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Fourier transformable measures with weak Meyer set support and their lift to the cut-and-project scheme

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Abstract. In this paper, we prove that given a cut-and-project scheme $(G, H, \mathcal{L})$ and a compact window $W \subseteq H$, the natural projection gives a bijection between the Fourier transformable measures on $G \times H$ supported inside the strip $\mathcal{L} \cap (G \times W)$ and the Fourier transformable measures on G supported inside $\curlywedge(W)$. We provide a closed formula relating the Fourier transform of the original measure and the Fourier transform of the projection. We show that this formula can be used to re-derive some known results about Fourier analysis of measures with weak Meyer set support.

1 Introduction

After the discovery of quasicrystals [39], it has become clear that we need to better understand the process of diffraction. Mathematically, the diffraction pattern of a solid can be viewed as the Fourier transform $\widehat{\gamma}$ of the autocorrelation measure γ of the structure (see [13] for the setup and the monographs and see [3, 4] for a general review of the theory). The measure γ is positive-definite, and therefore it is Fourier transformable as a measure [1, 8, 31] with positive Fourier transform $\widehat{\gamma}$. It is this measure $\widehat{\gamma}$, which models the diffraction of our solid.

Structures with pure point diffraction, that is, structures for which $\widehat{\gamma}$ is a pure point measure, are now very well understood. Building on the earlier work of Gil deLamadrid–Argabright [10], Solomyak [40, 41], Lee–Moody–Solomyak [20], Baake–Moody [7], Baake–Lenz [6], Gouere [11, 12], Moody–Strungaru [30], and Meyer [27], pure point diffraction was characterized in [22, 23]. The focus now shifted toward models with mixed diffraction spectra, especially those with a large pure point part.

The best mathematical models for Delone sets with a large pure point spectrum and (generic) positive entropy are Meyer sets. They have been introduced in the pioneering work of Meyer [26], and popularized in the area of Aperiodic Order by Moody [28, 29] and Lagarias [18, 19]. They are usually constructed via a cut-and-project scheme (or simply CPS) and can be characterized via harmonic analysis, discrete geometry, algebra, and almost periodicity [26, 29, 46]. The basic idea behind a

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CPS is to project points from a higher-dimensional lattice, which lie within a bounded strip of the real space, into the real space (see Definition 2.8 for the exact definition). If the cross section of the strip (called the window) is regular, then the resulting model set is pure point diffractive [7, 14, 35, 38]. Recent work proved pure point diffractivity for a larger class of weak model sets [5, 15–17, 48].

As subsets of regular model sets, Meyer sets still exhibit a large pure point spectrum [43–47, 49] and a highly ordered continuous spectrum [43, 45, 47, 49]. The long-range order of the spectrum of Meyer sets can be traced to that of a covering regular model set [47, 49], and can be derived from the Poisson summation formula for the lattice in the CPS [3, 25, 34, 35].

One would expect it to be possible to relate the diffraction of a Meyer set (or more generally a measure with Meyer set support) directly to the lattice $\mathcal{L}$ in the CPS. It is the goal of this paper to establish this connection. Let us briefly explain our approach.

Fix a CPS $(G, H, \mathcal{L})$ and a compact set $W \subseteq H$. It is easy to see that

$$\gamma = \sum_{x \in \curlywedge(W)} c_x \delta_x \qquad \longleftrightarrow \qquad \eta = \sum_{x \in \curlywedge(W)} c_x \delta_{(x,x^\star)}$$

establishes a bijection between translation bounded measures supported inside $\curlywedge(W)$ and translation bounded measures supported inside $\mathcal{L} \cap (G \times W)$. We first show in Proposition 3.6 that γ is positive-definite if and only if η is positive-definite. Since each Fourier transformable measure supported inside a Meyer set can be written as a linear combination of positive-definite measures supported inside a common model set, we establish in Theorem 4.1 that γ is Fourier transformable if and only if η is Fourier transformable, and relate their Fourier transform (see (4.1)).

We complete the paper by discussing in Section 5 how these results can be used to re-derive the known properties of diffraction for measures with weak Meyer set support, and potentially used to prove new results.

2 Definitions and notations

Throughout the paper, G denotes a second countable locally compact Abelian group (LCAG). By $C_{\mathsf{u}}(G)$, we denote the space of uniformly continuous and bounded functions on G. This is a Banach space with respect to the sup norm $\|.\|_\infty$. As usual, we denote by $C_0(G)$ the subspace of $C_{\mathsf{u}}(G)$ consisting of functions vanishing at infinity, and by $C_{\mathsf{c}}(G)$ the subspace of compactly supported continuous functions. Note that $C_{\mathsf{c}}(G)$ is not complete in $(C_{\mathsf{u}}(G), \|.\|_\infty)$.

In the spirit of [10], we denote by

$$K_2(G) := \operatorname{Span}\{\varphi * \psi : \varphi, \psi \in C_{\mathsf{c}}(G)\} .$$

Given two LCAG's G and H and two functions $g : G \to \mathbb{C}$, $h : H \to \mathbb{C}$, we denote by $g \otimes h : G \times H \to \mathbb{C}$ their tensor product

$$(g \otimes h)(x, y) = g(x)h(y) .$$

It is obvious that whenever $\varphi \in C_{\mathsf{c}}(G)$, $\psi \in C_{\mathsf{c}}(H)$ we have $\varphi \otimes \psi \in C_{\mathsf{c}}(G \times H)$. Moreover, if $\varphi \in K_2(G)$ and $\psi \in K_2(H)$, we have $\varphi \otimes \psi \in K_2(G \times H)$.

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In the rest of this section, we review some of the basic concepts which are important for this paper. For a more general review of these, we recommend [3, 4].

2.1 Measures

In the spirit of Bourbacki [9], by a measure, we understand a linear functional on $C_{\mathsf{c}}(G)$ which is continuous with respect to the inductive topology. This notion corresponds to the classical concept of a Radon measure (see [35, Appendix]). For the case $G = \mathbb{R}^d$, a clear exposition of this is given in [3].

Definition 2.1 A linear functional $\mu : C_{\mathsf{c}}(G) \to \mathbb{C}$ is called a *Radon measure* (or simply a *measure*) if for each compact set $K \subseteq G$ there exists a constant C_K such that, for all $\varphi \in C_{\mathsf{c}}(G)$ with $\operatorname{supp}(\varphi) \subseteq K$, we have

$$|\mu(\varphi)| \leq C_K \|\varphi\|_\infty \,.$$

We will often write $\int_G \varphi(t) d\mu(t)$ instead of $\mu(\varphi)$.

A measure μ is called *positive* if for all $\varphi \in C_{\mathsf{c}}(G)$ with $\varphi \geq 0$ we have $\mu(\varphi) \geq 0$.

By the Riesz representation theorem [37], a positive Radon measure is simply a positive regular Borel measure. Moreover, each Radon measure is a linear combination of (at most four) positive Radon measures [35, Appendix].

Next, we review the total variation of a measure.

Definition 2.2 Given a measure μ, we can define [32, 33, 35] a positive measure $|\mu|$, called the *total variation* of μ, such that, for all $\varphi \in C_{\mathsf{c}}(G)$ with $\varphi \geq 0$, we have

$$|\mu|\,(\varphi) = \sup\{|\mu(\psi)| : \psi \in C_{\mathsf{c}}(G),\ \text{with } |\psi| \leq \varphi\} \,.$$

We are now ready to introduce the concept of translation boundedness for measures and norm almost periodicity.

Definition 2.3 Let $A \subseteq G$ be a fixed precompact set with nonempty interior. We define the *A-norm* of μ via

$$\|\mu\|_A := \sup_{x \in G} |\mu|\,(x + A) \,.$$

A measure μ is called *translation bounded* if $\|\mu\|_A < \infty$.

Remark 2.4 ([7, 42]) Different precompact sets A_1, A_2 with nonempty interior define equivalent norms. Therefore, the definition of translation boundedness does not depend on the choice of A.

This allows us to define

$$\mathcal{M}^\infty(G) := \{\mu : \mu \text{ is a translation bounded measure}\} \,.$$

Then $(\mathcal{M}^\infty(G), \|.\|_A)$ is a normed space. It is in fact a Banach space [36].

Next, we review the definition of norm almost periodicity as introduced in [7].

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Definition 2.5 Let $A \subseteq G$ be a fixed precompact set with nonempty interior. A measure $\mu \in \mathcal{M}^{\infty}(G)$ is called *norm almost periodic* if, for each $\varepsilon > 0$, the set

$$P_{\varepsilon}^{A}(\mu) := \{t \in G : \|T_t\mu - \mu\|_A < \varepsilon\}$$

of *ε-norm almost periods* of μ is relatively dense.

As discussed above, different precompact sets define equivalent norms. This means that while the set of ε-norm almost periods on μ depends on the choice of A, the almost periodicity of μ is independent of this choice.

Any norm almost periodic measure is strongly almost periodic [7], and the two concepts are equivalent for measures with Meyer set support [7]. In general, norm almost periodicity is an uniform version of strong almost periodicity [42, Theorem 4.7]. The class of norm almost periodic pure point measure was studied in detail and characterized in [46].

Let us next recall positive-definiteness for functions and measures. For more details, we recommend [8, 31].

Definition 2.6 A function $f : G \longrightarrow \mathbb{C}$ is called *positive-definite* if, for all $n \in \mathbb{N}$ and all $x_1, \dots, x_n \in G$, the matrix $(f(x_k - x_l))_{k,l=1,\dots,n}$ is positive Hermitian. This is equivalent to

$$\sum_{k,l=1}^{n} \overline{c_l} f(x_k - x_l) c_k \geq 0 \qquad \forall n \in \mathbb{N}, x_1, \dots, x_n \in G, c_1, \dots, c_n \in \mathbb{C}\,.$$

A measure μ is called *positive-definite* if, for all $\varphi \in C_{\mathsf{c}}(G)$, we have

$$\mu(\varphi * \widetilde{\varphi}) \geq 0\,.$$

This is equivalent to $\mu * \varphi * \tilde{\varphi}$ being a positive-definite function for all $\varphi \in C_{\mathsf{c}}(G)$ [8, 31].

We complete the subsection by reviewing the notion of Fourier transformability for measures. For a more detailed review of the subject, we recommend [31].

Definition 2.7 A measure μ on G is called *Fourier transformable* if there exists a measure $\widehat{\mu}$ on $\widehat{G}$ such that, for all $\varphi \in K_2(G)$, we have $|\check{\varphi}| \in L^1(|\widehat{\mu}|)$ and

$$\int_G \varphi(t)\, \mathrm{d}\mu(t) = \int_{\widehat{G}} \check{\varphi}(\chi)\, \mathrm{d}\widehat{\mu}(\chi)\,.$$

2.2 Cut-and-project schemes and Meyer sets

In this part, we review some notions related to the cut-and-project formalism. For more details, we recommend [3, 28, 29].

Definition 2.8 By a *CPS*, we understand a triple $(G, H, \mathcal{L})$ consisting of a second countable LCAG G, an LCAG H, and a lattice $\mathcal{L} \subseteq G \times H$ such that:

(i) $\pi_H(\mathcal{L})$ is dense in H.
(ii) The restriction $\pi_G|_{\mathcal{L}}$ of the first projection π_G to $\mathcal{L}$ is one to one.

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Given a CPS $(G, H, \mathcal{L})$, we will denote by $L := \pi_G(\mathcal{L})$. Then, π_G induces a group isomorphism between $\mathcal{L}$ and L. Composing the inverse of this with the second projection π_H, we get a mapping

$$\star : L \to H,$$

which we will call the $\star$*-mapping*. We then have

$$\mathcal{L} = \{(x, x^\star) : x \in L\}\,.$$

Given a CPS $(G, H, \mathcal{L})$ and a subset $W \subseteq H$ we can define

$$\curlywedge(W) := \{x \in L : x^\star \in W\}\,.$$

When W is precompact, we will call $\curlywedge(W)$ a *weak model set*. If furthermore W has nonempty interior $\curlywedge(W)$ is called a *model set*.

Next, let us review the concept of a Meyer set, which plays a fundamental role in this paper.

Definition 2.9 A set $\Lambda \subseteq G$ is called a *Meyer set* if Λ is relatively dense and $\Lambda - \Lambda - \Lambda$ is uniformly discrete.

For equivalent characterizations of Meyer sets, see [18, 19, 26, 28, 46]. Of importance to us will be the following result.

Theorem 2.10 ([46]) *Let $\Lambda \subseteq G$ be relatively dense. Then Λ is Meyer if and only if it is a subset of a (weak) model set.*

Moreover, if Λ is Meyer, it is a subset of a weak model set in a CPS $(G, H, \mathcal{L})$ with metrizable and compactly generated H.

We should emphasize here that the key for all results below is that fact that a Meyer set is a subset of a model set, and relative denseness plays no role. Because of this, in the spirit of [49], we will refer to an arbitrary subset of a (weak) model set as a *weak Meyer set*. It is obvious that a subset of a weak Meyer set is a weak Meyer set and that a measure is supported inside a Meyer set if and only if its support is a weak Meyer set.

Given a CPS $(G, H, \mathcal{L})$, the map

$$L \ni x \to (x, x^\star) \in \mathcal{L} \tag{2.1}$$

is a group isomorphism, and hence it induces an isomorphism between the spaces of (bounded) functions on L and $\mathcal{L}$, respectively. Since $\mathcal{L}$ is a discrete group, the space of (translation bounded) measures on $\mathcal{L}$ can be identified with the space of (bounded) functions on $\mathcal{L}$. On another hand, L is typically dense in G, and many functions on L do not induce pure point measures on G.

For us, of interest will be measures supported inside weak model sets $\curlywedge(W)$. Since $\curlywedge(W)$ is uniformly discrete [28], the space of (translation bounded) measures on $\curlywedge(W)$ can be identified with the space of (bounded) functions on $\curlywedge(W)$, and corresponds via the above isomorphism with the spaces of (translation bounded) measures or (bounded) functions on $\mathcal{L}$, respectively, that are supported inside $G \times W$.

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Our focus in this paper is on these two spaces. We will study them as spaces of measures, and we will be interested in the relation between the Fourier theory of these two spaces, and the behavior of the Fourier transform with respect to the isomorphism induced by (2.1). For this reason, let us introduce the following notations.

Given a CPS $(G, H, \mathcal{L})$ and a compact set W, we denote by

$$\begin{aligned}\mathcal{M}^\infty(\curlywedge(W)) &:= \{\mu \in \mathcal{M}^\infty(G) : \operatorname{supp}(\mu) \subseteq \curlywedge(W)\};\\ \mathcal{M}^\infty(\mathcal{L}; W) &:= \{\nu \in \mathcal{M}^\infty(G\times H) : \operatorname{supp}(\nu) \subseteq (\mathcal{L} \cap (G \times W))\}.\end{aligned}$$

The isomorphism (2.1) induces a bijection $f : \curlywedge(W) \to \mathcal{L} \cap (G \times W)$. This induces a bijective map $\mathbb{L}_{G,H,\mathcal{L},W} : \mathcal{M}^\infty(\curlywedge(W)) \to \mathcal{M}^\infty(\mathcal{L}; W)$, taking a measure on $\curlywedge(W)$ into its pushforward via f, defined by

$$\mathbb{L}_{G,H,\mathcal{L},W}(\mu) = \sum_{(x,x^\star)\in\mathcal{L}} \mu(\{x\})\delta_{(x,x^\star)},$$

with inverse $\mathbb{P}_{G,H,\mathcal{L},W} : \mathcal{M}^\infty(\mathcal{L}; W) \to \mathcal{M}^\infty(\curlywedge(W))$

$$\mathbb{P}_{G,H,\mathcal{L},W}(\nu) = \sum_{(x,x^\star)\in\mathcal{L}} \nu(\{(x,x^\star)\})\delta_x .$$

Let us note here in passing that $\mathbb{P}_{G,H,\mathcal{L},W}$ is simply the pushforward via f^{-1}.

We will refer to these mappings as the *lift operator* and the *projection operator*, respectively. When the CPS and window are clear from the context, we will simply write $\mathbb{L}(\mu)$ and $\mathbb{P}(\nu)$, respectively, instead of $\mathbb{L}_{G,H,\mathcal{L},W}(\mu)$ and $\mathbb{P}_{G,H,\mathcal{L},W}(\nu)$, respectively.

The main results in this paper are that these operators are bijections between the subspaces of Fourier transformable (or cones of positive-definite) measures, and relate their Fourier transforms.

To understand the connection between the Fourier transforms, let us recall the notion of dual CPS. Given a CPS $(G, H, \mathcal{L})$, we can define

$$\mathcal{L}^0 := \{(\chi,\psi) \in \widehat{G} \times \widehat{H} : \chi(x)\psi(x^\star) = 1\, \forall x \in L\}.$$

Then, $(\widehat{G}, \widehat{H}, \mathcal{L}^0)$ is a CPS [5, 28, 29, 46]. We will refer to this as the CPS *dual to* $(G, H, \mathcal{L})$.

3 Positive-definite measures with weak Meyer set support

In this section, we show that $\mathbb{L}_{G,H,\mathcal{L},W}$ and $\mathbb{P}_{G,H,\mathcal{L},W}$ take positive-definite measures to positive-definite measures.

Let us start with the following obvious lemma, which follows immediately from Definition 2.6 and the fact that the function from (2.1) is a group isomorphism.

Lemma 3.1 *Let $(G, H, \mathcal{L})$ be a CPS, and let $f : L \to \mathbb{C}$ be a function. Define $g : \mathcal{L} \to \mathbb{C}$ via*

$$g(x, x^\star) := f(x).$$

Then f is positive-definite on L if and only if g is positive-definite on $\mathcal{L}$.

Let us recall now the following result of [24], which we will use often in the paper.

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Proposition 3.2 ([24, Proposition 2.4]) *Let G be a LCAG, let μ be a discrete measure on G, and let*

$$f(x) := \mu(\{x\}) .$$

Then, the following are equivalent:

(i) *The measure μ is a positive-definite measure on G.*
(ii) *The measure μ is a positive-definite measure on G_d.*
(iii) *The function f is a positive-definite function on G.*
(iv) *The function f is a positive-definite function on G_d.*

Next, we prove a slight generalization of [24, Lemma 2.10] and [3, Lemma 8.4].

Lemma 3.3 *Let γ be a positive-definite pure point measure on G, and let L be any subgroup of G. Then, the function $g : L \to \mathbb{C}$ defined via*

$$g(x) := \gamma(\{x\})$$

is a positive-definite function on L.

Proof Define $f : G \to \mathbb{C}$ via

$$f(x) := \gamma(\{x\}) .$$

Then f is a positive-definite function on G Proposition 3.2. Definition 2.6 immediately gives that the restriction $g = f|_L$ to the subgroup L is a positive-definite function on L. ■

We will also need the following result.

Lemma 3.4 *Let G be any group, and let $H \leq G$ be a subgroup. Let $f : H \to \mathbb{C}$ be a positive-definite functions. Then, the function $g : G \to \mathbb{C}$ defined via*

$$g(x) := \begin{cases} f(x), & \text{if } x \in H, \\ 0, & \text{otherwise} \end{cases}$$

is positive-definite on G.

Proof Let $n \in \mathbb{N}, x_1, \ldots, x_n \in G$ and $c_1, .., c_n \in \mathbb{C}$. Note that $g(x_k - x_l) = 0$ whenever $x_k - x_l \notin H$.

On G define the standard equivalence (mod H) as

$$x \equiv y \pmod H \Leftrightarrow x - y \in H .$$

This induces an equivalence relation on the set $\{x_1, \ldots, x_n\}$, and hence we can partition this set in equivalence classes $F_1, \ldots, F_m$.

To make the computation clearer, define $c : G \to \mathbb{C}$

$$c(x) := \begin{cases} c_j, & \text{if } x = x_j, \\ 0, & \text{otherwise.} \end{cases}$$

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Then,

$$\begin{aligned}\sum_{k,l=1}^{n} g(x_k - x_l) c_k \overline{c_l} &= \sum_{k,l=1}^{n} g(x_k - x_l) c(x_k) \overline{c(x_l)} \\ &= \sum_{i=1}^{m} \sum_{j=1}^{m} \left(\sum_{x \in F_i} \sum_{y \in F_j} g(x-y) c(x) \overline{c(y)} \right) \\ &= \sum_{i=1}^{m} \left(\sum_{x,y \in F_i} g(x-y) c(x) \overline{c(y)} \right).\end{aligned}$$

Now, fix some $1 \leq i \leq m$, and let $F_i := \{z_1, .., z_q\}$. Then,

$$\begin{aligned}\sum_{x,y \in F_i} g(x-y) c(x) \overline{c(y)} &= \sum_{r,s=0}^{q} g(z_r - z_s) c(z_r) \overline{c(z_s)} \\ &= \sum_{r,s=1}^{q} g(z_r - z_s) c(z_r) \overline{c(z_s)} \\ &= \sum_{r,s=1}^{q} f\left((z_r - z_1) - (z_s - z_1)\right) c(z_r) \overline{c(z_s)} \geq 0\end{aligned}$$

by the positive-definiteness of f applied to m; $y_1 := z_1 - z_1$; $y_2 := z_2 - z_1; \ldots; y_q := z_q - z_1 \in H$ and $c'_1 = c(z_1), \ldots, c'_q = c(z_q)$.

Therefore, for each i, we have $\sum_{x,y \in C_i} g(x-y) c(x) \overline{c(y)} \geq 0$, and hence

$$\sum_{k,l=1}^{n} g(x_k - x_l) c_k \overline{c_l} = \sum_{i=1}^{m} \left(\sum_{x,y \in C_i} g(x-y) c(x) \overline{c(y)} \right) \geq 0. \qquad \blacksquare$$

Remark 3.5 One can also prove Lemma 3.4 by using Fourier analysis. Indeed, since f is positive-definite, the measure $\mu := f\theta_{H_{\mathrm{d}}}$ is a positive-definite measure on the discrete group H_{d} [1, Corollary 4.3]. Then, it is Fourier transformable on H_{d} and its Fourier transform is positive [1, 8]. As H_{d} is closed in the discrete group G_{d}, by [1, Theorem 4.2], the measure $\nu := g\theta_{G_{\mathrm{d}}}$ is Fourier transformable on G_{d} and has positive Fourier transform. Then, μ is positive-definite [1, Theorem 4.1]. Therefore, by Proposition 3.2, g is positive-definite on G.

We are now ready to prove the following result.

Proposition 3.6 *Let $(G, H, \mathcal{L})$ be a CPS, let $\curlywedge(W)$ be a weak model set, and let $f : G \to \mathbb{C}$ be a function which vanishes outside $\curlywedge(W)$. Let*

$$\begin{aligned}\gamma &= \sum_{x \in \curlywedge(W)} f(x) \delta_x, \\ \eta &= \sum_{(x,x^\star) \in \mathcal{L}} f(x) \delta_{(x,x^\star)} = \mathbb{L}(\gamma).\end{aligned}$$

Then γ is a positive-definite measure on G if and only if η is a positive-definite measure on $G \times H$.

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Proof $\Rightarrow$: Denote as usual $L := \pi_G(\mathcal{L})$. Define $g : L \to \mathbb{C}$ via

$$g(x) := \gamma(\{x\}),$$

that is, $g = f|_L$.

Then, by Lemma 3.3, g is a positive-definite function on L and hence, by Lemma 3.1, the function $h : \mathcal{L} \to \mathbb{C}$

$$h(x, x^\star) = g(x)$$

is a positive-definite function on $\mathcal{L}$. Therefore, by Lemma 3.4, the function $j : G \times H \to \mathbb{C}$

$$j(z) := \begin{cases} h(z), & \text{if } z \in \mathcal{L}, \\ 0, & \text{otherwise} \end{cases}$$

is positive-definite on $G \times H$. The claim follows from Proposition 3.2.

$\Leftarrow$: Since η is positive-definite, by Lemma 3.3, the function $h : \mathcal{L} \to \mathbb{C}$ defined by

$$h(x, x^\star) := \eta(\{(x, x^\star)\}) = f(x),$$

is positive-definite on $\mathcal{L}$ and hence, by Lemma 3.1, the restriction $g = f|_L$ is positive-definite on L. As f is zero outside $\curlywedge(W) \subseteq L$, it follows from Lemma 3.4 that f is a positive-definite function on G. The claim follows now from Proposition 3.2. ∎

Remark 3.7 (a) In Proposition 3.6, the positive-definiteness of η and γ is equivalent to the positive-definiteness of the function f.

(b) Denoting by

$$\mathcal{PD}(\curlywedge(W)) := \{\mu \in \mathcal{M}^\infty(\curlywedge(W)) : \mu \text{ is positive-definite}\},$$
$$\mathcal{PD}(\mathcal{L}; W) := \{\nu \in \mathcal{M}^\infty(\mathcal{L}, W) : \nu \text{ is positive-definite}\}.$$

Proposition 3.6 says that

$$\mathbb{L}(\mathcal{PD}(\curlywedge(W))) = \mathcal{PD}(\mathcal{L}; W),$$
$$\mathbb{P}(\mathcal{PD}(\mathcal{L}; W)) = \mathcal{PD}(\curlywedge(W)).$$

4 The lift of Fourier transformable measures

We can now prove that, given a CPS $(G, H, \mathcal{L})$ and a compact set K, the lifting operator induces a bijection between the space of Fourier transformable measures supported inside $\curlywedge(W)$ and the space of Fourier transformable measures supported inside $\mathcal{L} \cap (G \times W)$.

Theorem 4.1 *Let $(G, H, \mathcal{L})$ be a CPS, and let $W \subseteq H$ be compact. Let γ be a translation bounded measure supported inside $\curlywedge(W)$, and let*

$$\eta := \mathbb{L}_{G,H,\mathcal{L},W}(\gamma).$$

Then γ is Fourier transformable if and only if η is Fourier transformable.

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Moreover, if $\varphi \in K_2(H)$ *is any function so that* $\varphi \equiv 1$ *on* W*, then, for all* $\psi \in C_c(\widehat{G})$*, we have* $\psi \otimes \hat{\varphi} \in L^1(\widehat{\eta})$ *and*

$$\widehat{\gamma}(\psi) = \widehat{\eta}(\psi \otimes \check{\varphi}) =: (\widehat{\eta})_{\check{\varphi}}(\psi)\,. \tag{4.1}$$

Proof $\Longrightarrow$ By [47, Lemma 8.3], there exist a compact set $W \subseteq K$ and four positive-definite measures $\omega_1, \omega_2, \omega_3, \omega_4$ supported inside $\curlywedge(K)$ such that

$$\gamma = \omega_1 - \omega_2 + i\omega_3 - i\omega_4\,.$$

Then, we have

$$\begin{aligned}\eta &= \mathbb{L}_{G,H,\mathcal{L},W}(\gamma) = \mathbb{L}_{G,H,\mathcal{L},K}(\gamma) = \mathbb{L}_{G,H,\mathcal{L},K}(\omega_1 - \omega_2 + i\omega_3 - i\omega_4)\\ &= \mathbb{L}_{G,H,\mathcal{L},K}(\omega_1) - \mathbb{L}_{G,H,\mathcal{L},K}(\omega_2) + i\mathbb{L}_{G,H,\mathcal{L},K}(\omega_3) - i\mathbb{L}_{G,H,\mathcal{L},K}(\omega_4)\,.\end{aligned}$$

Now, by Proposition 3.6, for all $1 \le j \le 4$, the measure $\mathbb{L}_{G,H,\mathcal{L},K}(\omega_j)$ is positive-definite. Therefore, as a linear combination of positive-definite measures, η is Fourier transformable.

$\Longleftarrow$. Our argument is similar to [34].

First, fix an arbitrary $\varphi \in K_2(H)$ so that $\varphi \equiv 1$ on W. We split the rest of the argument into two steps.

Step 1: We show that $(\widehat{\eta})_{\check{\varphi}}$ is a measure.

Let us first note that for all $\psi \in K_2(G)$, we have $\psi \otimes \varphi \in K_2(G \times H)$. Therefore, since η is Fourier transformable, we have

$$|\check{\psi} \otimes \check{\varphi}| \in L^1(|\widehat{\eta}|)\,. \tag{4.2}$$

We now show that for all $\phi \in C_c(\widehat{G})$, we have $|\phi \otimes \check{\varphi}| \in L^1(|\widehat{\eta}|)$ and that

$$(\widehat{\eta})_{\check{\varphi}}(\phi) := \widehat{\eta}(\phi \otimes \check{\varphi})$$

defines a measure.

Let $K \subseteq \widehat{G}$ be a fixed compact set. Then, there exists some $\psi \in K_2(G)$ such that $\check{\psi} \ge 1_K$ [8, 31].

Now, for all $\phi \in C_c(\widehat{G})$ with $\operatorname{supp}(\phi) \subseteq K$, we have

(4.3)

$$|\widehat{\eta}|\,(|\phi \otimes \check{\varphi}|) = \int_{\hat{G}\times\hat{H}} |\phi(s)| \cdot |\check{\varphi}(t)|\,\mathrm{d}|\widehat{\eta}|(s,t) \le \|\phi\|_\infty \int_{\hat{G}\times\hat{H}} |\check{\psi}(s)| \cdot |\widehat{\varphi}(t)|\,\mathrm{d}|\widehat{\eta}|(s,t) < \infty,$$

and hence $(\widehat{\eta})_{\check{\varphi}}$ is well defined.

Moreover, for all $\phi \in C_c(\widehat{G})$ with $\operatorname{supp}(\phi) \subseteq K$, it follows from (4.3) that

$$\left|(\widehat{\eta})_{\check{\varphi}}(\phi)\right| \le C_K \|\phi\|_\infty,$$

where

$$C_K := \int_{\hat{G}\times\hat{H}} \left|\check{\psi}(s)\right| \cdot |\widehat{\varphi}(t)|\,\mathrm{d}|\widehat{\eta}|(s,t) < \infty\,.$$

This shows that $(\widehat{\eta})_{\check{\varphi}}$ is a measure.

Step 2: We show that for all $\phi \in K_2(G)$, we have $|\check{\phi}| \in L^1(|(\widehat{\eta})_{\check{\varphi}}|)$ and

$$(\widehat{\eta})_{\check{\varphi}}(\check{\phi}) = \gamma(\phi)\,.$$

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Let $\phi \in K_2(G)$ be arbitrary.

Since G is second countable, so is $\widehat{G}$ [33]. In particular, $\widehat{G}$ is σ-compact [33]. Therefore, there exists a sequence K_n of compact sets with $K_n \subseteq (K_{n+1})^\circ$ such that

$$\widehat{G} = \bigcup_n K_n.$$

Let $\psi_n \in C_c(\widehat{G})$ be so that $1_{K_n} \le \psi_n \le 1_{K_{n+1}}$.

Then, $\psi_n \widehat{\phi} \in C_c(\widehat{G})$ and by the definition of $(\breve{\eta})_{\check{\varphi}}$, we have

$$(\widehat{\eta})_{\check{\varphi}}(\psi_n \check{\phi}) = \widehat{\eta}\left((\psi_n \check{\phi}) \otimes \check{\varphi}\right).$$

Now, for all n, we have by (4.2)

$$\left|(\psi_n \check{\phi}) \otimes \check{\varphi}\right| \le \left|\check{\phi}\right| \otimes |\check{\varphi}| \in L^1(|\widehat{\eta}|).$$

Therefore, by the dominated convergence theorem [33, Theorem 3.2.51], we have

$$\widehat{\eta}\left(\check{\phi} \otimes \check{\varphi}\right) = \lim_n \widehat{\eta}\left((\psi_n \check{\phi}) \otimes \check{\varphi}\right) = \lim_n (\widehat{\eta})_{\check{\varphi}}\left(\psi_n \check{\phi}\right). \tag{4.4}$$

Next, by the monotone convergence theorem [33], we have

$$|(\widehat{\eta})_{\check{\varphi}}|\left(|\check{\phi}|\right) = \lim_n |(\widehat{\eta})_{\check{\varphi}}|\left(|\psi_n \check{\phi}|\right).$$

Note that for each n, we have

$$\begin{aligned}
|(\widehat{\eta})_{\check{\varphi}}|\left(|\psi_n \check{\phi}|\right) &= \sup\{\left|(\widehat{\eta})_{\check{\varphi}}(\Psi)\right| : \Psi \in C_c(\hat{G}), |\Psi| \le |\psi_n \check{\phi}|\} \\
&= \sup\{|\widehat{\eta}\,(\Psi \otimes \check{\varphi})| : \Psi \in C_c(\hat{G}), |\Psi| \le |\psi_n \check{\phi}|\} \\
&= \sup\left\{\left|\int_{\hat{G}\times\hat{H}} \Psi(x)\check{\varphi}(y)\mathrm{d}\widehat{\eta}(x,y)\right| : \Psi \in C_c(\hat{G}), |\Psi| \le |\psi_n \check{\phi}|\right\} \\
&\le \sup\left\{\int_{\hat{G}\times\hat{H}} |\Psi(x)\check{\varphi}(y)|\mathrm{d}|\widehat{\eta}|(x,y) : \Psi \in C_c(\hat{G}), |\Psi| \le |\psi_n \check{\phi}|\right\} \\
&\le \int_{\hat{G}\times\hat{H}} |\psi_n \check{\phi}(x)\check{\varphi}(y)|\mathrm{d}|\widehat{\eta}|(x,y) \le \int_{\hat{G}\times\hat{H}} |\check{\phi}(x)\check{\varphi}(y)|\mathrm{d}|\widehat{\eta}|(x,y) \\
&= |\widehat{\eta}|\left(|\check{\phi} \otimes \check{\varphi}|\right).
\end{aligned}$$

Since $\check{\phi} \otimes \check{\varphi} \in L^1(|\widehat{\eta}|)$, we get

$$|(\widehat{\eta})_{\check{\varphi}}|\left(|\check{\phi}|\right) \le |\widehat{\eta}|\left(|\check{\phi} \otimes \check{\varphi}|\right) < \infty.$$

This shows that $|\check{\phi}| \in L^1(|(\widehat{\eta})_{\check{\varphi}}|)$. Therefore, $\psi_n \check{\phi}$ is dominated by $|\check{\phi}| \in L^1(|(\widehat{\eta})_{\check{\varphi}}|)$ and converges pointwise to $\check{\phi}$. Thus, by (4.4) and the dominated convergence theorem, we get

$$\widehat{\eta}\left(\check{\phi} \otimes \check{\varphi}\right) = \lim_n (\widehat{\eta})_{\check{\varphi}}\left(\psi_n \check{\phi}\right) = (\widehat{\eta})_{\check{\varphi}}\left(\check{\phi}\right).$$

Finally, by the Fourier transformability of η, we have

$$\widehat{\eta}\left(\check{\phi} \otimes \check{\varphi}\right) = \eta\,(\phi \otimes \varphi) = \gamma(\phi).$$

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Therefore, we proved that for all $\phi \in K_2(G)$, we have $\check{\phi} \in L^1(|(\widehat{\eta})_{\check{\varphi}}|)$ and

$$(\widehat{\eta})_{\check{\varphi}}\left(\check{\phi}\right) = \gamma(\phi)\,.$$

This proves that γ is Fourier transformable and

$$\widehat{\gamma} = (\widehat{\eta})_{\check{\varphi}}\,,$$

completing the proof. ■

Using the fact that $\mathbb{L}$ is a bijection with inverse $\mathbb{P}$, we get the following corollary.

Corollary 4.2 *Let $(G,H,\mathcal{L})$ be a CPS, and let $W \subseteq H$ be compact. Let η be a translation bounded measure supported inside $\mathcal{L} \cap (G \times W)$, and let $\gamma = \mathbb{P}_{G,H,\mathcal{L},W}(\eta)$. Then η is Fourier transformable if and only if γ is Fourier transformable.*

Moreover, if $\varphi \in K_2(H)$ is any function so that $\varphi \equiv 1$ on W, then, for all $\psi \in C_\mathsf{c}(\widehat{G})$, we have $\psi \otimes \hat{\varphi} \in L^1(\widehat{\eta})$ and (4.1) holds.

5 Applications

In this section, we will discuss the relation (4.1) and how can it be used to (re)derive some results from [47].

To make the things easier to follow, we introduce the notion of strongly admissible functions.

5.1 Strongly admissible functions for CPS

Let us start with the following definition.

Definition 5.1 Given a group H of the form $H = \mathbb{R}^d \times H_0$, with a LCAG H_0, a function $f : H \to \mathbb{C}$ is called *strongly admissible* if there exists $g \in C_\mathsf{u}(\mathbb{R}^d)$ and $\varphi \in C_\mathsf{c}(H_0)$ such that:

- $\|(1+|x|^{2d})g\|_\infty < \infty$.
- $f = g \otimes \varphi$.

Next, given a CPS $(G,H,\mathcal{L})$, we will denote by $\mathcal{M}_\mathcal{L}(G \times H)$, the space of $\mathcal{L}$-periodic measures on $G \times H$. Note that by [21, Proposition 6.1]

$$\mathcal{M}_\mathcal{L}(G \times H) \subseteq \mathcal{M}^\infty(G \times H)\,.$$

We will see below that given a Fourier transformable measure γ with weak Meyer set support, Theorem 4.1 can be used to create a CPS $(G, H = \mathbb{R}^d \times H_0, \mathcal{L})$, an $\mathcal{L}^0$-periodic measure $\rho(=\widehat{\eta})$ and a strongly admissible function f on $\widehat{H} = \mathbb{R}^d \times \widehat{H_0}$ such that, equation (4.1) yields

$$\gamma = (\rho)_f\,.$$

This motivates us to closely look at the properties of $(\rho)_f$, for a CPS $(G, H = \mathbb{R}^d \times H_0, \mathcal{L})$, $\rho \in \mathcal{M}_\mathcal{L}(G \times H)$ and strongly admissible functions f.

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Let us start with the following simple observation which also explains the name "strongly admissible."

Given a CPS $(G, H = \mathbb{R}^d \times H_0, \mathcal{L})$, a measure $\rho \in \mathcal{M}_{\mathcal{L}}(G \times H)$, and strongly admissible function *f*, it is obvious that the function f is admissible for $(G, H, \mathcal{L}, \rho)$ in the sense of [21, Definition 3.1]. Therefore, by [21, Proposition 6.3], we can define a translation bounded measure ρ_f on G via

$$\rho_f(\phi) := \rho(\phi \otimes f) \qquad \forall \phi \in C_c(G).$$

This measure is strongly almost periodic by [21, Theorem 3.1]. In fact, the strong admissibility of f immediately implies that ρ_f is norm almost periodic.

Indeed, let $(G, H = \mathbb{R}^d \times H_0, \mathcal{L})$, let $f = g \otimes \varphi$ be strongly admissible, and let $\rho \in \mathcal{M}_{\mathcal{L}}(G \times H)$. Pick any compact set $\operatorname{supp}(\varphi) \subseteq W \subseteq \widehat{H_0}$, and let $K, K_1 \subseteq G$ be compact sets in G with $K \subseteq K_1^\circ$. Then, a standard computation similar to [47, Lemma 5.2] shows that

$$\|(\rho)_f\|_K \leq C \|\varphi\|_\infty \|(1 + |x|^{2d}) g\|_\infty \|\rho\|_{K_1 \times [-\frac{1}{2}, \frac{1}{2}]^d \times W},$$

where

$$C := \left(\sum_{n \in \mathbb{Z}^d} \sup_{z \in n + [-\frac{1}{2}, \frac{1}{2}]^d} \frac{1}{1 + |z|^{2d}} \right) < \infty.$$

This immediately gives the following stronger version of [47, Lemma 5.2].

Fact 5.2 Let $(G, H = \mathbb{R}^d \times H_0, \mathcal{L})$ be a CPS, let $\rho \in \mathcal{M}_{\mathcal{L}}(G \times H)$, and let $f \in C_0(H)$ be strongly admissible. Then, ρ_f is a norm almost periodic measure.

5.2 Fourier transform of measures with weak Meyer set support

Fix an arbitrary Meyer set Λ and a Fourier transformable measure γ with $\operatorname{supp}(\gamma) \subseteq \Lambda$.

By Theorem 2.10 and the structure theorem of compactly generated groups, there exists a CPS $(G, \mathbb{R}^d \times \mathbb{Z}^m \times \mathbb{K}, \mathcal{L})$ with compact $\mathbb{K}$ and a compact $W \subseteq \mathbb{R}^d \times \mathbb{Z}^m \times \mathbb{K}$ such that

$$\Lambda \subseteq ⋏(W).$$

By eventually enlarging *W*, we can assume without loss of generality that

$$W = W_0 \times F \times \mathbb{K}$$

for compact $W_0 \subseteq \mathbb{R}^d$ and finite $F \subseteq \mathbb{Z}^m$.

Set $H_0 = \mathbb{Z}^m \times \mathbb{K}$. It is easy to see that we can find function $\varphi \in C_c^\infty(\mathbb{R}^d) \cap K_2(\mathbb{R}^d)$ and $\psi \in K_2(H_0)$ with the following properties:

- $\phi := \varphi \otimes \psi \equiv 1$ on W.
- $\widehat{\psi} \in C_c(\widehat{H_0})$.

It follows that

$$f := \check{\phi} = \check{\varphi} \otimes \check{\psi} \tag{5.1}$$

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is a strongly admissible function of $\widehat{H} = \mathbb{R}^d \times \widehat{H_0}$.

Next, define $\eta := \mathbb{L}_{G,\mathbb{R}^d \times H,\mathcal{L},W}(\gamma)$. Then, by Theorem 4.1, η is Fourier transformable. Moreover, since $\operatorname{supp}(\eta) \subseteq \mathcal{L}$, the measure $\rho = \widehat{\eta}$ is $\mathcal{L}^0$-periodic by [10, Proposition 6.1]. Finally, (4.1) gives

$$\widehat{\gamma} = (\rho)_f \,. \tag{5.2}$$

Fact 5.2 then gives the following result.

Corollary 5.3 ([47, Theorem 7.1]) *Let γ be a measure with weak Meyer set support. Then, $\widehat{\gamma}$ is norm almost periodic.*

5.3 Generalized Eberlein decomposition

In this subsection, we show a pseudo-compatibility of the mapping $\rho \to (\rho)_f$ of (5.3), for $\mathcal{L}$ periodic $\rho \in \mathcal{M}_{\mathcal{L}}(G \times H)$ and strongly admissible f, with respect to the Lebesgue decomposition. We explain this, as well as our meaning of "pseudo-compatibility" below.

First, it is easy to see that the map satisfies:

- if ρ is pure point, then $(\rho)_f$ is pure point;
- if ρ is absolutely continuous, then $(\rho)_f$ is absolutely continuous;
- if ρ is singular continuous, then $(\rho)_f$ can have all three spectral components;

and hence does not preserves the Lebesgue decomposition. On another hand, for each $\alpha \in \{\mathsf{pp}, \mathsf{ac}, \mathsf{sc}\}$, one can defined an operator P_α on the space $\mathcal{M}^\infty_{\mathcal{L}}(G \times H)$ with the property that for all strongly admissible f and all $\rho \in \mathcal{M}^\infty_{\mathcal{L}}(G \times H)$, we have

$$(P_\alpha(\rho))_f = \left((\rho)_f\right)_\alpha \,. \tag{5.3}$$

This can be done simply by first showing that

$$L_\alpha(\sum_{j=1}^m c_j \psi_j \otimes \phi_j) := \sum_{j=1}^m c_j \left(\rho_{\phi_j}\right)_\alpha (\psi_j)$$

for all $c_1, \ldots, c_m \in \mathbb{C}$, $\psi_1, \ldots, \psi_m \in C_\mathsf{c}(\widehat{G})$, $\phi_1, \ldots, \phi_m \in C_\mathsf{c}(\widehat{H})$ is well defined, linear, and continuous with respect to the inductive topology. Therefore, L_α can be uniquely extended to a measure $P_\alpha(\rho)$, which is $\mathcal{L}$ invariant and satisfies (5.3).

Now, exactly as above, let γ be a Fourier transformable measure supported inside a Meyer set Λ, and let $(G, H, \mathcal{L})$, $\eta, \varphi, \phi, \psi$ be as in Section 5.2. Let f be as in (5.1), and let $\rho = \widehat{\eta}$.

Then, for each $\alpha \in \{\mathsf{pp}, \mathsf{ac}, \mathsf{sc}\}$, the measure $P_\alpha(\rho)$ is the Fourier transform of some measure μ supported on $\mathcal{L}^0$ [36].

Define

$$\nu := \sum_{x \in L} \phi(x^\star) \mu(\{(x, x^\star)\}) \delta_x \,.$$

Then, $\operatorname{supp}(\nu) \subseteq \curlywedge(\operatorname{supp}(\phi))$ and, exactly as in the proof of Theorem 4.1, we get

$$\widehat{\nu} = (\widehat{\mu})_f = (P_\alpha(\widehat{\eta}))_f = ((\widehat{\eta})_h)_\alpha = (\widehat{\gamma})_\alpha \,.$$

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Therefore, we get the following corollary.

Corollary 5.4 ([47, Theorem 4.1]) *Let γ be a Fourier transformable measure supported inside a Meyer set Λ. Then, there exist a model set $\Gamma \supseteq \Lambda$ and three Fourier transformable measures $\gamma_{\mathsf{s}}, \gamma_{0\mathsf{s}}, \gamma_{0\mathsf{a}}$ supported inside Γ such that*

$$\gamma_{\mathsf{s}} = (\widehat{\gamma})_{\mathsf{pp}}\,,$$
$$\gamma_{0\mathsf{s}} = (\widehat{\gamma})_{\mathsf{sc}}\,,$$
$$\gamma_{0\mathsf{a}} = (\widehat{\gamma})_{\mathsf{ac}}\,.$$

5.4 Discussion

We have seen in this section that the Fourier transform of a measure γ with weak Meyer set support can be describe via (5.2) as the projection in the dual CPS of a $\mathcal{L}^0$-periodic measure via a strongly admissible function. We used this result to (re)derive properties of $\widehat{\gamma}$, and we expect that this connection will lead to some new applications in the future. Indeed, while now we know quite a few properties of the Fourier transform of measures with weak Meyer set support [2, 43–49], we know much more about fully periodic measures in LCAG (see, for example, [36]). Moreover, the strong admissibility of f is likely to transfer many properties from ρ to ρ_f. It is also worth pointing out that, while the strong admissibility of f was sufficient to derive the conclusions in this section, in fact f can be chosen of the form

$$f := g \otimes P \otimes \psi : \mathbb{R}^d \times \mathbb{T}^m \times \widehat{\mathbb{K}} \to \mathbb{C}\,,$$

where $g = \hat{\varphi} \in \mathcal{S}(\mathbb{R}^d)$ is the Fourier transform of some $\varphi \in C_{\mathsf{c}}^{\infty}(\mathbb{R}^d)$; P is a trigonometric polynomial, that is, a sum of characters, that is, $P = \sum_{j=1}^{m} \chi_j$ for some $\chi_1, \ldots, \chi_j \in \widehat{\mathbb{T}^m}$ and $\psi \in C_{\mathsf{c}}(\widehat{\mathbb{K}})$ is the characteristic function of $\{0\}$. These properties are much stronger than strong admissibility, and have the potential to lead to nice applications in the future.

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