^\beta|}{2^{(k_n-3)|\alpha|}} \left| \big(D^\alpha \varphi \big) (x/2^{k_n-3}) \right|.$$

With $x = 2^{k_n-3} y$, this gives

$$\left| x^\beta D^\alpha \psi(x) \right| = \frac{c_n 2^{(k_n-3)|\beta|} |y^\beta|}{2^{(k_n-3)|\alpha|}} \left| \big(D^\alpha \varphi \big) (y) \right| = c_n 2^{(k_n-3)(|\beta|-|\alpha|)} \left| y^\beta \big(D^\alpha \varphi \big) (y) \right|$$

$$\leqslant c_n 2^{(k_n-3)(|\beta|-|\alpha|)} \| \varphi \|_{\alpha,\beta} \leqslant C_{\alpha,\beta} \| \varphi \|_{\alpha,\beta}.$$

For any $x \in \mathbb{R}^d$, one of the two cases applies, and we thus always obtain the estimate $\left| x^\beta D^\alpha \psi(x) \right| \leqslant C_{\alpha,\beta} \| \varphi \|_{\alpha,\beta}$ and hence also

$$\| \psi \|_{\alpha,\beta} \leqslant C_{\alpha,\beta} \| \varphi \|_{\alpha,\beta}.$$

Since the multi-indices α and β were arbitrary, this estimate shows that $\psi \in \mathcal{S}(\mathbb{R}^d)$, thus completing the proof. $\square$

As a consequence, we obtain the following result, which is the key to relating our notions to the class of *positive* Radon measures.

Proposition 2.5. *Let μ be a positive Radon measure on $\mathbb{R}^d$ such that all $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with $\varphi \geqslant 0$ satisfy $\varphi \in L^1(\mu)$. Then, μ is slowly increasing.*

Proof. Set $A_0 = \{x \in \mathbb{R}^d : |x|_2 \leqslant 1\}$, and $A_j = \{x \in \mathbb{R}^d : 2^{j-1} \leqslant |x|_2 \leqslant 2^j\}$ for $j \in \mathbb{N}$. We now show that there are constants $c > 0$ and $a \geqslant 0$ such that

$$(1) \qquad\qquad \mu(A_j) \leqslant c\, 2^{aj} \quad \text{holds for all } j \geqslant 0.$$

Assume this to be false. Then, for all $c > 0$ and $a \geqslant 0$, there is some $j = j(c,a)$ with $\mu(A_j) > c\, 2^{aj}$. Setting $c = 1$, we get, for all $\ell \in \mathbb{N}_0$, some $k_\ell \in \mathbb{N}_0$ such that

$$(2) \qquad\qquad \mu(A_{k_\ell}) > 2^{\ell k_\ell}.$$

In fact, for each $\ell \in \mathbb{N}_0$, there must be infinitely many such k_ℓ. Indeed, assume to the contrary that there is some ℓ_0 for which

$$(3) \qquad\qquad \mu(A_k) > 2^{\ell_0 k}$$

holds only for finitely many k, say for $k_1, \ldots, k_j$. Then, for each $1 \leqslant i \leqslant j$, we can find some ℓ_i with $\mu(A_i) < 2^{\ell_i k_i}$. Consequently, for $\ell > \max\{\ell_0, \ell_1, \ldots, \ell_j\}$, one has

$$\mu(A_{k_i}) < 2^{\ell_i k_i} < 2^{\ell k_i}$$

for $1 \leqslant i \leqslant j$ together with $\mu(A_k) < 2^{\ell_0 k} < 2^{\ell k}$ for all $k \notin \{k_1, \ldots, k_j\}$. This shows that (2) holds for infinitely many integers k.

Consequently, for all $\ell \in \mathbb{N}_0$, there are infinitely many k with $\mu(A_k) > 2^{k\ell}$. We can then construct a sequence $4 < k_1 < k_2 < \cdots$ such that $k_{j+1} > k_j + 4$ holds for all $j \in \mathbb{N}$ together with $\mu(A_{k_j}) > 2^{jk_j}$.

Setting $c_j = 1/\mu(A_{k_j})$, we see that $c_j < 2^{-jk_j}$ and hence, for all $N \in \mathbb{N}$, we have

$$\limsup_{n\to\infty} c_n 2^{(k_n-3)N} \leqslant \limsup_{n\to\infty} 2^{(k_n-3)N - nk_n} = 0.$$

In particular, $c_n 2^{(k_n-3)N}$ is bounded for all $N \in \mathbb{N}$. It follows that k_n and c_n satisfy the conditions of Lemma 2.4. Consequently, there exists some non-negative function $\psi \in \mathcal{S}(\mathbb{R}^d)$ with $\psi(x) = 1/\mu(A_{k_n})$ for all $x \in A_{k_n}$. Since the sets A_{k_n} are pairwise disjoint, it follows that, for any $N \in \mathbb{N}$, we have

$$\mu(\psi) \geqslant \int_{\bigcup_{n=1}^N A_{k_n}} \psi(x)\, \mathrm{d}\mu(x) = \sum_{n=1}^N \int_{A_{k_n}} \psi(x)\, \mathrm{d}\mu(x) = \sum_{n=1}^N \int_{A_{k_n}} \frac{\mathrm{d}\mu(x)}{\mu(A_{k_n})} = N,$$

which contradicts the fact that $\psi \in L^1(\mu)$. So, our assumption is wrong. This shows that there are some constants $c > 0$ and $a \geqslant 0$ such that (1) holds for all $j \geqslant 0$. Also, after possibly replacing a by a larger number, we may assume $a \in \mathbb{N}$ without loss of generality.

Then, since we have $\mathbb{R}^d = \bigcup_{j=0}^\infty A_j$, where the A_j have disjoint interior but some common boundary for consecutive values of j, we also have

$$\int_{\mathbb{R}^d} \frac{\mathrm{d}\mu(x)}{1 + |x|_2^{a+1}} \leqslant \int_{A_0} \frac{\mathrm{d}\mu(x)}{1 + |x|_2^{a+1}} + \sum_{j=1}^\infty \int_{A_j} \frac{\mathrm{d}\mu(x)}{1 + |x|_2^{a+1}}.$$

Since A_0 is compact, we clearly have

$$I_0 := \int_{A_0} \frac{\mathrm{d}\mu(x)}{1 + |x|_2^{a+1}} \leqslant \mu(A_0) < \infty.$$

Next, for all $j \geqslant 1$, we get

$$I_j := \int_{A_j} \frac{\mathrm{d}\mu(x)}{1 + |x|_2^{a+1}} = \int_{2^{j-1} \leqslant |x|_2 \leqslant 2^{j+1}} \frac{\mathrm{d}\mu(x)}{1 + |x|_2^{a+1}}$$

$$\leqslant \frac{\mu(A_j)}{1 + 2^{(j-1)(a+1)}} \leqslant \frac{c\, 2^{aj}}{2^{(j-1)(a+1)}} = c\, 2^{-j} 2^{a+1}.$$

This shows that

$$\int_{\mathbb{R}^d} \frac{\mathrm{d}\mu(x)}{1 + |x|_2^{a+1}} \leqslant I_0 + c\, 2^{a+1} \sum_{j=1}^{\infty} 2^{-j} = I_0 + c\, 2^{a+1} < \infty,$$

which completes the proof. $\qquad\square$

At this point, we get the following result.

Theorem 2.6. *For a positive Radon measure μ on $\mathbb{R}^d$, the following properties are equivalent:*

(1) *μ is slowly increasing;*

(2) *μ is strongly tempered;*

(3) *one has $|\psi| \in L^1(\mu)$ for all $\psi \in \mathcal{S}(\mathbb{R}^d)$;*

(4) *one has $\psi \in L^1(\mu)$ for all $\psi \in \mathcal{S}(\mathbb{R}^d)$ with $\psi \geqslant 0$;*

(5) *μ is tempered.*

Proof. $(1) \Rightarrow (2)$ is the first claim of Lemma 2.3, while $(2) \Rightarrow (3)$ follows from Definition 2.1 and $(3) \Rightarrow (4)$ is obvious. Further, $(4) \Rightarrow (1)$ is Proposition 2.5, while $(2) \Rightarrow (5)$ is the second claim of Lemma 2.3.

To complete the proof, we could infer [15, p. 242] to obtain $(5) \Rightarrow (1)$, but we prefer to show $(5) \Rightarrow (4)$ as follows. Let ψ be any fixed, non-negative Schwartz function. Then, invoking a minor variant of the C^∞ partitions of unity [10, p. 299], there is a sequence of functions $\varphi_n \in C_{\mathsf{c}}^\infty(\mathbb{R}^d)$ with $\varphi_1 \leqslant \varphi_2 \leqslant \ldots$ such that $\varphi_n = 1$ on $\{|x| \leqslant n\}$ and $\varphi_n \psi \xrightarrow{n \to \infty} \psi$ in $\mathcal{S}(\mathbb{R}^d)$.

Since μ is tempered, there is some $T \in \mathcal{S}'(\mathbb{R}^d)$ such that $T(\varphi) = \mu(\varphi)$ holds for all $\varphi \in C_{\mathsf{c}}^\infty(\mathbb{R}^d)$. Then, by the monotone convergence theorem [10, Thm. 5.5], we have

$$\mu(\psi) = \lim_{n \to \infty} \mu(\varphi_n \psi) = \lim_{n \to \infty} T(\varphi_n \psi) = T(\psi) < \infty,$$

which completes the argument. $\qquad\square$

For the general situation, we now get the following result.

Theorem 2.7. *Let μ be a general Radon measure on $\mathbb{R}^d$. Then, the following properties are equivalent:*

(1) *μ is slowly increasing;*

(2) *$|\mu|$ is strongly tempered;*

(3) $|\mu|$ *is tempered;*

(4) $|\psi| \in L^1(|\mu|)$ *holds for all* $\psi \in \mathcal{S}(\mathbb{R}^d)$;

(5) μ *is strongly tempered.*

Further, if μ *is strongly tempered, it is tempered, while the converse need not hold.*

Proof. The equivalences of the first four conditions follow immediately from Theorem 2.6, while the implications

$$\mu \text{ slowly increasing} \implies \mu \text{ strongly tempered} \implies \mu \text{ tempered}$$

are the result of Lemma 2.3.

Finally, the fact that μ is strongly tempered implies $|\psi| \in L^1(|\mu|)$ for all $\psi \in \mathcal{S}(\mathbb{R}^d)$ follows directly from the definition.

[1, Prop. 7.1] provides an example of a Radon measure μ that is tempered, though $|\mu|$ is not. In particular, μ is not slowly increasing. $\qquad\square$

Let us now spell out the implication (4) $\implies$ (5) in Theorem 2.7 more explicitly as follows.

Corollary 2.8. *Let* μ *be a measure on* $\mathbb{R}^d$, *and assume that* $|\psi| \in L^1(|\mu|)$ *holds for all* $\psi \in \mathcal{S}(\mathbb{R}^d)$. *Then,* $\psi \mapsto \int_{\mathbb{R}^d} \psi(t)\, \mathrm{d}\mu(t)$ *defines a tempered distribution.* $\qquad\square$

Let us complete the section by another characterisation when a measure is slowly increasing.

Lemma 2.9. *A Radon measure* μ *is slowly increasing if and only if it is a linear combination of positive tempered measures.*

Proof. The direction $\Leftarrow$ is obvious. For the converse, consider the standard Hahn–Jordan decomposition [10, Cor. 3.6] of $\mu = \nu + \mathrm{i}\sigma$ with $\nu = \mathrm{Re}(\mu)$ and $\sigma = \mathrm{Im}(\mu)$, that is

$$\mu = (\nu_+ - \nu_-) + \mathrm{i}(\sigma_+ - \sigma_-),$$

where $\nu_\pm = \frac{1}{2}(|\nu| \pm \nu)$ are supported on disjoint sets, and analogously for $\sigma_\pm$. Then, one has

$$|\varrho| \leqslant |\mu| \quad \text{for all } \varrho \in \{\nu_+, \nu_-, \sigma_+, \sigma_-\}.$$

Since μ is slowly increasing, so are the four components. $\qquad\square$

Since the Example in [1, Prop. 7.1] is so crucial, but important details are skipped, let us next construct some examples of that type more explicitly.

3. Some tempered measures that are not slowly increasing

All the examples in this section fall into the class of tempered distributions with Fourier transforms in the sense of measures, which was studied in detail in [20]. Let us begin with an important technical step, which we present in all details for clarity and self-containedness.

Proposition 3.1. *For every* $A > 0$, *there exists a function* $g \in C_{\mathsf{c}}(\mathbb{R})$ *with the following three properties:* $\mathrm{supp}(g) \subseteq [-2, 2]$, $\|g\|_1 \geqslant A$, *and* $\|\widehat{g}\|_\infty \leqslant 1$.

Proof. Define the non-negative functions $f = 1_{[-1,1]}$ and $f_n = \frac{n}{2} 1_{[-\frac{1}{n}, \frac{1}{n}]}$, the latter for $n \in \mathbb{N}$. Clearly, for any $x \in \mathbb{R}$, one has

$$0 \leqslant (f * f_n)(x) = \int_{-1}^{1} f_n(x-y)\,\mathrm{d}y \leqslant \int_{\mathbb{R}} f_n(x-y)\,\mathrm{d}y = 1,$$

which implies $\|f * f_n\|_\infty \leqslant 1$. Also, since $\widehat{f_n}(t) = \mathrm{sinc}\left(\frac{2\pi t}{n}\right)$ with $\mathrm{sinc}(z) = \frac{\sin(z)}{z}$, one clearly has pointwise convergence $\widehat{f_n} \xrightarrow{n\to\infty} 1$ on $\mathbb{R}$.

For any $B > 0$, there is an $n \in \mathbb{N}$ such that

$$(4) \qquad \left\|\widehat{f_n * f}\right\|_1 = \left\|\widehat{f_n}\widehat{f}\right\|_1 \geqslant B.$$

To see this, observe that $\widehat{f}(t) = 2\,\mathrm{sinc}(2\pi t)$ is locally integrable, but gives $\left\|\widehat{f}\right\|_1 = \infty$ because, for any $N \in \mathbb{N}$, one has

$$\int_0^\infty |\mathrm{sinc}(2\pi t)|\,\mathrm{d}t \geqslant \sum_{n=0}^{N} \int_{\frac{n}{2}}^{\frac{n+1}{2}} \frac{|\sin(2\pi t)|}{2\pi t}\,\mathrm{d}t \geqslant \sum_{n=0}^{N} \frac{1}{\pi(n+1)} \int_{\frac{n}{2}}^{\frac{n+1}{2}} |\sin(2\pi t)|\,\mathrm{d}t = \frac{1}{\pi^2}\sum_{n=0}^{N} \frac{1}{n+1}$$

where the last term is the divergent harmonic series. In particular, we have

$$\int_{-\alpha}^{\alpha} \left|\widehat{f}(t)\right|\,\mathrm{d}t > 2B$$

for a suitable $\alpha > 0$.

Now, recall that $\widehat{f}(t)\widehat{f_n}(t) \xrightarrow{n\to\infty} \widehat{f}(t)$ holds for any $t \in \mathbb{R}$, where we also have $\left|\widehat{f_n}\right| \leqslant 1$. Thus, we have

$$\left|1_{[-\alpha,\alpha]}\widehat{f_n}\widehat{f}\right| \xrightarrow[n\to\infty]{\text{pointwise}} \left|1_{[-\alpha,\alpha]}\widehat{f}\right|$$

where $\left|1_{[-\alpha,\alpha]}\widehat{f_n}\widehat{f}\right|$ is dominated by $1_{[-\alpha,\alpha]}\left|\widehat{f}\right| \in L^1(\mathbb{R})$. By the dominated convergence theorem [10, Thm. 5.8], we thus get

$$\lim_{n\to\infty}\int_{-\alpha}^{\alpha} \left|\widehat{f_n}(t)\widehat{f}(t)\right|\,\mathrm{d}t = \int_{-\alpha}^{\alpha} \left|\widehat{f}(t)\right|\,\mathrm{d}t > 2B.$$

Consequently, there exists an $n \in \mathbb{N}$ with $\int_{-\alpha}^{\alpha} \left|\widehat{f_n}(t)\widehat{f}(t)\right|\,\mathrm{d}t > B$, which implies $\left\|\widehat{f_n}\widehat{f}\right\|_1 > B$ and thus (4).

Next, let $A > 0$ be fixed, and $C > 0$ some number that we shall specify later. If we choose some $B > AC$, there exists an $n \in \mathbb{N}$ such that $h = \widehat{f_n}\widehat{f}$ satisfies $\|h\|_1 \geqslant B > AC$. Then, for a suitable $a > 0$, we have $\int_{-a}^{a}|h(t)|\,\mathrm{d}t > AC$. With $h_1(t) := a h(at)$, we get

$$\int_{-1}^{1} |h_1(t)|\,\mathrm{d}t = \int_{-a}^{a}|h(u)|\,\mathrm{d}u > AC,$$

together with $\widehat{h_1}(t) = \widehat{h}(t/a) = (f_n * f)(-t/a)$, which also gives $\left\|\widehat{h_1}\right\|_\infty \leqslant 1$.

Fix some $\varphi \in C_c^\infty(\mathbb{R})$ with $\varphi \equiv 1$ on $[-1,1]$ and $\mathrm{supp}(\varphi) \subseteq [-2,2]$, and set $C := \|\widehat{\varphi}\|_1$, which clearly satisfies $C < \infty$. Now, set $g := C^{-1}\varphi h_1$, where $g \in C_c(\mathbb{R})$ is clear. Then,

$$\|g\|_1 = \frac{1}{C}\int_{\mathbb{R}} |\varphi(t)\,h_1(t)|\,\mathrm{d}t \geqslant \frac{1}{C}\int_{-1}^{1} |\varphi(t)\,h_1(t)|\,\mathrm{d}t = \frac{1}{C}\int_{-1}^{1} |h_1(t)|\,\mathrm{d}t > A.$$

This shows $\|g\|_1 \geqslant A$, and we also have $\mathrm{supp}(g) \subseteq \mathrm{supp}(\varphi) \subseteq [-2,2]$.

Finally, we have

$$\|\widehat{g}\|_\infty = C^{-1}\big\|\widehat{\varphi \cdot h_1}\big\|_\infty = C^{-1}\big\|\widehat{\varphi} * \widehat{h_1}\big\|_\infty \leqslant C^{-1}\big\|\widehat{\varphi}\big\|_1\big\|\widehat{h_1}\big\|_\infty = \big\|\widehat{h_1}\big\|_\infty \leqslant 1,$$

which proves the claim.   □

Remark 3.2. It is important to note that the function g, once A is large enough, cannot be a positive function. If it were, we would get $\widehat{g}(0) = \|g\|_1 > A$, in contradiction to $\|\widehat{g}\|_\infty \leqslant 1$. The analogous comment also applies to the function h_1 constructed in the proof.   ◇

This has the following consequence, which is also part of [1, Prop. 7.1].

Corollary 3.3. *There is a sequence $(g_n)_{n\in\mathbb{N}}$ of functions $g_n \in C_c(\mathbb{R})$ with the following three properties:* $\mathrm{supp}(g_n) \subseteq \big[-\frac{1}{n+1}, \frac{1}{n+1}\big]$, $\|g_n\|_1 \geqslant (n^2+1)^n$, *and* $\|\widehat{g_n}\|_\infty \leqslant 2^{-n}$.

Proof. Let $A > 0$ and g be as in Proposition 3.1, and let $\beta, \gamma > 0$ be arbitrary. If we set $h_{\beta,\gamma}(x) := \frac{1}{\beta} g(\gamma x)$, we get $\mathrm{supp}(h_{\beta,\gamma}) \subseteq \big[-\frac{2}{\gamma}, \frac{2}{\gamma}\big]$ together with

$$\big\|h_{\beta,\gamma}\big\|_1 = \beta^{-1} \int_\mathbb{R} |g(\gamma x)| \, \mathrm{d}x = \frac{\|g\|_1}{\beta\gamma} \geqslant \frac{A}{\beta\gamma}$$

and $\widehat{h_{\beta,\gamma}}(t) = \frac{1}{\beta\gamma}\widehat{g}(t/\gamma)$, hence $\big\|\widehat{h_{\beta,\gamma}}\big\|_\infty \leqslant \frac{1}{\beta\gamma}$. Choosing $\gamma = 2(n+1)$ and $\beta = 2^{n-1}/(n+1)$, the claim follows with $g_n = h_{\beta,\gamma}$ and $A = 2^n(n^2+1)^n$.   □

Let us next prove a result that is the key to go beyond the example of [1, Prop. 7.1]. First, let us recall that a sequence $(\mu_n)_{n\in\mathbb{N}}$ of measures is said to *converge vaguely* to a measure μ if, for all $\varphi \in C_c(\mathbb{R}^d)$, we have $\mu_n(\varphi) \xrightarrow{n\to\infty} \mu(\varphi)$.

Lemma 3.4. *Let $A > 0$ and let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of finite measures on $\mathbb{R}^d$ with the following three properties:* $\mathrm{supp}(\nu_n) \subseteq [-A, A]^d$, $|\nu_n|(\mathbb{R}^d) \geqslant (n^2+1)^n$, *and* $\|\widehat{\nu_n}\|_\infty \leqslant 2^{-n}$. *Set $v = (4A, 0, \ldots, 0)$ and define*

$$\mu_n = \sum_{m=1}^{n} \delta_{mv} * \nu_m.$$

Then, $(\mu_n)_{n\in\mathbb{N}}$ converges vaguely to a measure μ that is tempered but not slowly increasing.

Proof. Let $C_\mathsf{u}(\mathbb{R}^d)$ denote the space of bounded, uniformly continuous functions on $\mathbb{R}^d$ and consider $H_n = \widehat{\mu_n}$ as defined by

$$\widehat{\mu_n}(x) = \sum_{m=1}^{n} \mathrm{e}^{-2\pi\mathrm{i}\,mv\cdot x}\, \widehat{\nu_m}(x),$$

where $v \cdot x = 4Ax_1$, which clearly satisfies $H_n \in C_\mathsf{u}(\mathbb{R}^d)$. As $\|\widehat{\nu_n}\|_\infty < 2^{-n}$, the sequence $(H_n)_{n\in\mathbb{N}}$ converges, in $\big(C_\mathsf{u}(\mathbb{R}^d), \|.\|_\infty\big)$, to some $H \in C_\mathsf{u}(\mathbb{R}^d)$.

To continue, it follows immediately from the definition of μ_n that, for any $\varphi \in C_c(\mathbb{R}^d)$, there is an integer $N = N(\varphi)$ such that $\mu_N(\varphi) = \mu_{N+k}(\varphi)$ holds for all $k \geqslant 1$. Consequently,

$(\mu_n)_{n \in \mathbb{N}}$ is vaguely Cauchy, and hence vaguely convergent so some Radon measure μ on $\mathbb{R}^d$. Moreover, for any $\varphi \in C_{\mathsf{c}}(\mathbb{R}^d)$, one has

$$\mu(\varphi) \ = \ \mu_n(\varphi) \qquad \text{for all } n \geqslant N(\varphi),$$

with the $N(\varphi)$ from above. Next,

$$T_n(\psi) \ := \ \int_{\mathbb{R}^d} \psi(-t) \, H_n(t) \, \mathrm{d}t \quad \text{and} \quad T(\psi) \ := \ \int_{\mathbb{R}^d} \psi(-t) \, H(t) \, \mathrm{d}t$$

define tempered distributions, with $T_n \xrightarrow{n \to \infty} T$ in $\mathcal{S}'(\mathbb{R})$.

Further, by [13, Lemma 4.9.14] and [1, Prop. 3.1], we have

$$\mu_n(\varphi) \ = \ \int_{\mathbb{R}^d} \check{\varphi}(t) \, \mathrm{d}\widehat{\mu_n}(t) \ = \ T_n(\widehat{\varphi})$$

for all $\varphi \in C_{\mathsf{c}}^\infty(\mathbb{R}^d)$. This gives

$$\widehat{T}(\varphi) \ = \ T(\widehat{\varphi}) \ = \ \lim_{n \to \infty} T_n(\widehat{\varphi}) \ = \ \lim_{n \to \infty} \mu_n(\varphi) \ = \ \mu(\varphi),$$

which shows that μ is indeed tempered.

Finally, let $n \geqslant 3$ and let $\varphi \in C_{\mathsf{c}}(\mathbb{R}^d)$ satisfy $\mathrm{supp}(\varphi) \subseteq nv + \left[-A, A\right]^d$. Then, due to our construction, we have

$$\mu_m(\varphi) \ = \ \int_{\mathbb{R}^d} \varphi(x) \, \mathrm{d}\big(\delta_{nv} * \nu_n\big)(x) \qquad \text{for all } m > n,$$

and thus also $\mu(\varphi) = \int_{\mathbb{R}^d} \varphi(x) \, \mathrm{d}\big(\delta_{nv} * \nu_n\big)(x)$. This shows that

$$\mu\big|_{nv + [-A,A]^d} \ = \ \delta_{nv} * \nu_n.$$

In particular, for any fixed $P \in \mathbb{R}[x_1, \ldots, x_d]$, we get

$$\int_{\mathbb{R}^d} \frac{\mathrm{d}|\mu|(x)}{1 + |P(x)|} \ \geqslant \ \int_{[-A,A]^d} \frac{\mathrm{d}|\nu_n|(x_1, \ldots, x_d)}{1 + |P(x_1 - 4An, x_2, \ldots, x_d)|} \ \xrightarrow{n \to \infty} \ \infty,$$

where the last claim follows immediately from $|\nu_n|\big([-A, A]^d\big) \geqslant (n^2 + 1)^n$. This shows that μ cannot be slowly increasing. $\qquad\square$

Remark 3.5. By construction, the measures μ_n and μ from Lemma 3.4 satisfy the following simple property. For each compact set $K \subset \mathbb{R}^d$, there exists some integer $N = N(K)$ such that, for all $n > N$, we have $\mu|_K = \mu_n|_K$, where $\mu|_K$ denotes the restriction of μ to K. $\qquad\Diamond$

Now, setting $\nu_n = g_n \lambda_{\mathrm{L}}$ with g_n as in Corollary 3.3 and λ_{L} denoting Lebesgue measure, Proposition 3.4 gives the following concrete version of [1, Prop. 7.1].

Proposition 3.6. *Let g_n be as in Corollary 3.3 and consider the measures defined by*

$$\mu_n(\varphi) \ := \ \sum_{j=1}^n \int_{\mathbb{R}} \varphi(x) \, g_j(x + j) \, \mathrm{d}x.$$

Then, the sequence $(\mu_n)_{n \in \mathbb{N}}$ converges vaguely to a signed Radon measure μ that is tempered but not slowly increasing. $\qquad\square$

Later, in Example 4.7, we shall provide an example of a tempered measure with locally finite support that is not slowly increasing. Before we can do this, we need to discuss the case of measures with uniformly discrete support more generally.

4. Radon measures with uniformly discrete support

Here, we consider the important special case of measures with uniformly discrete support, for which the three key notions turn out to be equivalent. This class is particularly relevant in the theory of aperiodic order, with several applications to mathematical quasicrystals and Meyer sets; see [4, 8, 11, 12, 14, 18, 17] and references therein.

Note first that, if μ is tempered, strongly tempered, or slowly increasing, the same property holds for $\overline{\mu}$. This has the following immediate consequence.

Fact 4.1. *If μ is a Radon measure on $\mathbb{R}^d$, one has*

(1) μ *is tempered* $\iff$ $\mathrm{Re}(\mu)$ *and* $\mathrm{Im}(\mu)$ *are tempered;*

(2) μ *is slowly increasing* $\iff$ $\mathrm{Re}(\mu)$ *and* $\mathrm{Im}(\mu)$ *are slowly increasing.* □

To continue, we need a simple separation result as follows.

Lemma 4.2. *Let $U, V \subset \mathbb{R}^d$ be such that $U \cup V$ is uniformly discrete and $U \cap V = \varnothing$. Then, there exists a function $f \in C^\infty(\mathbb{R}^d)$ such that f and all its derivatives are bounded, together with $f(x) = 1$ for all $x \in U$ and $f(y) = 0$ for all $y \in V$.*

Proof. Let $r > 0$ be such that, for all $x, y \in U \cup V$ with $x \neq y$, we have $B_r(x) \cap B_r(y) = \varnothing$. Let $\varphi \in C_{\mathsf{c}}^\infty(\mathbb{R}^d)$ be so that $\varphi(0) = 1$ together with $\mathrm{supp}(\varphi) \subseteq B_r(0)$ and $\varphi(x) \in [0, 1]$ for all x, which exists by standard arguments.

Define $f = \sum_{u \in U} T_u \varphi$, where $(T_u \varphi)(x) = \varphi(x - u)$. Since $\mathrm{supp}(T_u \varphi) \subseteq B_r(u)$, where the $B_r(u)$ are pairwise disjoint open sets, it is immediate that f has the desired properties. □

Next, let us recall the following standard result, which we prove for convenience.

Fact 4.3. *Let $f \in C^\infty(\mathbb{R}^d)$ and $T \in \mathcal{S}'(\mathbb{R}^d)$. If f and all its derivatives are bounded, the mapping $\varphi \mapsto (fT)(\varphi) := T(f\varphi)$ defines a tempered distribution.*

Proof. Define $F \colon \mathcal{S}(\mathbb{R}^d) \longrightarrow C_{\mathsf{u}}(\mathbb{R}^d)$ via

$$F(\varphi) := f\varphi.$$

Let α, β be arbitrary multi-indices, with ordering defined componentwise, and set

$$\binom{\alpha}{\beta} = \prod_{i=1}^{d} \binom{\alpha_i}{\beta_i}.$$

Then, via the multivariate derivation formula of Leibniz, we have

$$\left|x^\beta D^\alpha(f\varphi)\right| = \left|x^\beta \sum_{\gamma \leqslant \alpha} \binom{\alpha}{\gamma}(D^\gamma f)(D^{\alpha-\gamma}\varphi)\right| \leqslant \sum_{\gamma \leqslant \alpha}\binom{\alpha}{\gamma}\left|(D^\gamma f)\,x^\beta(D^{\alpha-\gamma}\varphi)\right|$$
$$\leqslant \sum_{\gamma \leqslant \alpha}\binom{\alpha}{\gamma}\|D^\gamma f\|_\infty\,\|\varphi\|_{\beta,\alpha-\gamma}$$

for every $x \in \mathbb{R}^d$. This shows that $F\big(\mathcal{S}(\mathbb{R}^d)\big) \subseteq \mathcal{S}(\mathbb{R}^d)$ and that $F \colon \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{S}(\mathbb{R}^d)$ is continuous with respect to the Schwartz topology. In particular, $fT = T \circ F \in \mathcal{S}'(\mathbb{R}^d)$. $\quad\square$

The equivalence of the key notions in this case can now be stated as follows.

Theorem 4.4. *Let μ be a Radon measure on $\mathbb{R}^d$ with uniformly discrete support. If μ is tempered, it is also slowly increasing.*

Proof. Since μ is a tempered measure with uniformly discrete support, so are $\mathrm{Re}(\mu)$ and $\mathrm{Im}(\mu)$. If we show the latter to be slowly increasing, μ is slowly increasing by Facy 4.1. Thus, without loss of generality, we may assume μ to be a signed measure. Define

$$\Lambda_\pm = \{x \in \mathbb{R}^d : \mu(\{x\}) \gtrless 0\}.$$

Then, the set $\Lambda = \{x \in \mathbb{R}^d : \mu(\{x\}) \neq 0\}$, which is the support of μ and uniformly discrete by assumption, satisfies $\Lambda = \Lambda_+ \cup \Lambda_-$ together with $\Lambda_+ \cap \Lambda_- = \varnothing$.

If μ is tempered, there is a $T \in \mathcal{S}'(\mathbb{R}^d)$ such that $\mu(\varphi) = T(\varphi)$ holds for all $\varphi \in C_{\mathsf{c}}^\infty(\mathbb{R}^d)$. Let $f \in C^\infty(\mathbb{R}^d)$ be a function such that f and all its derivatives are bounded with $f|_{\Lambda_+} \equiv 1$ and $f|_{\Lambda_-} \equiv 0$, which is guaranteed to exist by Lemma 4.2, and set $g = 1 - f$, so also $g \in C^\infty(\mathbb{R}^d)$ and g and all its derivatives are bounded.

Setting $\mu_+ = f \cdot \mu$ and $\mu_- = (-g) \cdot \mu$, we get $\mu = \mu_+ - \mu_-$ where μ_+ and μ_- are positive Radon measures by construction. Further, for all $\varphi \in C_{\mathsf{c}}^\infty(\mathbb{R}^d)$, we have

$$\mu_+(\varphi) = \big(f\mu\big)(\varphi) = \mu(f\varphi) = T(f\varphi) = \big(fT\big)(\varphi).$$

Since $T \in \mathcal{S}'(\mathbb{R}^d)$ with $f \in C^\infty(\mathbb{R}^d)$ and f and all its derivatives are bounded, we have $fT \in \mathcal{S}'(\mathbb{R}^d)$ by Fact 4.3. Therefore, μ_+ is a positive, tempered measure, hence also slowly increasing.

In the same way, one gets $\mu_-(\varphi) = (gT)(\varphi)$, hence μ_- is slowly increasing as well. $\quad\square$

Explicitly, we can summarise the situation as follows.

Corollary 4.5. *Let μ be a Radon measure on $\mathbb{R}^d$ with uniformly discrete support. Then, the following properties are equivalent:*

    (1) *μ is slowly increasing;*

    (2) *μ is strongly tempered;*

    (3) *one has $|\psi| \in L^1(\mu)$ for all $\psi \in \mathcal{S}(\mathbb{R}^d)$;*

    (4) *one has $\psi \in L^1\big(|\mu|\big)$ for all $\psi \in \mathcal{S}(\mathbb{R}^d)$ with $\psi \geqslant 0$;*

    (5) *μ is tempered.* $\quad\square$

Remark 4.6. If $U \cup V$ is locally finite, looking at the the proof of Lemma 4.2, we can still select radii $r_u > 0$ for the points $u \in U$ such that $B_{r_u}(u) \cap (U \cup V) = \{u\}$. Further, we can find functions $\varphi_u \in C_c^\infty(\mathbb{R}^d)$ so that $\varphi_u(u) = 1$ together with $\mathrm{supp}(\varphi_u) \subseteq B_r(u)$ and $\varphi_u(x) \in [0, 1]$ for all x. Then, via $f = \sum_{u \in U} \varphi_u$, we get a function $f \in C^\infty(\mathbb{R}^d)$ that is bounded.

However, if $U \cup V$ is locally finite but *not* uniformly discrete, the radii r_u get arbitrarily close to zero. This forces the derivatives of f to become unbounded. Consequently, in the proof of Theorem 4.4, fT is a distribution that need no longer be tempered. This shows that our proof of Theorem 4.4 cannot be extended to general measures with locally finite support. In fact, we shall see in the next example that there exist tempered pure point measures with locally finite support that are not slowly increasing. $\diamond$

Employing the construction of [9], we now show that Theorem 4.4 does not hold for measures with locally finite support.

Example 4.7. For distinct, positive numbers $a, b \in \mathbb{R}$, consider $\mu_{a,b} := \delta_0 + \delta_a + \delta_b - \delta_{a+b}$. Then, as observed in [9], we have $\|\mu_{a,b}\| = 4$ and

$$(5) \qquad\qquad \|\widehat{\mu_{a,b}}\| \leqslant 2\sqrt{2},$$

because a simple calculation with $z = \mathrm{e}^{-2\pi\mathrm{i}a\cdot x}$ and $w = \mathrm{e}^{-2\pi\mathrm{i}b\cdot x}$ shows that

$$\|\widehat{\mu_{a,b}}\|^2 = (1 + z + w - zw)(1 + \overline{z} + \overline{w} - \overline{zw}) = 4 - \overline{zw} + z\overline{w} + w\overline{z} - zw,$$

which is a positive number, so we get $\|\widehat{\mu_{a,b}}\|^2 \leqslant 8$ via the triangle inequality.

Next, for each n, select numbers $a_1, \ldots, a_n, b_1, \ldots, b_n \in (0, 1]$ that are linearly independent over $\mathbb{Q}$. Then, the elements $k_1 a_1 + \ldots + k_n a_n + \ell_1 b_1 + \ldots + \ell_n b_n$ are distinct for all 2^{2n} choices of $k_1, \ldots, k_n$ and $\ell_1, \ldots, \ell_n$ in $\{0, 1\}$.

Now, consider

$$\nu_n := \mathop{\text{\Huge{*}}}_{i=1}^{n} \mu_{a_i, b_i}.$$

A simple computation shows that ν_n has the form

$$(6) \qquad \nu_n = \sum_{k_1,\ldots,k_n,\ell_1,\ldots,\ell_n \in \{0,1\}} s(k_1, \ldots, k_n, \ell_1, \ldots, \ell_n)\, \delta_{k_1 a_1 + \ldots + k_n a_n + \ell_1 b_1 + \ldots + \ell_n b_n}$$

with

$$s(k_1, \ldots, k_n, \ell_1, \ldots, \ell_n) = (-1)^{\mathrm{card}\{i:1\leqslant i\leqslant n, k_i=\ell_i=1\}} = \pm 1.$$

Since the Dirac measures on the RHS of (6) have pairwise disjoint supports, we get

$$\|\nu_n\| = \sum_{k_1,\ldots,k_n,\ell_1,\ldots,\ell_n \in \{0,1\}} |s(k_1, \ldots, k_n, \ell_1, \ldots, \ell_n)| = 2^{2n}.$$

Moreover, Eq. (5) implies

$$\|\widehat{\nu_n}\|_\infty = \Big\|\prod_{i=1}^{n} \widehat{\mu_{a_i,b_i}}\Big\|_\infty \leqslant 2^{\frac{3n}{2}}.$$

Now, for each $m \in \mathbb{N}$, pick some n such that $2^{\frac{n}{2}} \geqslant 2^m(m^2+1)^m$, and consider

$$\omega_m := \frac{\nu_n}{2^m \|\widehat{\nu_n}\|_\infty}.$$

Then, we get

$$\|\omega_m\| = \frac{\|\nu_n\|}{2^m \|\widehat{\nu_n}\|_\infty} \geqslant 2^{\frac{n}{2}-m} \geqslant (m^2+1)^m \qquad \text{and} \qquad \|\widehat{\omega_m}\|_\infty = 2^{-m}.$$

Further, by construction, $\mathrm{supp}(\omega_m) \subseteq [0,2] \subseteq [-2,2]$. Therefore, by Lemma 3.4, the measure

$$\mu = \sum_{m=1}^{\infty} \delta_{8m} * \omega_m$$

is tempered, but not slowly increasing. Moreover, since each ω_m has finite support, μ has locally finite support by Remark 3.5.

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